Backward Invariance for Linear Differential Algebraic Equations

Stefano Tognazzi[1], Mirco Tribastone[1], Max Tschaikowski[2] and Andrea Vandin[3]

***Abstract*— Differential-algebraic equations (DAEs) are a widespread dynamical model that describes continuously evolving quantities defined with differential equations, subject to constraints expressed through algebraic relationships. As such, DAEs arise in many fields ranging from physics, chemistry, and engineering. In this paper we focus on linear DAEs, and develop a theory for their minimization up to an equivalence relation. We present *backward invariance*, which relates DAE variables that have equal solutions at all time points (thus requiring them to start with equal initial conditions) and extends the line of research on backward-type bisimulations developed for Markov chains and ordinary differential equations. We apply our results to the electrical engineering domain, showing that backward invariance can explain symmetries in certain networks as well as analyze DAEs which could not be originally treated due to their size.**

I. Introduction

Differential-algebraic equations (DAEs) are a popular model of dynamical systems across many branches of science and engineering. They often arise when describing physical quantities that evolve continuously and deterministically according to an ordinary differential equation (ODE), together with constraints that express properties such as conservation of energy. An example is *RLC* electric circuits, where components such as capacitors and inductors are associated with first-order linear differential equations while the algebraic constraints impose Kirchhoff's conservation laws for voltages in closed loops and currents at each junction [1]. A more general approach that leads to DAEs as the underlying dynamics is that of *bond graphs* (e.g., [2]), a unifying model for networked physical systems such as electrical, mechanical, and hydraulic networks.

Solving DAEs is difficult [3]. Compared to ODEs, the initial condition may not be chosen arbitrarily (provided that the solution exists), but it has to be *consistent*, i.e. it has to satisfy the algebraic constraints. This leads to expensive implicit schemes that require the solution of a system of equations at each time step, thus motivating a large body of literature on model reduction techniques for DAEs (cf. [4], [5], [6] and references therein). Such techniques are appealing for numerical purposes since they were proven to be effective for reducing models of real large-scale systems. However, the reduction is approximate in general and the reduced model may not carry physical intelligibility (because its variables represent a linear transformation of the original state space). This limits their use when the modeler strives to gain mechanistic insights from the model, for example to explain symmetric behavior in a design [7], [8].

In this paper we present a technique for reducing linear DAE systems which is both exact and *structure-preserving*, in the sense that the reduction is obtained through a quotienting up to an equivalence that relates variables that have the same solutions at all time points. Our line of research can be related to aggregation of Markov chains [9], [10] and backward differential equivalence (BDE) for ODE systems [11]. Unfortunately, while the notion of BDE can be generalized to *algebraic backward invariance* (ABI) by working on the equivalent linear ODE system into which a linear DAE system can be transformed (see Section III), ABI requires the solution of an ill-conditioned problem [12], [13].

We circumvent this issue by developing in Section IV a criterion for backward equivalence over the *numerical solution scheme* of the DAE system, instead of the DAE system itself. The idea is to find a *numeric ABI* by relating numerical solutions of variables that are equal at all steps, through inspection of the explicit law that updates the solution at each time step. Using the popular backward differentiation formulae as the numerical solver [14], we prove that a numeric ABI is an ABI of the original DAE system, independently of the discretization time step used in the numerical solver. Moreover, we show that computing the coarsest numeric ABI through a partition refinement algorithm reduces to computing the coarsest BDE of a linear ODE system that is related to the explicit update law of the numerical solver (hence, it can be computed in polynomial time [15], [16]).

We apply our theory to benchmark DAEs from electric circuit theory by showing that numeric ABI is i) effective in discovering symmetries in the topology of the electrical networks and ii) allows one to analyze circuits that would otherwise issue out-of-memory errors.

Further related work. In [17], the author proposes a technique for the minimization of polynomial ODE systems in the spirit of BDE but which does not apply to DAE systems. Instead, the recent work [18] extends classic bisimulation relations for linear systems [19], [20], [21] to linear DAE systems. While similar in style, [18] is not comparable to ABI. This is because [18] cannot be used to show that certain variables enjoy identical solutions at all time points; conversely, unlike [18], ABI cannot be used to show that a linear observation map (i.e., $y = Cx$ for some matrix C) coincides the solution of the reduced system. The symmetry property underlying ABI and BDE can be related to Φ-related vector fields [22]. However, ABI and BDE impose constraints on initial conditions and establish a reduction of a system instead of relating two given systems, see Section VI.

[1] S. Tognazzi and M. Tribastone are with IMT Lucca, Italy: name.surname@imtlucca.it

[2] M. Tschaikowski is with TU Wien, Austria: max.tschaikowski@tuwien.ac.at

[3] A. Vandin is with DTU Compute, Denmark: anvan@dtu.dk

Paper outline. Section II sets the scene by presenting background knowledge on BDE and linear DAEs. Building on that, Section III introduces ABI, while Section IV introduces numeric ABI and presents an efficient algorithm for its computation. A thorough evaluation of numeric ABI follows in Section V where it is applied to benchmark models from practice. After providing further discussion on related work in Section VI, the paper concludes in Section VII.

II. Preliminaries

Notation. The time derivative of a variable x_i is denoted by $\dot{x}_i$. The set of variables is given by $\{x_i \mid i \in \mathcal{S}\}$, where $\mathcal{S}$ is some finite index set; partitions of $\mathcal{S}$ are denoted by $\mathcal{H}$ and $\mathcal{G}$. A partition $\mathcal{H}$ of $\mathcal{S}$ refines a partition $\mathcal{G}$ if and only if each block of $\mathcal{H}$ is a subset of some block of $\mathcal{G}$. We write B^A for the set of functions from A to B. Following the standard notation, $\mathbb{C}$ and x^T denote the complex numbers and the transpose of a vector $x \in \mathbb{C}^{\mathcal{S}}$, respectively; instead, for an $i \in \mathcal{S}$, $e_i \in \mathbb{R}^{\mathcal{S}}$ is such that $e_i(j) = 1$ if $i = j$ and 0 otherwise. For a given function $f : A \to B$ and a set $C \subseteq A$, $f(C)$ defines the image of f under C, i.e., $f(C) = \{f(x) \mid x \in C\}$. The subspace spanned by the columns of $C \in \mathbb{R}^{\mathcal{S}\times\mathcal{S}}$ is denoted by $\langle C \rangle$.

Backward Differential Equivalence. The following notion has been introduced in [11].

Definition 1. *Fix a linear ODE system $\dot{x} = Ax + b$ with $A \in \mathbb{R}^{\mathcal{S}\times\mathcal{S}}$ and $b \in \mathbb{R}^{\mathcal{S}}$. We call*

- *$x \in \mathbb{R}^{\mathcal{S}}$ uniform on $\mathcal{H}$ when $x_i = x_j$ for all $i, j \in H$ and $H \in \mathcal{H}$;*
- *$\mathcal{U}_{\mathcal{H}}$ the subspace of all vectors that are uniform on $\mathcal{H}$;*
- *$\mathcal{H}$ a BDE partition if, for any initial condition $x(0) \in \mathcal{U}_{\mathcal{H}}$, the ODE solution yields $x(t) \in \mathcal{U}_{\mathcal{H}}$ for all $t > 0$.*

Informally, a partition $\mathcal{H}$ of ODE variables $\mathcal{S}$ is a BDE if any initial condition $x(0) \in \mathbb{R}^{\mathcal{S}}$ which is uniform on $\mathcal{H}$ gives rise to a solution that is uniform on $\mathcal{H}$. For the sake of presentation, we recast it now to linear ODE systems using the concept of invariant spaces as has been done in [23].

BDE can be characterized in terms of *backward invariance*.

Theorem 1 (Backward Invariance)**.** *Given an ODE system $\dot{x} = Ax + b$, a partition $\mathcal{H}$ of $\mathcal{S}$ is called backward invariance (BI) whenever $A(\mathcal{U}_{\mathcal{H}}) + b \subseteq \mathcal{U}_{\mathcal{H}}$. It holds that $\mathcal{H}$ is a BDE if and only if $\mathcal{H}$ is a backward invariance.*

The following result follows from [15], [16] and allows for an efficient computation of the coarsest backward invariance.

Theorem 2. *Fix a linear ODE system $\dot{x} = Ax + b$ and let $\mathcal{G}$ be some partition of $\mathcal{S}$. Then, the coarsest BI partition $\mathcal{H}$ that refines $\mathcal{G}$ exists and can be computed in polynomial time via a partition refinement algorithm.*

Linear DAEs. Systems of linear differential algebraic equations (DAEs) satisfy the form $E\dot{x} = Ax + b$ where $E, A \in \mathbb{R}^{\mathcal{S}\times\mathcal{S}}$ and $b \in \mathbb{R}^{\mathcal{S}}$. In case E is invertible, a linear DAE system can be directly recast into a linear ODE system via $\dot{x} = E^{-1}Ax + E^{-1}b$. If E is not invertible, however, the transformation corresponds to the computation of the Kronecker normal form, an instance of index reduction [14], [24], which is ill-conditioned in general [13], unless additional assumptions [12] are imposed.

We summarize several facts from the theory of DAEs [14].

Definition 2. *A DAE system $E\dot{x} = Ax + b$ is called regular if, for every initial condition $x(0) \in \mathbb{R}^{\mathcal{S}}$, there exists either a unique solution or no solution at all.*

Theorem 3. *A DAE system $E\dot{x} = Ax + b$ is regular if and only if $A - \lambda E$ is invertible for at least one $\lambda \in \mathbb{C}$. The set of initial conditions for which a regular DAE admits solutions, $\mathcal{D}$, is an affine subspace of $\mathbb{R}^{\mathcal{S}}$. For any regular DAE system $E\dot{x} = Ax + b$, there exist $\hat{A} \in \mathbb{R}^{\mathcal{S}\times\mathcal{S}}$ and $\hat{b} \in \mathbb{R}^{\mathcal{S}}$ such that for any $x(0) \in \mathcal{D}$, the solutions of $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}$ and $E\dot{x} = Ax + b$ coincide when subject to $x(0)$.*

The above results ensure that any linear DAE system $E\dot{x} = Ax + b$ can be expressed as a linear ODE system $\dot{x} = \hat{A}x + \hat{b}$. As mentioned before, the computation of $\hat{A}$, $\hat{b}$ and the domain $\mathcal{D}$ relies on the Kronecker normal form [12], [13].

Example 1. *Consider the DAE system $E\dot{x} = Ax$ where*

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ -3 & 0 & -9 \end{pmatrix} \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix} \tag{1}$$

Then, it can be proven that $E\dot{x} = Ax$ admits a unique solution only when $x(0) \in \mathcal{D} = \langle (1,0,0)^T, (0,-2,1)^T \rangle$. Moreover, it can be shown that the corresponding solution satisfies the linear ODE system $\dot{x} = \hat{A}x$, where $\hat{A} = ((3,2,-1)^T, (0,0,0)^T, (2,0,0)^T)$.

III. Algebraic Backward Invariance

We start by lifting the notion of BDE to linear DAEs.

Definition 3. *Let $E\dot{x} = Ax + b$ be a regular DAE. A partition $\mathcal{H}$ of $\mathcal{S}$ is called a BDAE of $E\dot{x} = Ax + b$ if $\mathcal{U}_{\mathcal{H}} \cap \mathcal{D} \neq \emptyset$ and, for any initial condition $x(0) \in \mathcal{U}_{\mathcal{H}} \cap \mathcal{D}$, it holds that $x(t) \in \mathcal{U}_{\mathcal{H}} \cap \mathcal{D}$ for all $t > 0$.*

It is interesting to note that, in contrast to BDE, the notion of BDAE has to account for the domain $\mathcal{D}$. This is because only initial conditions $x(0)$ from $\mathcal{D}$ induce a unique solution of $E\dot{x} = Ax + b$. Moreover, it already suffices to ask for $x(t) \in \mathcal{U}_{\mathcal{H}}$ for all $t > 0$ instead of $x(t) \in \mathcal{U}_{\mathcal{H}} \cap \mathcal{D}$ for all $t > 0$. This is because any solution x of a regular DAE system $E\dot{x} = Ax + b$ has to be contained in $\mathcal{D}$, see Theorem 3.

Example 2. *Consider the running example (1) and let $\mathcal{H} = \{\{1,2\},\{3\}\}$. Then, for any initial condition $(2\alpha, 2\alpha, -\alpha) \in \mathcal{U}_{\mathcal{H}} \cap \mathcal{D} = \langle (2,2,-1)^T \rangle$, where $\mathcal{D}$ is as in Example 1, the solution of the DAE system is given by $x(t) = e^{2t}(2\alpha, 2\alpha, -\alpha)$. This can be verified by plugging x into $E\dot{x} = Ax$. Hence, $\mathcal{H}$ is a BDAE of (1).*

Since a linear DAE $E\dot{x} = Ax + b$ can be expressed in terms of the underlying ODE system $\dot{x} = \hat{A}x + \hat{b}$, it is tempting to call a partition $\mathcal{H}$ of $E\dot{x} = Ax + b$ algebraic backward invariance whenever $\mathcal{H}$ is a backward invariance of $\dot{x} = \hat{A}x + \hat{b}$. While this can be shown to be a sufficient

condition for BDAE, however, we next observe that it does not characterize BDAE.

Example 3. *Consider the running example (1). From Example 2, we know that $\mathcal{H} = \{\{1,2\},\{3\}\}$ is a BDAE. With $\hat{A}$ being as in Example 1, however, it can be easily seen that $\hat{A}(\mathcal{U}_\mathcal{H}) \not\subseteq \mathcal{U}_\mathcal{H}$. At the same time, it is interesting to note that $\hat{A}(\mathcal{U}_\mathcal{H} \cap \mathcal{D}) \subseteq \mathcal{U}_\mathcal{H}$.*

We hence define algebraic backward invariance as follows.

Definition 4. *Let $E\dot{x} = Ax + b$ be a regular DAE and let $\dot{x} = \hat{A}x + \hat{b}$ be the corresponding ODE system. We call $\mathcal{H}$ algebraic backward invariance (ABI) of $E\dot{x} = Ax + b$ whenever $\hat{A}(\mathcal{U}_\mathcal{H} \cap \mathcal{D}) + \hat{b} \subseteq \mathcal{U}_\mathcal{H}$ and $\mathcal{U}_\mathcal{H} \cap \mathcal{D} \neq \emptyset$.*

The following generalization of Theorem 1 states that ABI characterizes BDAE.

Theorem 4. *Let $E\dot{x} = Ax+b$ be a regular DAE and let $\dot{x}=\hat{A}x+\hat{b}$ be the underlying ODE system. A partition $\mathcal{H}$ is an ABI if and only if it is a BDAE.*

Note that the constraint $\mathcal{U}_\mathcal{H} \cap \mathcal{D} \neq \emptyset$ in Definition 4 ensures that there actually exist initial conditions to which an ABI partition can be applied.

We proceed by providing the notion of a reduced model underlying an ABI.

Definition 5. *Given a regular DAE system $E\dot{x} = Ax + b$, let $\dot{x} = \hat{A}x + \hat{b}$ be the underling ODE system and let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be a BDAE partition. For $H \in \mathcal{H}$, let $i_H \in H$ be the representative of H and $\mathcal{S}_0 = \{i_H \mid H \in \mathcal{H}\}$. Set further $e_H \in \mathbb{R}^\mathcal{S}$ by $e_H(i) = 1$ if $i \in H$ and zero otherwise.*

Then, the reduced DAE system $\underline{E}\,\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}$ and the reduced ODE system $\dot{\underline{x}} = \hat{\underline{A}}\,\underline{x} + \hat{\underline{b}}$ are given by $\underline{E} = S_l E S_r$, $\underline{A} = S_l A S_r$, $\underline{b} = S_l b$, $\hat{\underline{A}} = S_l \hat{A} S_r$ and $\hat{\underline{b}} = S_l \hat{b}$, where $\underline{E}, \underline{A}, \hat{\underline{A}} \in \mathbb{R}^{\mathcal{S}_0 \times \mathcal{S}_0}$ and $\underline{b}, \hat{\underline{b}} \in \mathbb{R}^{\mathcal{S}_0}$ and the transformation matrices $S_l \in \mathbb{R}^{\mathcal{S}_0 \times \mathcal{S}}$, $S_r \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}_0}$ are defined via $S_l = (e_{i_{H_1}}, \ldots, e_{i_{H_m}})^T$ and $S_r = (e_{H_1}, \ldots, e_{H_m})$. The domain of the reduced DAE system is $\underline{\mathcal{D}} = \{S_l x \mid x \in \mathcal{U}_\mathcal{H} \cap \mathcal{D}\}$.

The following can be shown.

Theorem 5. *Given a regular DAE system $E\dot{x} = Ax + b$ and let $\dot{x} = \hat{A}x + \hat{b}$ be the underlying ODE system. Then, for a BDAE $\mathcal{H}$, the reduced models $\underline{E}\,\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}$ and $\dot{\underline{x}} = \hat{\underline{A}}\,\underline{x} + \hat{\underline{b}}$ can be computed in polynomial time. Moreover, for any $x(0) \in \mathcal{U}_\mathcal{H} \cap \mathcal{D}$, the solution of $E\dot{x} = Ax + b$ is uniquely related to the solution of $\underline{E}\,\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}$ and $\dot{\underline{x}} = \hat{\underline{A}}\,\underline{x} + \hat{\underline{b}}$ via $\underline{x}(0) = S_l x(0)$. In particular, $\underline{x}(t) = S_l x(t)$ and $x(t) = S_r \underline{x}(t)$ for all $t \geq 0$.*

Example 4. *From Example 2, we know that $\mathcal{H} = \{H_1, H_2\} = \{\{1,2\},\{3\}\}$ is a BDAE of (1). Together with $i_{H_1} = 1$ and $i_{H_2} = 3$, we conclude that*

$$\underline{E} = \begin{pmatrix} 0 & 1 \\ -3 & -9 \end{pmatrix} \quad \underline{A} = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \quad \hat{\underline{A}} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

$$S_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S_r = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{\mathcal{D}} = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rangle$$

Since the solution of the running example is given by $x(t) = e^{2t}(2\alpha, 2\alpha, -\alpha)^T$ for all $(2\alpha, 2\alpha, -\alpha)^T \in \mathcal{U}_\mathcal{H} \cap \mathcal{D}$, it holds that $S_l x(t) = e^{2t}(2\alpha, -\alpha)^T = \underline{x}(t)$. A direct computation confirms that $\underline{x}$ solves $\underline{E}\,\underline{x} = \underline{A}\underline{x}$ and $\underline{x} = \hat{\underline{A}}\,\underline{x}$ for all $\underline{x}(0) = (2\alpha, -\alpha)^T \in \underline{\mathcal{D}}$. Moreover, it can be easily seen that $x(t) = S_r \underline{x}(t)$ for all $\underline{x}(0) \in \underline{\mathcal{D}}$.

IV. Numeric Algebraic Backward Invariance

While ABI provides a generalization of BI to linear DAEs, it has to be noted that checking whether a given partition is ABI requires the knowledge of $\mathcal{D}$. Since the computation of $\mathcal{D}$ relies on the solution of an ill-conditioned problem [13], this section introduces numeric ABI. Numeric ABI focusses on numerical solutions of linear DAE systems.

Backward Differentiation Formulae. A popular family of numerical schemes to solve linear DAE systems are the so-called backward differential formulae (BDF) [14]. The general idea is, as in the case of all numerical schemes, to approximate the true solution of a given linear DAE system $E\dot{x} = Ax + b$ on an interval $[0; T]$ by a sequence of points $x[0], x[1], \ldots, x[M]$ such that $x[i]$ approximates $x(t_i)$, where $\Delta t = T/M$ and $t_i = i\Delta t$ for all $0 \leq i \leq M$.

The next result [14, Theorem 5.24] ensures 1) that the BDF solution $(x[i])_i$ is well-defined if the time step Δt is sufficiently small and; 2) that the numeric solution $(x[i])_i$ converges to the solution of $E\dot{x} = Ax+b$ if $x(0) = x[0] \in D$.

Theorem 6. *Given a regular linear DAE system $E\dot{x}=Ax+b$ and an initial condition $x(0) \in \mathbb{R}^\mathcal{S}$, fix $1 \leq k \leq 6$ and set $x[-6] = x[-5] = \ldots = x[-1] = 0$. Then, there exist $\Delta t_0 > 0$ and $C > 0$ that do not depend on $\Delta t > 0$ and which satisfy the following.*

1) *For all $0 < \Delta t \leq \Delta t_0$ there exists a unique $x[i]$ solving the linear system of equations*

$$(A - \tfrac{\alpha(k,0)}{\Delta t} E)x[i] = E\Big(\sum_{l=1}^{k} \tfrac{\alpha(k,l)}{\Delta t} x[i-l]\Big) - b$$

   for all $1 \leq i \leq M$, where the k-BDF coefficients $\alpha(k,\cdot)$ are given in [14];
2) *if x denotes the solution subject to a consistent initial condition $x(0) \in D$, then it holds that $\max_{0 \leq i \leq M} \|x(t_i) - x[i]\|_\infty \leq C\Delta t^k$ for all $0 < \Delta t \leq \Delta t_0$.*

Remark 1. *The BDF schemes apply an auto-correction in the case an inconsistent initial condition $x(0)$ is provided. In particular, if $x(0) \notin D$ and $\nu \geq 1$ denotes the DAE index [14] of $E\dot{x} = Ax + b$, Theorem 6 remains true and it holds [14, Remark 5.25] that $x[i] \in \mathcal{D}$ for all $\nu \leq i \leq M$.*

Armed with the BDF scheme and the convergence result of Theorem 6, we introduce numeric BDAE and numeric ABI. Intuitively, numeric BDAE and numeric ABI take the roles of BDAE and ABI, respectively, if Δt is sufficiently small. This is because the ground truth given in terms of the DAE system $E\dot{x} = Ax+b$ and the underlying analytical solution x is replaced, in practical computations, with the BDF scheme of Theorem 6 and the numerical solution $(x[i])_i$.

Definition 6. *Fix a regular DAE system $E\dot{x} = Ax + b$, time step $\Delta t > 0$ and $1 \leq k \leq 6$ such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible, set $\Phi(x) := (A - \frac{\alpha(k,0)}{\Delta t}E)^{-1}Ex - (A - \frac{\alpha(k,0)}{\Delta t}E)^{-1}b$. A partition $\mathcal{H}$ is called*

- *numeric BDAE if, for any $x[0] \in \mathcal{U}_{\mathcal{H}}$, it holds that $x[i] \in \mathcal{U}_{\mathcal{H}}$ for all $1 \leq i \leq M$;*
- *numeric ABI if $\Phi(\mathcal{U}_{\mathcal{H}}) \subseteq \mathcal{U}_{\mathcal{H}}$.*

Note that $x[i] = \Phi(\sum_{l=1}^{k} \frac{\alpha(k,l)}{\Delta t}x[i-l])$ for all $1 \leq i \leq M$. As for the algebraic notions, the following result can be stated for the numeric ones.

Theorem 7. *Given a regular DAE system $E\dot{x} = Ax + b$, $1 \leq k \leq 6$ and $\Delta t > 0$ such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible, let Φ be as in Definition 6.*

- *$\mathcal{H}$ is numeric BDAE if and only if $\mathcal{H}$ is numeric ABI.*
- *For any partition $\mathcal{G}$, the coarsest numeric ABI $\mathcal{H}$ which refines $\mathcal{G}$ is the coarsest BI of $\dot{x} = \Phi(x)$ that refines $\mathcal{G}$; $\mathcal{H}$ can be computed in polynomial time.*

Theorem 7 states that a numeric ABI of the DAE system $E\dot{x} = Ax + b$ is a BI of the ODE system $\dot{x} = \Phi(x)$. Hence, for any partition $\mathcal{G}$ of $\mathcal{S}$, we can use the polynomial time algorithm of Theorem 2 to compute the coarsest numeric ABI that refines $\mathcal{G}$.

Numeric ABI allows to speed up numerical computations by invoking the reduced numerical mapping.

Definition 7. *Given a regular DAE system $E\dot{x} = Ax + b$, $1 \leq k \leq 6$ and $\Delta t > 0$ such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible, assume that $\mathcal{H}$ is a numeric ABI. Then, the underlying reduced mapping $\underline{\Phi} : \mathbb{R}^{\mathcal{S}_0} \to \mathbb{R}^{\mathcal{S}_0}$ is defined via*

$$x \mapsto \big(S_l(A - \tfrac{\alpha(k,0)}{\Delta t}E)^{-1}ES_r\big)x - S_l(A - \tfrac{\alpha(k,0)}{\Delta t}E)^{-1}b$$

with S_l, S_r and $\mathcal{S}_0$ as in Definition 5.

Similarly to ABI and BDAE, the following can be shown.

Theorem 8. *Given a regular DAE system $E\dot{x} = Ax + b$, $1 \leq k \leq 6$, $\Delta t > 0$ such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible and some partition $\mathcal{G}$, the coarsest numeric ABI $\mathcal{H}$ refining $\mathcal{G}$ and the underlying reduced numerical mapping $\underline{\Phi}$ from Definition 7 can be computed in polynomial time. Moreover, with $\underline{x}[i] = \underline{\Phi}(\sum_{l=1}^{k} \frac{\alpha(k,l)}{\Delta t}\underline{x}[i-l])$, it holds that $x[i] = S_r\underline{x}[i]$ for all $1 \leq i \leq M$, provided that $x[0] = S_r\underline{x}[0]$.*

On the Relation of Numeric and Algebraic Notions. Despite numeric BDAE and numeric ABI provide an efficient model reduction technique for practical computations, it is interesting to ask whether the convergence of the BDF scheme toward the true analytical solution, ensured by Theorem 6, can be used to tie ABI to numeric ABI.

The following auxiliary result is pivotal.

Proposition 1. *Given a regular DAE system $E\dot{x} = Ax + b$ and $1 \leq k \leq 6$, let $\Delta t_0 > 0$ be such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible for all $0 < \Delta t \leq \Delta t_0$. Then, for all $\lambda \geq \lambda_0 = \frac{\alpha(k,0)}{\Delta t_0}$, the matrices $R_\lambda = (A - \lambda E)^{-1}E$ and $S_\lambda = (A - \lambda E)^{-1}$ are well-defined and satisfy the Riccati matrix differential equations $\dot{R}_\lambda = R_\lambda^2$ and $\dot{S}_\lambda = R_\lambda S_\lambda$ on $[\lambda_0; \infty)$.*

With this, we are in a position to show the following.

Proposition 2. *Given a regular DAE system $E\dot{x} = Ax + b$ and $1 \leq k \leq 6$, let $\Delta t_0 > 0$ be such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible for all $0 < \Delta t \leq \Delta t_0$. Then, $\mathcal{H}$ is numeric ABI for all $0 < \Delta t \leq \Delta t_0$ whenever $\mathcal{H}$ is numeric ABI for Δt_0.*

Proposition 2 states that the variable equivalences identified in the BDF scheme by a numeric ABI do not depend on the chosen time step Δt. Armed with this, the next major result establishes that numeric ABI implies ABI.

Theorem 9. *Given a regular DAE system $E\dot{x} = Ax + b$ and $1 \leq k \leq 6$, let $\Delta t_0 > 0$ be such that $(A - \frac{\alpha(k,0)}{\Delta t}E)$ is invertible for all $0 < \Delta t \leq \Delta t_0$. If $\mathcal{H}$ is a numeric ABI, then it is also an ABI.*

Remark 2. *Numeric ABI is only a sufficient condition for ABI in general. Indeed, for (1), the matrix $(A - \lambda E)$ is invertible if and only if $\lambda^2 - 3\lambda + 2 \neq 0$. Hence, $(A - \lambda E)$ is invertible for all $\lambda \geq 3$, meaning that λ_0 and Δt_0 can be set to 3 and $\frac{1}{3}$, respectively. A computation of $(A - \frac{1}{3}E)^{-1}E$ reveals that the coarsest numeric ABI for Δt_0 (hence, by Theorem 2, for all $0 < \Delta t \leq \Delta t_0$) is the trivial partition $\{\{1\}, \{2\}, \{3\}\}$. At the same time, however, Example 2 establishes that $\{\{1, 2\}, \{3\}\}$ is an ABI of (1).*

Even though we expect numeric ABI to coincide with ABI for most practical models, the above remark demonstrates that numeric ABI is in general stricter than ABI. At the same time, the coarsest numeric ABI partition can be computed in polynomial time, while ABI relies on the solution of an ill-conditioned problem [13].

V. Evaluation

We hereby apply numeric ABI to DAE benchmarks from the electrical engineering literature using a Matlab prototype. We compare the solution runtimes of the original model against those of the reduced model (including minimization runtime) using the `ode15s` solver, which is capable of processing linear (and nonlinear) DAE systems with DAE index at most one. Experiments have been performed on a 2.6 GHz Intel 2016b; the Matlab command `ode15s` have been invoked with standard settings.

H-tree model. We consider a power distribution network from [25] which has been adapted by removing inductances. The network has a hierarchical tree topology (H-tree), depicted in Figure 1. At each depth i, all branches have equal resistances and capacitances R_i and C_i, respectively, whose values are given in Table II. For depths $i \leq 4$, the values were taken from [25], while values for depths $i \geq 5$ have been extrapolated. An H-tree model of depth N results in a DAE system with $2^N + 1$ variables, given by the voltages across each capacitor and by the voltage V_{in}. Here, we assume that V_{in} has constant value 5.0V.

Table I considers H-tree networks with increasing depth N (column $|\mathcal{S}|$ gives the state space size). Numeric ABI confirms the symmetry discussed in [25], namely that the voltages across the capacitors of same depth are equal at all time points, resulting in $N + 2$ partition blocks (column $|\mathcal{H}|$).

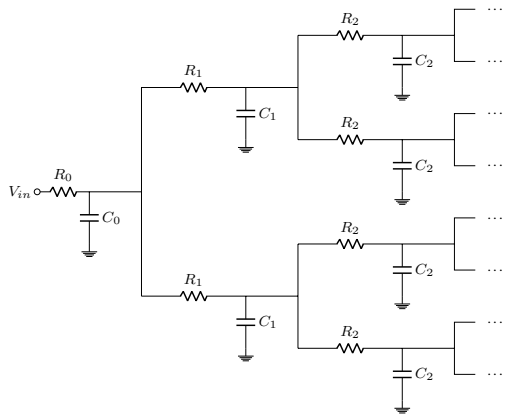

Fig. 1: H-tree network adapted from [25].

Model			*Num. ABI*	
N	$\lvert\mathcal{S}\rvert$	*Sol. (s)*	$\lvert\mathcal{H}\rvert$	*Time (s)*
3	9	8.86.E–2	5	1.01E–2
4	17	1.44E–1	6	1.12E–2
5	33	1.61E–1	7	1.30E–2
6	65	1.72E–1	8	1.47E–2
7	129	1.77E–1	9	1.63E–2
8	257	2.39E–1	10	3.31E–2
9	513	2.86E–1	11	8.73E–2
10	1 025	1.04E+0	12	2.93E–1
11	2 049	5.05E+0	13	6.46E–1
12	4 097	3.77E+1	14	2.45E+0
13	8 193	6.27E+2	15	8.89E+0
14	16 385	*O.M.*	16	4.34E+1

TABLE I: Numerical results for H-trees.

i	2	3	4	5	6	7	8	9	10	11	12	13	14
R_i	6.37	12.75	25.50	50	100	200	400	800	1 600	3 200	6 400	12 800	25 600
C_i	0.300	0.130	0.140	0.070	0.070	0.035	0.035	0.018	0.018	0.009	0.009	0.005	0.005

TABLE II: Resistances and capacitances of the H-tree networks at depths i.

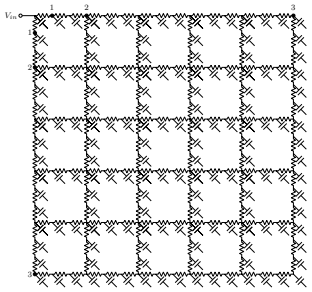

Fig. 2: Mesh networks adapted from [26].

Model		*Num. ABI*	
N	$\lvert\mathcal{S}\rvert$	$\lvert\mathcal{H}\rvert$	*Time (s)*
4	106	56	2.48E–1
5	157	82	3.06E–1
6	218	113	3.51E–1
7	289	149	6.41E–1
8	370	190	1.05E+0
9	461	236	1.70E+0
10	561	287	4.26E+0

TABLE III: Numerical results for meshes.

Column *Sol.* provides the computation time of the original model via ode15s in the case where the time horizon was set to 2 time units. To allow for a fair comparison, column *Time* provides the cumulative runtime for the computation of the coarsest numeric ABI and the solution of the underlying reduced DAE system via ode15s. Numeric ABI always led to better performance. In addition, it allowed us to analyze the case $N = 14$, whose full model could not be solved by ode15s due to out of memory errors.

Mesh networks. Figure 2 shows a square transmission line mesh, adapted from [26], where each line has $N = 5$ segments of 3 resistor/capacitor series each.[1] The variables of the underlying DAE system represent the voltages at each capacitor.

Table III considers seven variants of the network for N ranging from 4 to 10 for resistances and capacitances equal to 1 and 0.1, respectively. Columns $|\mathcal{S}|$ and $|\mathcal{H}|$ are as in Table I, while *Time* accounts for the computation of the coarsest numeric ABI only. This is because both the original and the reduced DAE systems had index higher than one, which cannot be handled by ode15s. We wish to point out, however, that the reduced numerical model from Theorem 8 could be, in principle, derived. It is interesting to note that, for each N, the size of the largest numeric ABI is $N + 1 + (|\mathcal{S}| - N)/2$. This has a geometric interpretation in that it relates (voltages of) capacitor nodes placed symmetrically with respect to the main diagonal of the network (Figure 2 marks nodes within representative equivalence classes with the same number).

Further benchmarks. Table IV considers further benchmarks of real-world electrical circuits available at `https://sites.google.com/site/rommes/software`. All models are given in the form $E\dot{x}(t) = Ax(t) + Bu(t)$ where $x(t) \in \mathbb{R}^n$ is the system's state, $u(t)$ represents some m-dimensional input and $B \in \mathbb{R}^{n\times m}$. In order to obtain symmetries that are valid for any input signal u, we started from an input partition $\mathcal{G}$ whose blocks of variables are *affected in the same way* by the input, that is, two variable indices i, j were put into the same block of $\mathcal{G}$ if and only if $e_i^T(Bu) = e_j^T(Bu)$ for all u (which is equivalent to saying that i, j were put into the same block if and only if $e_i^T(Be_k) = e_j^T(Be_k)$ for all $1 \le k \le m$). For the experiments, we considered the input $b = B\mathbf{1}$, with $\mathbf{1}$ being the m-dimensional vector whose coordinates are all one. This led to systems of the form $E\dot{x} = Ax + b$.

As in the case of Table III, the column *Time* in Table IV refers to the computation of the coarsest numeric ABI only because the original and reduced DAE systems had index greater than one. We remark that all instances could be reduced to about 70% of their original size.

VI. Further Discussion

Since ABI is an extension of BDE, we next relate BDE and bisimulation [19], [20], [21]. Essentially, BDE provides a reduction of a *single* system, while bisimulation relates *two* given systems. These two notions can be compared when

[1] Every capacitor is connected to the ground, which is not explicitly drawn to avoid clutter.

Model			Num. ABI	
Name	*Ref.*	$\lvert\mathcal{S}\rvert$	$\lvert\mathcal{H}\rvert$	*Time (s)*
bips98_606	[27]	7 135	5 656	5.29E+3
bips98_1142	[27]	9 735	7 225	1.08E+4
bips98_1450	[27]	11 305	8 115	1.73E+4
nopss_11k	[28]	11 685	8 015	1.95E+4
mimo46x46_system	[29]	13 250	9 132	5.35E+4
bips07_1693	[27]	13 275	9 073	7.59E+4
mimo8x8_system	[30]	13 309	9 070	5.14E+4

TABLE IV: Further circuit benchmarks: numerical results.

the observation maps are not restricted to linear functions. In this case, the original system and its BDE reduction can be shown to be bisimilar. For instance, $\{\{1,2\}\}$ is a BDE partition of $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$ because $x_1(t) = x_2(t)$ for all $t > 0$ if $x_1(0) = x_2(0)$. The corresponding BDE reduction from Definition 5 is $\dot{x}_1 = x_1$. By fixing the quotient system $\dot{y}_1 = y_1$ and the observation maps $(x_1, x_2) \mapsto x_1$, $y_1 \mapsto y_1$, it can be shown that $\mathcal{B} = \{((x_1, x_2), y_1) \mid x_1 = x_2 = y_1\}$ is a bisimulation. However, for this comparison to be made, one needs the appropriate observation maps, which are induced by the BDE reduction. Moreover, the coarsest BDE reduction can be computed in polynomial time also in the case of nonlinear ODE systems [15], [16].

VII. Conclusion

In this paper we have developed a theory for the reduction of differential-algebraic equations (DAEs) based on a notion of backward equivalence, relating variables when they have equal solutions at all time points. Unlike related work on Markov chains and ordinary differential equations, equivalences on DAEs require a careful treatment of the domain on which the DAE is defined. This led to a different notion of equivalence called numerical algebraic backward invariance (ABI). Model reductions underlying numeric ABI can be computed in polynomial time and have been obtained for a number of benchmark models of electrical circuits. Developing approximate notions is part of future work.

Acknowledgments

The authors thank the anonymous reviewers for helpful comments. Mirco Tribastone is supported by a DFG Mercator Fellowship (SPP 1593, DAPS2 Project). Max Tschaikowski is supported by a Lise Meitner Fellowship which is funded by the Austrian Science Fund (FWF) under grant number M 2393-N32 (COCO). Most of the work has been conducted while Max Tschaikowski was with IMT Lucca.

References

[1] C.-W. Ho, A. Ruehli, and P. Brennan, "The modified nodal approach to network analysis," *IEEE Trans. Circuits Syst.*, vol. 22, no. 6, pp. 504–509, 1975.

[2] P. J. Gawthrop and G. P. Bevan, "Bond-graph modeling," *IEEE Control Systems*, vol. 27, no. 2, pp. 24–45, 2007.

[3] U. M. Ascher and L. R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. SIAM, 1988.

[4] V. Mehrmann and T. Stykel, "Balanced Truncation Model Reduction for Large-Scale Systems in Descriptor Form," in *Dimension Reduction of Large-Scale Systems*, 2005, pp. 83–115.

[5] T. Reis and T. Stykel, "Balanced truncation model reduction of second-order systems," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 14, no. 5, pp. 391–406, 2008.

[6] P. Benner and T. Stykel, *Model Order Reduction for Differential-Algebraic Equations: A Survey*. Springer, 2017, pp. 107–160.

[7] M. Sunnaker, G. Cedersund, and M. Jirstrand, "A method for zooming of nonlinear models of biochemical systems," *BMC Systems Biology*, vol. 5, no. 1, p. 140, 2011.

[8] M. Apri, M. de Gee, and J. Molenaar, "Complexity reduction preserving dynamical behavior of biochemical networks," *Journal of Theoretical Biology*, vol. 304, pp. 16 – 26, 2012.

[9] A. Abate, L. Brim, M. Ceska, and M. Z. Kwiatkowska, "Adaptive aggregation of markov chains: Quantitative analysis of chemical reaction networks," in *CAV*, 2015, pp. 195–213.

[10] P. Buchholz, "Exact and Ordinary Lumpability in Finite Markov Chains," *Journal of Applied Probability*, vol. 31, no. 1, pp. 59–75, 1994.

[11] L. Cardelli, M. Tribastone, M. Tschaikowski, and A. Vandin, "Symbolic computation of differential equivalences," in *POPL*, 2016.

[12] J. Demmel and B. Kågström, "The Generalized Schur Decomposition of an Arbitrary Pencil A-λB: Robust Software with Error Bounds and Applications. Part I: Theory and Algorithms," *ACM Trans. Math. Softw.*, vol. 19, p. 160174, 1993.

[13] P. Kunkel and V. Mehrmann, "A new look at pencils of matrix valued functions," *Linear Algebra and its Applications*, vol. 212-213, pp. 215 – 248, 1994.

[14] ——, *Differential-Algebraic Equations. Analysis and Numerical Solution.* Zurich, Switzerland: EMS Publishing House, 2006.

[15] L. Cardelli, M. Tribastone, M. Tschaikowski, and A. Vandin, "Efficient syntax-driven lumping of differential equations," in *TACAS*, vol. 9636, 2016, pp. 93–111.

[16] ——, "Maximal aggregation of polynomial dynamical systems," *Proceedings of the National Academy of Sciences (PNAS)*, vol. 114, no. 38, pp. 10 029–10 034, 2017.

[17] M. Boreale, "Algebra, coalgebra, and minimization in polynomial differential equations," in *FOSSACS*, 2017, pp. 71–87.

[18] N. Y. Megawati and A. van der Schaft, "Bisimulation equivalence of differential-algebraic systems," *Int. J. Control*, vol. 91, no. 1, pp. 45–56, 2018.

[19] A. J. van der Schaft, "Equivalence of dynamical systems by bisimulation," *IEEE Transactions on Automatic Control*, vol. 49, 2004.

[20] G. J. Pappas, "Bisimilar linear systems," *Automatica*, vol. 39, no. 12, pp. 2035–2047, 2003.

[21] A. Girard and G. Pappas, "Approximate Bisimulations for Nonlinear Dynamical Systems," in *CDC*, 2005, pp. 684–689.

[22] G. J. Pappas, G. Lafferriere, and S. Sastry, "Hierarchically consistent control systems," *IEEE TAC*, vol. 45, no. 6, pp. 1144–1160, 2000.

[23] L. Cardelli, M. Tribastone, M. Tschaikowski, and A. Vandin, "Comparing chemical reaction networks: A categorical and algorithmic perspective," in *LICS*, 2016.

[24] C. C. Pantelides, "The Consistent Initialization of Differential-Algebraic Systems," *SIAM Journal on Scientific and Statistical Computing*, vol. 9, no. 2, pp. 213–231, 1988.

[25] J. Rosenfeld and E. G. Friedman, "Design Methodology for Global Resonant H-Tree Clock Distribution Networks," *IEEE Trans. VLSI Syst.*, vol. 15, no. 2, pp. 135–148, 2007.

[26] B. N. Sheehan, "ENOR: model order reduction of RLC circuits using nodal equations for efficient factorization," in *Proceedings 1999 Design Automation Conference*, 1999, pp. 17–21.

[27] F. D. Freitas, J. Rommes, and N. Martins, "Gramian-based reduction method applied to large sparse power system descriptor models," *IEEE Transactions on Power Systems*, vol. 23, no. 3, pp. 1258–1270, 2008.

[28] J. Rommes, N. Martins, and F. Freitas, "Computing rightmost eigenvalues for small-signal stability assessment of large scale power systems," in *IEEE PES General Meeting*, July 2010, pp. 1–1.

[29] N. Martins, P. C. Pellanda, and J. Rommes, "Computation of transfer function dominant zeros with applications to oscillation damping control of large power systems," *IEEE Trans. Power Syst.*, vol. 22, no. 4, pp. 1657–1664, 2007.

[30] J. Rommes and N. Martins, "Efficient computation of multivariable transfer function dominant poles using subspace acceleration," *IEEE Transactions on Power Systems*, vol. 21, no. 4, pp. 1471–1483, 2006.

[31] M. Tschaikowski and M. Tribastone, "Approximate reduction of heterogenous nonlinear models with differential hulls," *IEEE Trans. Automat. Contr.*, vol. 61, no. 4, pp. 1099–1104, 2016.