

# *DC-Servomechanism*

**Alberto Bemporad**

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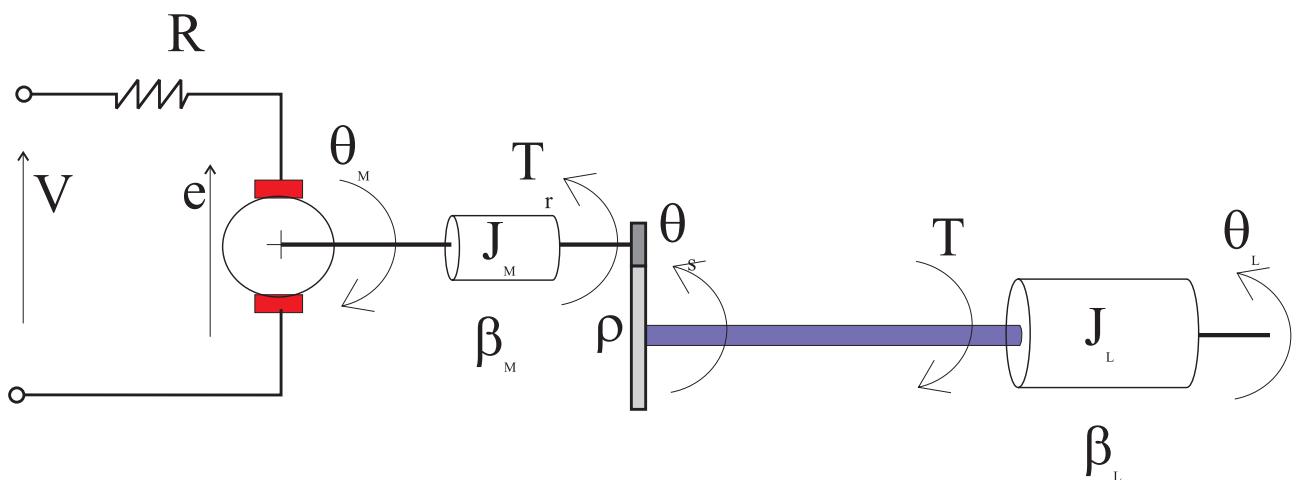


Figure 1: Servomechanism model (*modello del servomeccanismo*).

# 1 Servomechanism Model

The position servomechanism is schematically described in Fig. 1. It consists of a DC-motor (*motore in corrente continua*), gearbox (*scatola di riduzione*), elastic shaft (*albero di trasmissione elastico*) and load (*carico meccanico*). We derive the equations of the system, in order to obtain a model suitable for control tasks. To this aim, let us consider each physical component of the system.

## 1.1 DC Motor - Armature Control (*Controllo in Tensione di Armatura*)

1.

$$V = Ri + e$$

where  $V$  = applied armature voltage (*tensione di armatura*),  $i$  = armature current (*corrente di armatura*),  $R$  = resistance,  $e$  = back e.m.f. (*forza contro-elettromotrice*)

2.

$$e = k_b \omega_M$$

where  $\omega_M = \dot{\theta}_M$  =angular velocity of motor-shaft,  $k_b$  =constant  
3.

$$\phi = k_f i_f$$

where  $\phi$  =air gap flux,  $i_f$  =field current (*corrente di campo*),  
 $k_f$  =constant

4.

$$T_M = k_1 \phi i$$

where  $T_M$  =torque (*coppia* ) developed by the motor,  $k_1$   
=constant

5. By steady-state power balance (*equilibrio della potenza in condizioni di regime*)

$$T_M \omega_M = ei$$

and therefore  $k_1 k_f i_f = k_b = k_T$ ,  $k_T$  =motor constant  
(costante del motore)

6.

$$J_M \dot{\omega}_M = T_M - \beta_M \omega_M - T_r$$

where  $J_M$  =equivalent moment of inertia of motor,  $\beta_M$  =equivalent viscous friction coefficient (*coefficiente di attrito viscoso*) of motor,  $T_r$  =other torques.

In summary:

$$V = Ri + k_T \omega \quad (1)$$

$$J_M \dot{\omega}_M = k_T i - \beta_M \omega_M - T_r \quad (2)$$

## 1.2 Gear Box (*Scatola di Riduzione*)

Consider the gear box depicted in Fig. 2.

1. Same distance traveled by the two wheels

$$\theta_1 r_1 = \theta_2 r_2$$

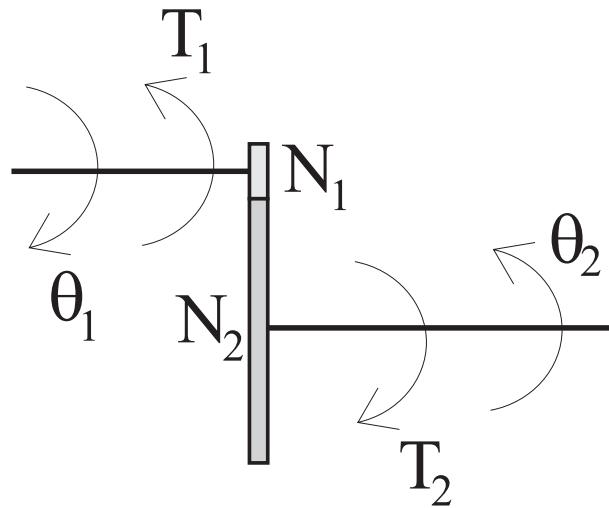


Figure 2: Gear box.

where  $r_1, r_2$  = wheel radii (*raggi delle ruote dentate*)

2. and therefore

$$\frac{\theta_1}{\theta_2} = \frac{r_2}{r_1} = \frac{N_2}{N_1} = \rho$$

where  $N_1, N_2$  = number of teeth (*numero di denti*),  $\rho$  = gear ratio (*rapporto di riduzione*).

3. By differentiation,

$$\omega_1 = \rho \omega_2$$

4. Power transmission (no loss) (*potenza trasmessa, trascu-*

rando le perdite)

$$T_1\omega_1 = T_2\omega_2$$

and therefore

$$\frac{T_1}{T_2} = \frac{1}{\rho}$$

By referring again to Fig. 1, we have for  $\theta_1 = \theta_M$ ,  $\theta_2 = \theta_s$ ,  $T_1 = -T_r$ ,  $T_2 = T$

$$\theta_s = \frac{1}{\rho}\theta_M \quad (3)$$

$$T_r = -\frac{1}{\rho}T \quad (4)$$

where  $\theta_M$  =angular displacement of motor shaft (*posizione angolare dell'angolo motore*),  $\theta_s$  =angular displacement of load-side gear (*posizione angolare a valle della scatola di riduzione*),  $T$  =torque acting on the load (*coppia agente sul carico*).

### 1.3 Elastic Shaft (*Albero di trasmissione elastico*)

The shaft has finite torsional rigidity (*rigidità torsionale*)  $k_\theta$ :

$$T = k_\theta(\theta_L - \theta_s) \quad (5)$$

where  $\theta_L$  =angular displacement of load (*posizione angolare del carico*).

### 1.4 Load Dynamics (*Dinamica del Carico Meccanico*)

$$J_L \dot{\omega}_L = -\beta_L \omega_L - T \quad (6)$$

where  $\omega_L = \dot{\theta}_L$  =angular load velocity (*velocità angolare del carico*),  $J_L$  =equivalent moment of inertia of load,  $\beta_L$  =equivalent viscous friction coefficient of load.

## 1.5 Differential Equations of the System

By collecting the previous relation, we can write the

$$\dot{\omega}_L = -\frac{k_\theta}{J_L} \left( \theta_L - \frac{\theta_M}{\rho} \right) - \frac{\beta_L}{J_L} \omega_L \quad (7)$$

$$\dot{\omega}_M = \frac{k_T}{J_M} \left( \frac{V - k_T \omega_M}{R} \right) - \frac{\beta_M \omega_M}{J_M} + \frac{k_\theta}{\rho J_M} \left( \theta_L - \frac{\theta_M}{\rho} \right) \quad (8)$$

## 1.6 State-Space Model (*Modello in Forma Spazio di Stato*)

By setting  $x_p \triangleq [\theta_L \ \dot{\theta}_L \ \theta_M \ \dot{\theta}_M]'$ , the model can be described by the following state-space form

$$\begin{aligned} \dot{x}_p &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_\theta}{J_L} & -\frac{\beta_L}{J_L} & \frac{k_\theta}{\rho J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_\theta}{\rho J_M} & 0 & -\frac{k_\theta}{\rho^2 J_M} & -\frac{\beta_M + k_T^2/R}{J_M} \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_T}{R J_M} \end{bmatrix} V \\ \theta_L &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_p \\ \dot{\theta}_L &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x_p \\ T &= \begin{bmatrix} k_\theta & 0 & -\frac{k_\theta}{\rho} & 0 \end{bmatrix} x_p \end{aligned}$$

Since the steel shaft (*albero in acciaio*) has finite shear strength (*resistenza al taglio*), determined by a maximum admissible  $\tau_{adm} = 50 N/mm^2$ , the torsional torque  $T$  must satisfy the constraint

$$|T| \leq 78.5398 \text{ Nm} \quad (9)$$

Moreover, the input DC voltage  $V$  has to be constrained within the range

$$|V| \leq 220 \text{ V} \quad (10)$$

The model is transformed in discrete time by sampling (*campionando*) every  $T_s = 0.1s$  and using a zero-order holder on the input voltage.

## 1.7 Parameters

The parameters of the system are reported in Table 1.

## 2 Esercizio

1. Descrivere con uno script Matlab il modello del sistema (tempo continuo)
2. Calcolare la risposta di  $\theta_L$ ,  $\dot{\theta}_L$  e  $T$  dovuta ad un gradino di tensione  $V = 120$  V
3. Calcolare la risposta all'impulso
4. Calcolare l'evoluzione libera da condizione iniziale  $\theta_L(0) = \theta_M(0) = 0$ ,  $\dot{\theta}_L = 1$  rad/s,  $\dot{\theta}_M = 20$  rad/s
5. Ottenere un modello a tempo discreto del sistema (tempo di campionamento  $T_s = 0.1$  s)
6. Confrontare la risposta al gradino del sistema discretizzato e del sistema tempo continuo.

Table 1: Model parameters

Symbol	Value (MKS)	Meaning
$L_S$	1.0	shaft length
$d_S$	0.02	shaft diameter
$J_S$	negligible	shaft inertia
$J_M$	0.5	motor inertia
$\beta_M$	0.1	motor viscous friction coefficient
$R$	20	resistance of armature
$k_T$	10	motor constant
$\rho$	20	gear ratio
$k_\theta$	1280.2	torsional rigidity
$J_L$	$50J_M$	nominal load inertia
$\beta_L$	25	load viscous friction coefficient
$T_s$	0.1	sampling time