UNCONSTRAINED NONLINEAR OPTIMIZATION

Reference:

J.Nocedal and S.J. Wright, "*Numerical Optimization*," 2006. Chapter 3

UNCONSTRAINED NONLINEAR OPTIMIZATION METHODS

- For an arbitrary smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we want to to solve the **unconstrained nonlinear programming** problem

$$\min_x f(x)$$

- There are fundamentally two classes of iterative methods:
  - **line search methods** choose a descent direction p_k, search a suitable scalar $\alpha_k > 0$ such that $f(x_k + \alpha_k p_k) < f(x_k)$, and set $x_{k+1} = x_k + \alpha_k p_k$
  - **trust region methods** compute a quadratic approximation $q(x)$ of f around x_k, solve

$$p_k = \arg \min_{p:\ \|p\|_2 \le \Delta} q(x_k + p)$$

    where the size Δ of the "trust-region" of the model is shrunk until $f(x_k + p_k) < f(x_k)$, and set and set $x_{k+1} = x_k + p_k$
- The above methods converge to a local minimum (a global one if f convex)

LINE SEARCH METHODS: STEEPEST DESCENT

- **Steepest descent** is the most obvious method, as it picks up p_k orthogonal to the level sets of f

$$p_k = -\nabla f(x_k)$$

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

- From Taylor's theorem

$$f(x_k + \alpha p_k) = f(x_k) - \alpha \|\nabla f(x_k)\|_2^2 + \alpha^2 p_k' \nabla^2 f(x_k + t p_k) p_k,\ t \in (0, \alpha)$$

- Note that the Hessian of f is not required to compute p_k

- The method can be very slow to converge

LINE SEARCH METHODS: NEWTON'S METHOD

- **Newton's method** chooses $p_k = -(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$ (**Newton's direction**)

$$x_{k+1} = x_k - \alpha_k(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$$

- Newton's direction provides the minimum of the quadratic Taylor's approximation q of f at x_k:

$$q(x_k + p) = f(x_k) + \nabla f(x_k)'p + \frac{1}{2}p'\nabla^2 f(x_k)p$$

- If $\nabla^2 f(x_k) \succ 0$ then for some $\sigma_k > 0$

$$\nabla f(x_k)'p_k = -\nabla f(x_k)'(\nabla^2 f(x_k))^{-1}\nabla f(x_k) \leq -\sigma_k \|p_k\|_2^2$$

so from Taylor's theorem we have $f(x_k + \alpha p_k) < f(x_k)$ for α small enough

LINE SEARCH METHODS: NEWTON'S METHOD

- The method converges very fast, especially close to x^*, where the function f and its quadratic approximation tend to coincide

- For $\alpha_k \equiv 1$ we have **pure Newton's method**. However line search over α is required to ensure convergence

- In case $\nabla^2 f(x_k)$ is not positive definite, a possibility is to use instead $\nabla^2 f(x_k) + \mathrm{diag}(\delta_k)$.

  For example δ_k can be computed during a Cholesky factorization to make intermediate diagonal entries $\geq \epsilon$ for some $\epsilon > 0$

LINE SEARCH METHODS: QUASI NEWTON METHODS

- Newton's method requires computing $\nabla^2 f(x_k)$, which could be expensive

- **Quasi-Newton methods** replace $\nabla^2 f(x_k)$ with a matrix B_k which is easier to compute, satisfying the **secant equation**

$$B_{k+1}s_k = y_k, \text{ where } s_k = x_{k+1} - x_k,\ y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$$

  and set $p_k = -B_k^{-1}\nabla f(x_k)$

- The **BFGS formula** (Broyden, Fletcher, Goldfarb, and Shanno) updates

$$B_{k+1} = B_k - \frac{B_k s_k s_k' B_k}{s_k' B_k s_k} + \frac{y_k y_k'}{y_k' s_k}$$

  where $B_k \succ 0$ if $B_0 \succ 0$ and $s_k' y_k > 0$ for all k

LINE SEARCH METHODS: QUASI-NEWTON METHODS

- Since B_{k+1} differs from B_k by two one-rank updates, we can update a factorization of B_k recursively.

- In alternative, one can avoid B_k and directly update $H_k = B_k^{-1}$

$$H_{k+1} = H_k + \frac{s_k' y_k + y_k' B_k^{-1} y_k}{(s_k' y_k)^2} s_k s_k' - \frac{B_k^{-1} y_k s_k' + s_k (B_k^{-1} y_k)}{s_k' y_k}$$

- For large-scale problems, **limited-memory BFGS** only stores a finite number m of past values of (s_k, y_k) (usually $m < 10$) and directly computes the descent direction $p_k = -H_k \nabla f(x_k)$ without storing H_k

LINE SEARCH METHODS: NONLINEAR CONJUGATE-GRADIENT

(Fletcher, Reeves, 1964)

- **Nonlinear conjugate gradient methods** (CG) update the search direction p_k as follows:

$$\begin{aligned} \beta_{k+1} &= \frac{\nabla f(x_{k+1})'\nabla f(x_{k+1})}{\nabla f(x_k)'\nabla f(x_k)} \\ p_{k+1} &= -\nabla f(x_{k+1}) + \beta_{k+1} p_k \end{aligned}$$

  where the scalar β_k is chosen so that p_k and p_{k-1} are **conjugate**

- The method does not requires the storage of matrices
- The method is almost as simple as steepest descent but usually more efficient (although it does not converge as fast as (quasi-)Newton methods)
- As for steepest descent, nonlinear CG may be sensitive to problem **scaling**

LINE SEARCH

- Given a descent direction p_k, ideally one should choose $x_{k+1} = x_k + \alpha_k p_k$ with

$$\alpha_k = \arg\min_{\alpha>0} f(x_k + \alpha p_k)$$

- Such a scalar nonlinear optimization may be difficult to solve and require a lot of evaluations of f, so we look for simpler methods
- Simply imposing $f(x_k + \alpha_k p_k) < f(x_k)$ may not work, as the improvement may become smaller and smaller as k grows

- Sufficient decrease is provided by satisfying **Armijo condition**

$$f(x_k+\alpha p_k) \leq f(x_k)+c_1\alpha\nabla f(x_k)'p_k \quad c_1 \in (0,1)$$

where usually c_1 is small (e.g., $c_1 = 10^{-4}$)

LINE SEARCH

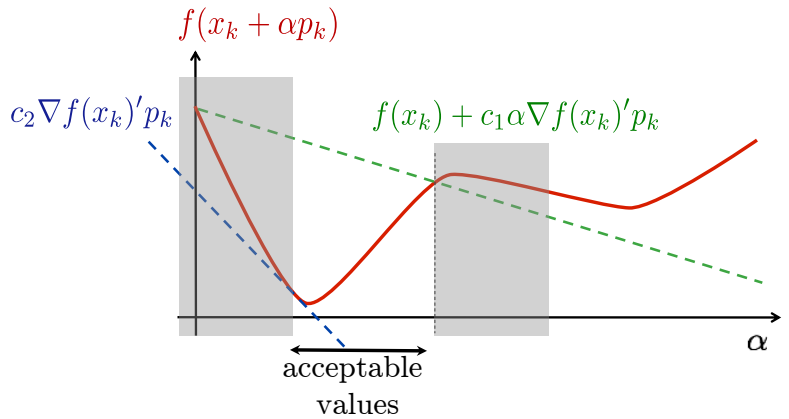
- **Wolfe conditions** include Armijo condition + the **curvature condition**

$$\frac{df(x_k+\alpha p_k)}{d\alpha} = \nabla f(x_k+\alpha p_k)' p_k \geq c_2 \nabla f(x_k)' p_k \quad c_2 \in (c_1, 1)$$

(the condition is **strong** if $|\nabla f(x_k+\alpha p_k)' p_k| \leq c_2 |\nabla f(x_k)' p_k|$ is imposed)

- The curvature condition avoid values of α that are too small, when f is still decaying fast (=very negative derivative)

- Usually $c_1 = 0.9$ in (quasi-)Newton methods and $c_1 = 0.1$ in the nonlinear CG method



- It is possible to prove that if f is continuously differentiable and f bounded below along the descent direction $x_k+\alpha p_k, \alpha \geq 0$, the (strong) Wolfe conditions can be satisfied

BACKTRACKING LINE SEARCH

- The following is a simple practical algorithm for selecting a step size α_k satisfying Armijo formula:

Choose $\bar{\alpha} > 0, \rho \in (0,1), c \in (0,1)$. Set $\alpha = \bar{\alpha}$.

Repeat until $f(x_k + \alpha p_k) \leq f(x_k) + c\alpha \nabla f(x_k)' p_k$

$\alpha \leftarrow \rho\alpha$

end repeat

Set $\alpha_k = \alpha$.

- Possible choices are $\bar{\alpha} = 1$ (e.g., in Newton's method) and $\rho = \frac{1}{2}$ (bisection)

LINE SEARCH - CONVERGENCE RESULT

THEOREM (ZOUTENDIJK)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded below and differentiable in an open set $\mathcal{N}$ containing the level set $\mathcal{L} = \{x : \ f(x) \leq f(x_0)\}$, and let ∇f Lipschitz continuous on $\mathcal{N}$, that is

$$\|\nabla f(x_1) - \nabla f(x_2)\| \leq L\|x_1 - x_2\|, \ \forall x_1, x_2 \in \mathcal{N}$$

for some $L > 0$. Any line search method with p_k a descent direction and α_k satisfying the Wolfe conditions is such that

$$\sum_{k=0}^{\infty} \cos^2(\theta_k)\|\nabla f(x_k)\|^2 < \infty, \quad \cos(\theta_k) = \frac{-\nabla f(x_k)' p_k}{\|\nabla f(x_k)\|\|p_k\|}$$

- If we choose p_k such that $\cos\theta_k \geq \delta > 0, \forall k \geq 0$, then $\lim_{k\to\infty} \|\nabla f(x_k)\| = 0$.

LINE SEARCH - CONVERGENCE RATE

- The condition $\cos\theta_k \geq \delta > 0, \forall k \geq 0$, holds for the steepest descent method

- It also holds for (quasi-)Newton methods when $B_k \succ 0$ with uniformly bounded condition number

- The convergence result show that the algorithm converges to a stationary point $\nabla f(x) = 0$

CONVERGENCE RATES

- In analyzing the speed of convergence of iterative algorithms, we refer to **convergence rates**. Let $x_k : \mathbb{N} \to \mathbb{R}^n$ be a converging sequence, $\lim_{k\to\infty} x_k = x^*$. We define

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = r,\ r \in (0,1) \qquad \textbf{linear convergence}$$

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = r_k,\ \lim_{k\to\infty} r_k = 0 \qquad \textbf{superlinear convergence}$$

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} > 0 \qquad \textbf{quadratic convergence}$$

- Convergence only relates to the **asymptotic behavior** of the algorithm. The transient is often more relevant, especially stopping tolerances are not small

LINE SEARCH - CONVERGENCE RATE

- When f is twice differentiable and $\nabla^2 f(x^*) \succ 0$ we can show that **steepest descent** has the **linear convergence rate**

$$f(x_{k+1}) - f(x^*) \leq r^2 (f(x_k) - f(x^*)), \quad \frac{\lambda\max - \lambda\min}{\lambda\max + \lambda\min} < r < 1$$

where $\lambda\max, \lambda\min$ are the max/min eigenvalues of $\nabla^2 f(x^*)$

- When f is twice differentiable and $\nabla^2 f(x^*) \succ 0$ and x_0 is sufficiently close to x^* **Newton's method** has the **quadratic convergence rate**

$$\|x_{k+1} - x^*\| \leq \tilde{L} \|x_k - x^*\|^2, \quad \|\nabla f(x_{k+1})\| \leq 2L \|\nabla^2 f(x^*)^{-1}\|^2 \|\nabla f(x_k)\|^2$$

while **quasi-Newton methods** have a **superlinear convergence rate**

LINE SEARCH METHODS: COORDINATE DESCENT

- **Coordinate descent** consists of successively optimizing **only one coordinate** x_{i_k} at each step k, $i_k \in \{1, \ldots, n\}$
- The index i_k can be selected **cyclically** $i_{k+1} = [i_k \bmod n] + 1$ or **randomly**
- In case f is differentiable, the update is

$$x_{k+1} = x_k - \alpha_k \frac{\partial f(x_k)}{\partial x_{i_k}} e_{i_k}, \quad \alpha_k > 0, \quad e_i = i\text{th column of } I$$

- In case of perfect line search $x_{k+1} = \arg\min_\alpha f(x_k + \alpha e_{i_k})$
- The method can be applied even if f is nonsmooth and some x_i discrete
- Although $f(x^{k+1}) \leq f(x^k)$ the method may not converge to a local minimum
- The method stops if there is no improvement in $f(x^k)$ after one full cycle

NONLINEAR LEAST SQUARES AND GAUSS-NEWTON METHOD

- We want to solve the **nonlinear least-squares** problem

$$\min_x \frac{1}{2}\sum_{j=1}^{m} r_j^2(x) \qquad = \frac{1}{2}\|r(x)\|_2^2$$

  where each **residual** $r_j : \mathbb{R}^n \to \mathbb{R}$ is smooth, $\forall j = 1, \ldots, m$ (assume $m \geq n$)
- Let $J(x)$ be the **Jacobian** associated with $r(x)$

$$J(x) = \begin{bmatrix} \nabla r_1(x)' \\ \vdots \\ \nabla r_m(x)' \end{bmatrix}$$

- The gradient $\nabla f(x) = \sum_{j=1}^m r_j(x)\nabla r_j(x) = J(x)'r(x)$, the Hessian is

$$\nabla^2 f(x) = \sum_{j=1}^{m} \nabla r_j(x)\nabla r_j(x)' + r_j(x)\nabla^2 r_j(x) = J(x)'J(x) + \sum_{j=1}^{m} r_j(x)\nabla^2 r_j(x)$$

- Gauss-Newton method approximates $\nabla^2 f(x_k) \approx J(x_k)'J(x_k)$

NONLINEAR LEAST SQUARES AND GAUSS-NEWTON METHOD

- Gauss-Newton does not require computing the Hessian matrices $\nabla^2 r_j(x_k)$

$$x_{k+1} = x_k + \alpha_k p_k, \quad p_k = -(J(x_k)'J(x_k))^{-1} J(x_k)'r(x_k)$$

- In many problems $J(x_k)'J(x_k)$ dominates over the neglected term $\sum_{j=1}^m \nabla r_j(x_k)\nabla^2 r_j(x_k)$ close to x^*, so convergence speed can get very close to Newton method

- When $J(x_k)$ is full rank, p_k is a descent direction:

$$p_k'\nabla f(x_k) = p_k' J(x_k)'r(x_k) = -p_k'(J(x_k)'J(x_k))p_k = -\|J(x_k)p_k\|_2^2$$

- Note that p_k can be obtained by solving the least-squares problem

$$p_k = \arg\min_p \frac{1}{2}\|J(x_k)p + r(x_k)\|_2^2$$

which is the linearized version of the problem at x_k, $r(x) \approx r(x_k) + J(x_k)p$

NONLINEAR LEAST SQUARES AND GAUSS-NEWTON METHOD

- Any technique can be used to solve each least-squares problem
- The Gauss-Newton (GN) method converges under mild assumptions

  (Nocedal, Wright, 2006, Th. 10.1)

$$\lim_{k\to\infty} \nabla f(x_k) = \lim_{k\to\infty} J(x_k)'r(x_k) = 0$$

- The **Levenberg-Marquardt** (LM) method is a damped version of GN, based on selecting p_k by solving the regularized system

$$p_k = -(\rho_k I + J(x_k)'J(x_k))^{-1} J(x_k)'r(x_k)$$

The parameter ρ_k can be selected at each iteration by simple rules.

Note that LM $\approx$GN for $\rho_k \ll 1$, LM $\approx$ gradient descent for $\rho_k \gg 1$.

The LM method can be reinterpreted also as a **trust-region** method.

GAUSS-NEWTON METHOD - EXAMPLE

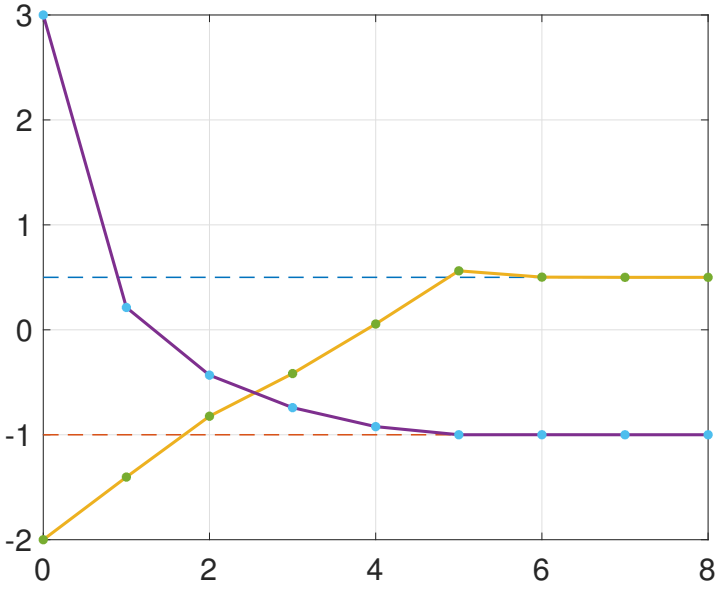
- We have a data set of $N = 10000$ samples (u_k, y_k)

$$y_k = x_1^2 u_{1k} + x_1 x_2 u_{2k} - x_2 u_{3k}^2 + n_k$$

where $x_1 = 0.5, x_2 = -1$ are unknown and noise $n_k \sim N(0, \sigma^2), \sigma = 0.01$

- We want to estimate the parameter vector x by minimizing

$$\min_x \frac{1}{2} \sum_{i=1}^{N} \| y_k - x_1^2 u_{1k} + x_1 x_2 u_{2k} - x_2 u_{3k}^2 \|_2^2$$



- Gauss-Newton method converges in 6 ms[1] after 8 iterations, with stopping tolerance $\|\nabla f(x_k)\| \leq 10^{-4}$

[1] Macbook 3 GHz Intel Core i7, MATLAB R2016b

NONLINEAR LEAST SQUARES - FITTING AN EPIDEMIC MODEL

- The spread of Coronavirus COVID-19 can be modeled by the **logistic model**[2]

$$n(t) = \frac{K}{1 + Ae^{-rt}}$$

where $n(t)$ = number of confirmed infected at time t and K = final epidemic size

- We want to fit (K, r, A) to data available for different countries[3,4]

$$\min_{K,r,A} \frac{1}{2} \sum_{j=1}^{m} \left\| n(t_j) - \frac{K}{1 + Ae^{-rt_j}} \right\|_2^2$$

nonlinear least squares

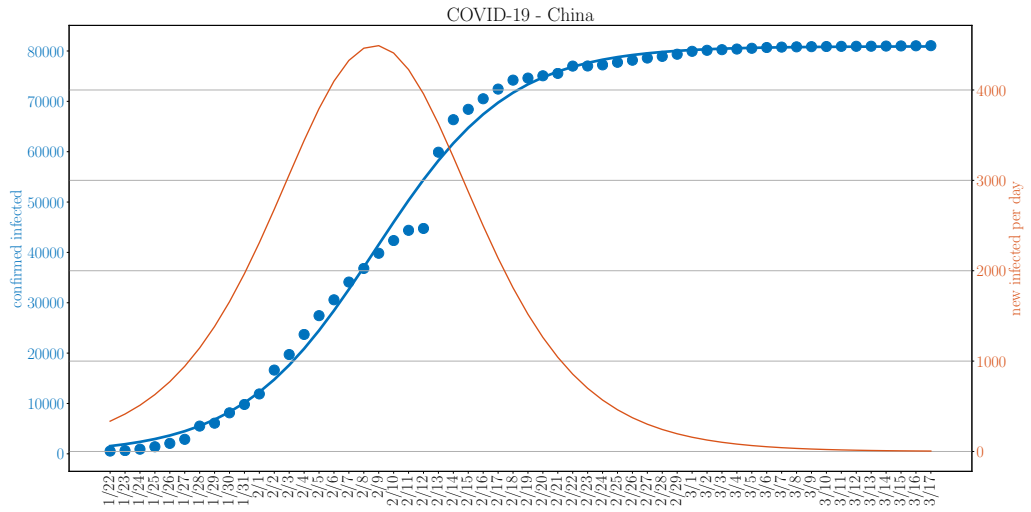
- Here $r_j = n(t_j) - \frac{K}{1+Ae^{-rt_j}}$, $\nabla r_j = \frac{1}{(1+Ae^{-rt_j})^2} \begin{bmatrix} 1+Ae^{-rt_j} \\ -Ke^{-rt_j} \\ KAt_je^{-rt_j} \end{bmatrix}$

[2] See also (Batista, 2020) https://www.researchgate.net/publication/339240777

[3] World data available at https://github.com/CSSEGISandData/COVID-19/raw/master/csse_covid_19_data/csse_covid_19_time_series/time_series_19-covid-Confirmed.csv

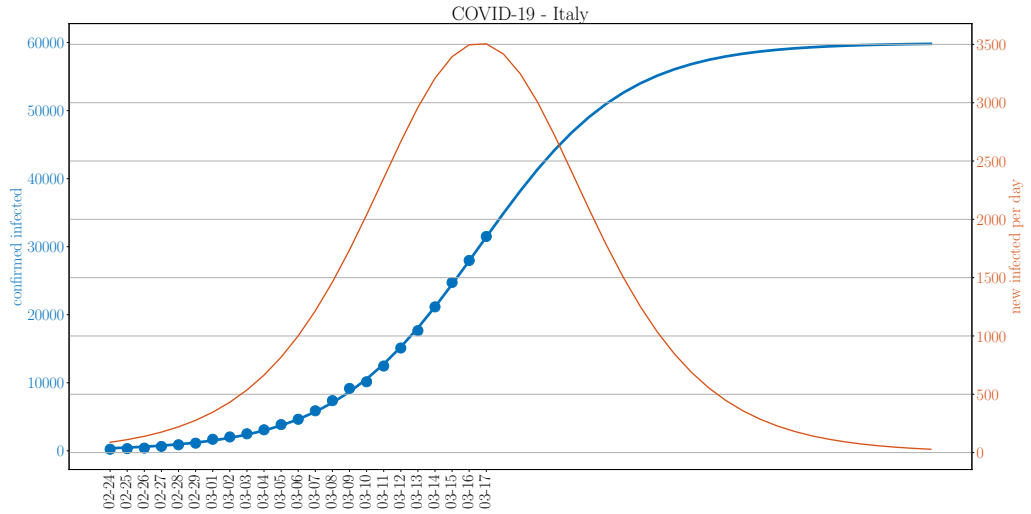
[4] Data for Italy available at https://github.com/pcm-dpc/COVID-19

NONLINEAR LEAST SQUARES - FITTING AN EPIDEMIC MODEL



Results for China:

$$K = 80917.6142$$
$$r = 0.2221$$
$$A = 51.6394$$



Results for Italy:

$$K = 59939.3989$$
$$r = 0.2344$$
$$A = 157.5456$$

[4] Problem solved using derivative-free Particle Swarm Optimization (Eberhart, Kennedy, 1995) via the `pyswarm` interface `https://pythonhosted.org/pyswarm/`

SEQUENTIAL QUADRATIC PROGRAMMING

Reference:

J.Nocedal and S.J. Wright, "*Numerical Optimization*," 2006. Chapter 18

EQUALITY-CONSTRAINED NLP

- We consider the equality-constrained NLP problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & h(x) = 0 \end{aligned}$$

with $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ smooth functions.

- The Lagrangian function and its derivatives are

$$\begin{aligned} \mathcal{L}(x,\nu) &= f(x) + \nu' h(x) \\ \nabla_x \mathcal{L}(x,\nu) &= \nabla f(x) + A'(x)\nu, \qquad A'(x) = \left[\, \nabla h_1(x) \;\ldots\; \nabla h_m(x) \,\right] \\ \nabla^2_{xx} \mathcal{L}(x,\nu) &= \nabla^2 f(x) + \sum_{i=1}^{m} \nu_i \nabla^2 h_i(x) \end{aligned}$$

- Assume $A(x)$ full row rank, $d' \nabla^2_{xx} \mathcal{L}(x,\nu) d > 0, \forall d \neq 0$ such that $A(x)d = 0$

QUADRATIC APPROXIMATION

- For all $\nu \in \mathbb{R}^m$, the original problem is equivalent to solving

$$\begin{array}{ll} \min & f(x) + \nu' h(x) = \mathcal{L}(x, \nu) \\ \text{s.t.} & h(x) = 0 \end{array}$$

- Consider a pair (x_k, ν_k) and the quadratic approximation of the problem around x_k

$$\begin{array}{ll} \min & f(x_k) + \nu_k' h(x_k) + (\nabla f(x_k) + A'(x_k)\nu_k)'p + \frac{1}{2} p' \nabla^2_{xx} \mathcal{L}(x_k, \nu_k) p \\ \text{s.t.} & h(x_k) + A(x_k)p = 0 \end{array}$$

- By exploiting $\nu_k'(h(x_k) + A(x_k)p) = 0$, the QP is equivalent to

$$\begin{array}{ll} \min & \frac{1}{2} p' \nabla^2_{xx} \mathcal{L}(x_k, \nu_k) p + \nabla f(x_k)' p \\ \text{s.t.} & h(x_k) + A(x_k)p = 0 \end{array}$$

SQP FOR EQUALITY-CONSTRAINED NLP

- The optimality conditions for the QP are

$$\begin{bmatrix} \nabla^2_{xx}\mathcal{L}(x_k,\nu_k) & A'(x_k) \\ A(x_k) & 0 \end{bmatrix} \begin{bmatrix} p_k \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} -\nabla f(x_k) \\ -h(x_k) \end{bmatrix}$$

- From the solution p_k we set $x_{k+1} = x_k + \alpha_k p_k$ (Newton's step)

- Also, we decided to update ν_{k+1} as the vector of Lagrange multipliers of the approximated QP

- **Sequential quadratic programming** (SQP) for equality-constrained NLP's iterates the above steps from an initial pair (x_0, ν_0) until convergence

- Note that in case $h(x) = Ax - b$ we have $\nabla^2_{xx}\mathcal{L}(x,\nu) = \nabla^2 f(x)$

SQP FOR NLP WITH EQUALITY AND INEQUALITY CONSTRAINTS

- A similar reasoning applies to general NLP problems

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0,\ i \in I \\ & g_j(x) = 0,\ j \in E \end{aligned}$$

with $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ smooth functions, $\forall i = 1, \ldots, m$.

- The quadratic approximation is

$$\begin{aligned} \min \quad & \tfrac{1}{2} p' \nabla^2_{xx} \mathcal{L}(x_k, \nu_k) p + \nabla f(x_k)' p \\ \text{s.t.} \quad & g_i(x_k) + \nabla g_i(x_k)' p \leq 0,\ i \in I \\ & g_j(x_k) + \nabla g_j(x_k)' p = 0,\ j \in E \end{aligned}$$

- We solve the QP, get the primal-dual solution (p_k, ν_{k+1}), and update $x_{k+1} = x_k + \alpha_k p_k$
- Several variants of the SQP method exist (including quasi-Newton methods)

INTERIOR-POINT METHODS

References:

S. Boyd, "Convex Optimization," lecture notes, `http://ee364a.stanford.edu`

J. Nocedal and S.J. Wright, "*Numerical Optimization*," 2006. Chapter 19

INTERIOR-POINT METHODS FOR CONVEX PROGRAMS

- Consider the **convex programming** problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0,\ i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- Assumptions:
  - f, g_i convex and twice continuously differentiable
  - $A \in \mathbb{R}^{p \times n}$ has $\text{rank}\, A = p$
  - an optimizer x^* exists and $f^* = f(x^*) \in \mathbb{R}$
  - the problem is strictly feasible

$$\exists x \,\text{dom}\, f : \ g_i(x) < 0,\ \forall i = 1, \ldots, m,\ Ax = b$$

  which ensures strong duality, i.e., $f(x^*) = q(\lambda^*, \nu^*)$, q = dual function

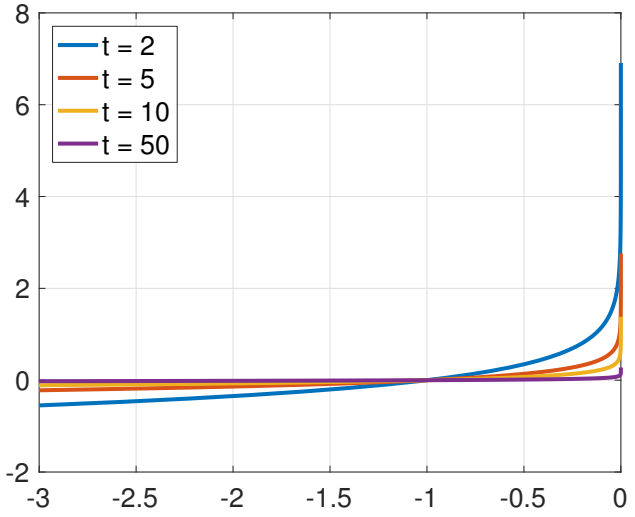
LOGARITHMIC BARRIER

- Denote by $I : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ the **indicator function** of the negative reals ($I(\alpha) = 0$ if $\alpha \leq 0, I(\alpha) = +\infty$ if $\alpha > 0$). The problem can be rewritten as

$$\begin{aligned} \min \quad & f(x) + \sum_{i=1}^{m} I(g_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- If we approximate $I(\alpha)$ with the smooth **logarithmic barrier** function $-\frac{1}{t}\log(-\alpha), t > 0$, we get

$$\begin{aligned} \min \quad & f(x) - \frac{1}{t}\sum_{i=1}^{m} \log(-g_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$



- The larger t the better the approximation

LOGARITHMIC BARRIER FUNCTION

- The **logarithmic barrier function**

$$\phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$$

has the following properties:

- $\operatorname{dom}\phi = \{x : \ g_i(x) < 0, \ i = 1, \ldots, m\}$
- ϕ is convex, since $-\log$ is monotonic and g_i is convex
- ϕ is twice continuously differentiable and

$$\begin{aligned} \nabla\phi(x) &= -\sum_{i=1}^{m} \frac{1}{g_i(x)} \nabla g_i(x) \\ \nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{g_i(x)^2} \nabla g_i(x) \nabla g_i(x)' - \sum_{i=1}^{m} \frac{1}{g_i(x)} \nabla^2 g_i(x) \end{aligned}$$

CENTRAL PATH

- For $t \geq 0$ let $x^*(t)$ be the optimizer of the approximated problem

$$\begin{aligned} \min \quad & tf(x) + \phi(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

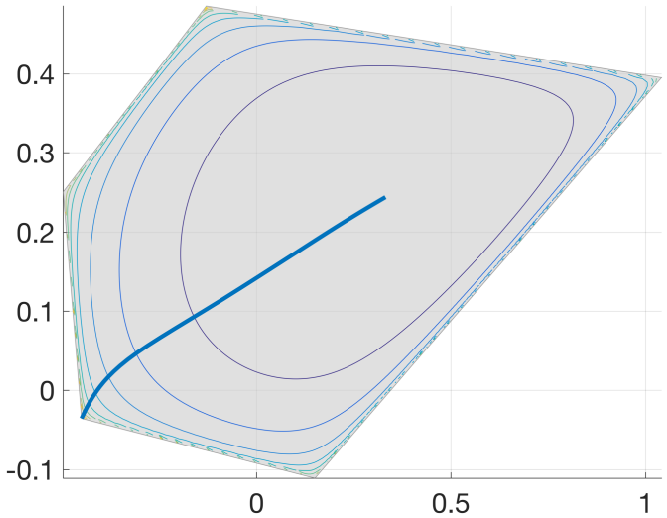
(assume for now that $x^*(t)$ is unique for all t)

- We call **central path** the curve $\{x^*(t)\}_{t>0}$

- Example: central path for the linear program

$$\begin{aligned} \min \quad & c'x \\ \text{s.t.} \quad & Gx \leq g,\ g \in \mathbb{R}^5 \end{aligned}$$

and level sets of $\phi(x)$



OPTIMALITY CONDITIONS

- The original problem satisfies the optimality conditions

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + A'\nu^* = 0$$
$$\lambda^* \geq 0,\ Ax^* = b,\ g_i(x^*) \leq 0, \lambda_i^* g_i(x^*) = 0$$

- The approximated optimizer $x^*(t)$ satisfies the optimality conditions

$$\nabla f(x^*(t)) - \sum_{i=1}^{m} \frac{1}{t g_i(x^*(t))} \nabla g_i(x^*(t)) + \frac{1}{t} A' w^*(t) = 0$$
$$Ax^*(t) = b$$

where $w^*(t)$ is the corresponding vector of Lagrange multipliers

OPTIMALITY CONDITIONS

- If we set $\lambda_i^*(t) \triangleq -\dfrac{1}{t g_i(x^*(t))}$, $\nu^*(t) = \frac{1}{t} w^*(t)$, for all $t > 0$ we have

$$\nabla f(x^*(t)) + \sum_{i=1}^{m} \lambda_i^*(t) \nabla g_i(x^*(t)) + A' \nu^*(t) = 0, \quad Ax^*(t) = b$$

$$\lambda_i^*(t) \geq 0, \quad \lambda_i^* g_i(x^*(t)) = -\frac{1}{t}$$

- These are the same KKT conditions of the original problem, except for the relaxation of the complementary slackness condition to $\lambda_i^* g_i(x^*(t)) = -\frac{1}{t}$

OPTIMALITY CONDITIONS

- The dual function q of the original problem evaluated at $\lambda^*(t), \nu^*(t)$ is

$$\begin{aligned} q(\lambda^*(t),\nu^*(t)) &= \min_x \overbrace{f(x) + \sum_{i=1}^m \lambda_i^*(t) g_i(x) + (Ax-b)'\nu^*(t)}^{\mathcal{L}(x,\lambda^*(t),\nu^*(t))} \\ &= f(x^*(t)) - \frac{m}{t} \end{aligned}$$

  as $x^*(t)$ also satisfies the optimality condition $\nabla_x \mathcal{L}(x,\lambda^*(t),\nu^*(t)) = 0$

- Since $q(\lambda^*(t),\nu^*(t)) \leq f(x^*)$, and since $f(x^*) \leq f(x^*(t))$ as $x^*(t)$ is feasible, we get

$$f(x^*(t)) - \frac{m}{t} \leq f(x^*) \leq f(x^*(t))$$

  which confirms the intuition $f(x^*(t)) \to f(x^*)$ as $t \to +\infty$

FINDING A FEASIBILE POINT (PHASE I)

- Consider the **feasibility problem**

$$\text{find } x \text{ such that } g_i(x) \leq 0,\ i = 1, \ldots, m, \quad Ax = b$$

- The **basic phase I method** consists of solving the following convex problem with $n+1$ variables

$$\begin{aligned}(x_0^*, s_0^*) = \arg\min_{x,s} \quad & s \\ \text{s.t.} \quad & g_i(x) - s \leq 0,\ i = 1, \ldots, m \\ & Ax = b\end{aligned}$$

  from any initial guess x_0 such that $Ax_0 = b$, $s_0 > \max g_i(x_0)$

- If $s_0^* < 0$ then x_0^* is strictly feasible for the original problem

BARRIER METHOD

- **Barrier method**: Given an initial strictly feasible x, execute:

  0. Let $t_0 > 0, t = t_0, \beta > 1$, tolerance $\epsilon > 0$

  1. Compute $x \leftarrow \arg\min \quad tf(x) + \phi(x)$ **(centering step)**
     $\text{s.t.} \quad Ax = b$
  2. If $\frac{m}{t} \leq \epsilon$ stop
  3. Otherwise increase $t \leftarrow \beta t$ and go to 1

- Newton's method solves the **centering step**, with the last x as initial guess
- Tradeoff: a large β makes fewer centering steps but more Newton iterations at each step. Typically $\beta = 10 \div 20$
- The algorithm terminates with $f(x) - f^*(x) \leq \frac{m}{t} \leq \epsilon$ in exactly $\left\lceil \frac{\log \frac{m}{\epsilon t_0}}{\log \beta} \right\rceil$ centering steps + computation of initial $x^*(t_0)$

PRIMAL-DUAL INTERIOR-POINT METHODS

- Consider the general NLP problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0 \end{array}$$

with $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m, h : \mathbb{R}^n \to \mathbb{R}^p$ smooth functions.

- The optimality condition for the NLP with slacks can be written as

$$\begin{array}{rcl} \nabla f(x) + \nabla g(x)'z + \nabla h(x)'y & = & 0 \\ g(x) + s & = & 0 \\ h(x) & = & 0 \\ SZe & = & 0 \\ s, z & \geq & 0 \end{array} \qquad \begin{array}{rcl} S & = & \text{diag}(s) \\ Z & = & \text{diag}(z) \\ e & = & [1 \ \ldots \ 1]' \end{array}$$

PRIMAL-DUAL INTERIOR-POINT METHODS

- Let us now relax the optimality conditions as

$$\begin{aligned} \nabla f(x) + \nabla g(x)'z + \nabla h(x)'y &= 0 \\ g(x) + s &= 0 \\ h(x) &= 0 \qquad \mu \geq 0 \\ SZe &= \mu e \\ s, z &\geq 0 \end{aligned}$$

- Let $x^*(\mu), s^*(\mu), y^*(\mu), z^*(\mu)$ be the solution of the relaxed KKT equations
- For $\mu > 0$ the curve $(x^*(\mu), s^*(\mu), y^*(\mu), z^*(\mu))$ is the **primal-dual central path**
- Note that $\mu = s_i^*(\mu) z_i^*(\mu) = -g_i(x^*(\mu)) z_i^*(\mu) = \frac{1}{t}$
- For $\mu \to 0$, under suitable assumptions, the central path converges to the primal/dual optimizer (x^*, s^*, y^*, z^*)

PRIMAL-DUAL INTERIOR-POINT METHODS

- Primal-dual interior point methods apply a Newton step to solve the system of relaxed KKT equations with decreasing values of μ
- They are more efficient than barrier method when high accuracy is needed
- Often exhibit superlinear asymptotic convergence
- They can start at infeasible points

PRIMAL-DUAL INTERIOR-POINT METHOD FOR LP

- Let us consider the LP

$$\begin{aligned} \min_x \quad & c'x \\ \text{s.t.} \quad & Ax \le b \\ & Ex = f \end{aligned}$$

- By introducing the slack vector $s = b - Ax$, the KKT conditions

$$\begin{aligned} c + A'z + E'y &= 0 \\ Ax + s &= b \\ Ex &= f \\ z_i s_i &= 0,\ i = 1, \dots, m \\ z, s &\ge 0 \end{aligned}$$

can be rewritten as

$$F(x,z,y,s) = \begin{bmatrix} A'z + E'y + c \\ Ax + s - b \\ Ex - f \\ ZSe \end{bmatrix} = 0, \quad z, s \ge 0$$

where $Z = \text{diag}(z_1, \dots, z_m), S = \text{diag}(s_1, \dots, s_m), e = [1 \ \dots \ 1]'$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR LP

- We want to solve the nonlinear system $F(x,z,y,s)=0$ by Newton's method
- Starting from a candidate solution $z>0, s>0, x, y$, Newton's step $\Delta x, \Delta z, \Delta y, \Delta s$ is given by solving the linear system

$$0 = F(x,z,y,s) + \nabla F(x,z,y,s) \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta y \\ \Delta s \end{bmatrix}$$

- Let $\begin{bmatrix} r^c \\ r^b \\ r^f \end{bmatrix} = \begin{bmatrix} A'z+E'y+c \\ Ax+s-b \\ Ex-f \end{bmatrix}$. The linear system to solve is

$$\begin{bmatrix} 0 & A' & E' & 0 \\ A & 0 & 0 & I \\ E & 0 & 0 & 0 \\ 0 & S & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r^c \\ -r^b \\ -r^f \\ -ZSe \end{bmatrix}$$

- To preserve $z, s \geq 0$ we set $\begin{bmatrix} x \\ z \\ y \\ s \end{bmatrix} \leftarrow \begin{bmatrix} x \\ z \\ y \\ s \end{bmatrix} + \alpha \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta y \\ \Delta s \end{bmatrix}$ with α sufficiently small

PRIMAL-DUAL INTERIOR-POINT METHOD FOR LP

- To prevent excessively small α, given the current $x_k, s_k, \lambda_k, x_k, s_k > 0$, primal-dual interior-point method solve instead the relaxed system

$$\begin{bmatrix} 0 & A' & E' & 0 \\ A & 0 & 0 & I \\ E & 0 & 0 & 0 \\ 0 & S_k & 0 & Z_k \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta z_k \\ \Delta y_k \\ \Delta s_k \end{bmatrix} = \begin{bmatrix} -r_k^c \\ -r_k^b \\ -r_k^f \\ -Z_k S_k e + \sigma_k \mu_k e \end{bmatrix}$$

  where $\mu_k = \frac{1}{m} z_k' s_k$ is the current **duality measure** and $\sigma_k \in [0, 1]$ is the **centering parameter**, that is the factor we want to reduce the current μ_k

- The performance of the method depends on how α_k and σ_k are chosen
- **Mehrotra's predictor-corrector algorithm** is one of the most used IP methods for LP (Mehrotra, 1992)
- **Homogeneous and self-dual formulations** are useful to easily recognize infeasibility and unboundedness (Yu, Todd, Mizuno, 1994) (Xu, Hung, Ye, 1996)

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

- Consider the convex QP

$$\begin{array}{ll} \min_x & \frac{1}{2}x'Qx + c'x \\ \text{s.t.} & Ax \le b \\ & Ex = f \end{array} \qquad Q = Q' \succeq 0$$

- By introducing the slack vector $s = b - Ax$, the KKT conditions

$$\begin{aligned} Qx + c + E'y + A'z &= 0 \\ Ex &= f \\ Ax + s &= b \\ z_i s_i &= 0,\ i = 1, \dots, m \\ z, s &\ge 0 \end{aligned}$$

can be rewritten as

$$0 = F(x, z, y, s) = \begin{bmatrix} Qx + E'y + A'z + c \\ Ex - f \\ Ax + s - b \\ ZSe \end{bmatrix} \triangleq \begin{bmatrix} r_Q \\ r_E \\ r_A \\ r_S \end{bmatrix}, \quad z, s \ge 0$$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998)

- Start from a candidate solution $z > 0, s > 0, x, y$
- As for LP, we want to solve $F(x, z, y, s) = 0$ by Newton's method
- We use a variant of Mehrotra's predictor-corrector algorithm (Mehrotra, 1992)
- First, we solve the linear system (**predictor step**)

$$\underbrace{\begin{bmatrix} Q & E' & A' & 0 \\ E & 0 & 0 & 0 \\ A & 0 & 0 & I \\ 0 & 0 & S & Z \end{bmatrix}}_{\nabla F} \begin{bmatrix} \Delta x_{\text{aff}} \\ \Delta y_{\text{aff}} \\ \Delta z_{\text{aff}} \\ \Delta s_{\text{aff}} \end{bmatrix} = \underbrace{\begin{bmatrix} -r_Q \\ -r_E \\ -r_A \\ -r_S \end{bmatrix}}_{-F}$$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998)

- Next, we solve the linear system (**centering-corrector step**)

$$\begin{bmatrix} Q & E' & A' & 0 \\ E & 0 & 0 & 0 \\ A & 0 & 0 & I \\ 0 & 0 & S & Z \end{bmatrix} \begin{bmatrix} \Delta x_{\rm cc} \\ \Delta y_{\rm cc} \\ \Delta z_{\rm cc} \\ \Delta s_{\rm cc} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\Delta S_{\rm aff}\Delta Z_{\rm aff} e + \sigma\mu e \end{bmatrix}$$

where the **centering parameter** $\sigma \in [0,1)$ is chosen as

$$\begin{aligned} \alpha_{\rm aff} &= \arg\max_\alpha \{\alpha \in [0,1] : \begin{bmatrix} z+\alpha\Delta z_{\rm aff} \\ s+\alpha\Delta s_{\rm aff} \end{bmatrix} \geq 0\} \\ \mu_{\rm aff} &= (z+\alpha_{\rm aff}\Delta z_{\rm aff})'(s+\alpha_{\rm aff}\Delta s_{\rm aff})/m \\ \mu &= z's/m \qquad \leftarrow \text{duality gap} \\ \sigma &= (\mu_{\rm aff}/\mu)^3 \end{aligned}$$

- **Note**: the same left-hand-side matrix is used to solve both linear systems. So such a matrix can be factorized just once at each IP iteration

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998)

- Now set

$$\Delta x = \Delta x_{\rm aff} + \Delta x_{\rm cc}, \quad \Delta y = \Delta y_{\rm aff} + \Delta y_{\rm cc}$$
$$\Delta z = \Delta z_{\rm aff} + \Delta z_{\rm cc}, \quad \Delta s = \Delta s_{\rm aff} + \Delta s_{\rm cc}$$

  and choose $\alpha_{\max} = \arg\max\{\alpha \in [0,1] : \begin{bmatrix} z+\alpha\Delta z \\ s+\alpha\Delta s \end{bmatrix} \geq 0\}$, so that z, s remain nonnegative

- The actual step-length is chosen as $\alpha = \gamma\alpha_{\max}$, with the step-factor $\gamma \in (0,1)$ close to 1, see (Mehrotra, 1992)
- For even better choices of the step-length α see (Curtis, Nocedal, 2007)
- For reducing the number of factorizations, execute multiple corrections steps (Gondzio, 1996)
- Given a starting point $\bar{x}, \bar{y}, \bar{z}, \bar{s}$, a good initial guess is to solve for $\Delta z_{\rm aff}, \Delta s_{\rm aff}$ and set $x_0 = \bar{x}, y_0 = \bar{y}, z_0 = \max\{1, |\bar{z} + \Delta z_{\rm aff}|\}, s_0 = \max\{1, |\bar{s} + \Delta s_{\rm aff}|\}$

PRIMAL-DUAL INTERIOR-POINT METHOD FOR QP

(Nocedal, Wright, 2006) (Rao, Wright, Rawlings, 1998) (Gondzio, Terlaki, 1994)

- Let $\Delta\tilde{z} = Z^{-1}\Delta z$. We can eliminate $\Delta s = Z^{-1}r_S - S\Delta\tilde{z}$ and get the system

$$\begin{bmatrix} Q & E' & A'Z \\ E & 0 & 0 \\ A & 0 & -S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\tilde{z} \end{bmatrix} = \begin{bmatrix} -r_Q \\ -r_E \\ Z^{-1}r_S - r_A \end{bmatrix}$$

- The above system can be made symmetric by multiplying the last rows by Z
- We can further easily eliminate $Z^{-1}\Delta z = S^{-1}(A\Delta x + r_A - Z^{-1}r_S)$ and get

$$\begin{bmatrix} Q + A'ZS^{-1}A & E' \\ E & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -r_Q + A'S^{-1}(r_S - Zr_A) \\ -r_E \end{bmatrix}$$

- Note that $Z^{-1}S$ is positive and diagonal.

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI

- **IPOPT** (**I**nterior **P**oint **OPT**imizer[5]) is a software package based an IP method to solve the NLP (Wächter, Biegler, 2006)

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g_\ell \leq g(x) \leq g_u \\ & x_\ell \leq x \leq x_u \end{aligned}$$

- **CasADi**[6] is a modeling language for NLP problems. It implements **automatic differentiation** for computing gradients (Andersson, Gillis, Horn, Rawlings, Diehl, 2019)

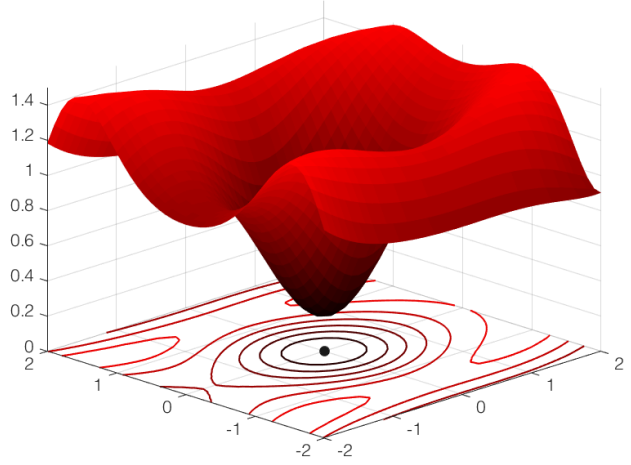
- CasADi + IPOPT greatly simplifies formulating and solving nonlinear optimization problems via interior-point methods in MATLAB, python, or C++

[5] https://coin-or.github.io/Ipopt/
[6] https://web.casadi.org/

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI

- Let us minimize the course-logo function

$$f(x,y) = -e^{-(x^2+y^2)} + 0.3\sin\left(\frac{1}{10}x^3 + y^2\right) + 1.2$$




```
import casadi.*
x=SX.sym('x');
y=SX.sym('y');
f=-exp(-(x^2+y^2))+.3*sin(x^3/10+y^2)+1.2;
P=struct('f',f,'x',[x;y]);
F=nlpsol('F','ipopt',P);
r=F('x0',[-1;-1]);
xopt=full(r.x);
fopt=full(r.f);
```

MATLAB

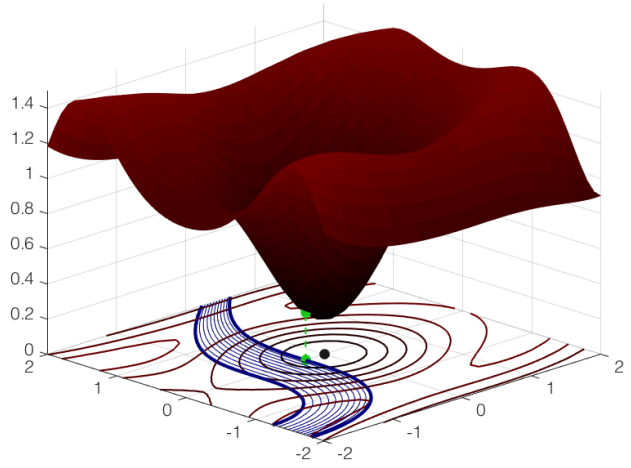
```
from casadi import *
x=SX.sym('x')
y=SX.sym('y')
f=-exp(-(x**2+y**2))+.3*sin(x**3/10+y**2)+1.2
P=dict(x=vertcat(x,y), f=f)
F=nlpsol('F','ipopt',P)
r=F(x0=[-1,-1])
xopt=r['x'].full()
fopt=r['f'].full()
```

python

- Optimizer $x^* = 0, y^* = 0$, optimum $f^* = 0.2$

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI

```
Number of objective function evaluations             = 18
Number of objective gradient evaluations             = 11
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 10
```



- Let's add the constraint

$$1 \leq (x+2)^2 - \frac{1}{2}y^3 \leq 3$$

```
g=(x+2)^2-y^3/2;
P=struct('f',f,'x',[x;y],'g',g);
F=nlpsol('F','ipopt',P);
r=F('x0',[-1;-1],'ubg',3,'lbg',1);
xopt=full(r.x);
fopt=full(r.f);
lam_g_opt = full(r.lam_g);
```

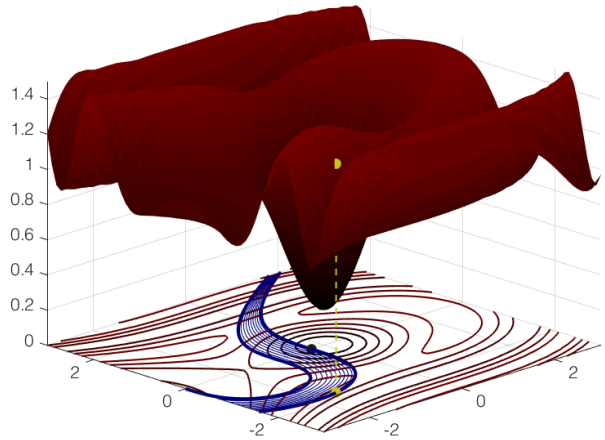
MATLAB

```
g= (x+2)**2-y**3/2
P=dict(x=vertcat(x,y), f=f, g=g)
F=nlpsol('F','ipopt',P)
r=F(x0=[-1,-1],ubg=3,lbg=1))
xopt=r['x'].full()
fopt=r['f'].full()
lam_g_opt=r['lam_g'].full()
```

python

- New optimizer $x^* = -0.2679, y^* = 0$, optimum $f^* = 0.2687$

EXAMPLE: NLP SOLUTION VIA IPOPT & CASADI



- **Caveat**: the NLP is non convex.

- If we start from $x_0 = -2, y_0 = -2$ we get the different local minimum

  $x^* = -1.5078, y^* = -1.7668$, optimum $f^* = 1.3019$!

- If function/constraints are not convex, one may need to test different initial conditions, or switch to **global optimization** methods