MODEL PREDICTIVE CONTROL

HYBRID MODELS FOR MPC

Alberto Bemporad

http://cse.lab.imtlucca.it/~bemporad/mpc_course.html

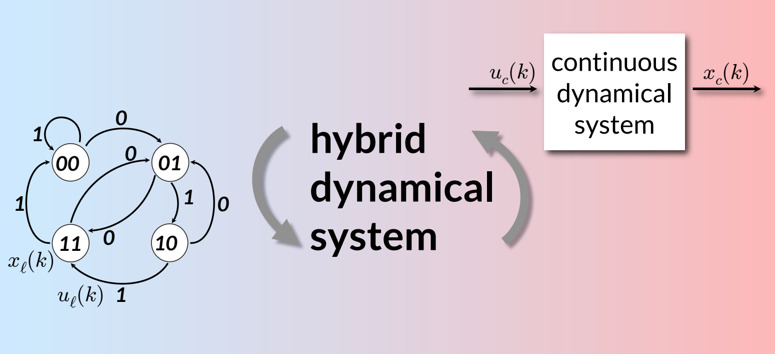
IMT SCHOOL FOR ADVANCED STUDIES LUCCA

COURSE STRUCTURE

- ✔ **Basic concepts** of model predictive control (MPC) and **linear MPC**
- ✔ **Linear time-varying** and **nonlinear** MPC
- ✔ **Quadratic programming** (QP) and **explicit MPC**
- **Hybrid** MPC
- **Stochastic** MPC
- **Learning-based** MPC

HYBRID MODELS

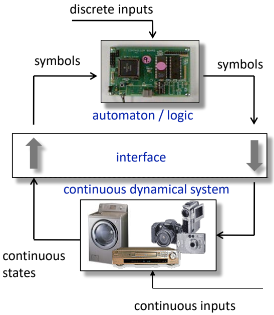
HYBRID DYNAMICAL SYSTEMS

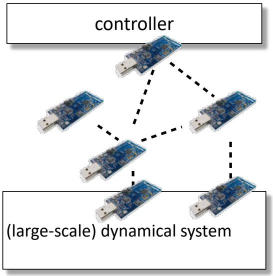


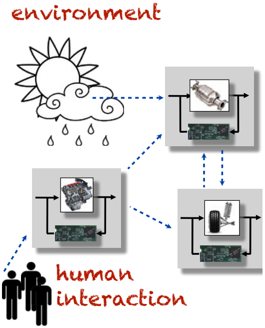
- Variables are **binary-valued**
  $x_\ell \in \{0,1\}^{n_\ell}$, $u_\ell \in \{0,1\}^{m_\ell}$
- Dynamics = **finite state machine**
- **Logic constraints**

- Variables are **real-valued**
  $x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^{m_c}$
- **Difference/differential equations**
- **Linear inequality** constraints

TECHNOLOGICAL PUSH FOR STUDYING HYBRID SYSTEMS







embedded systems

networked control systems

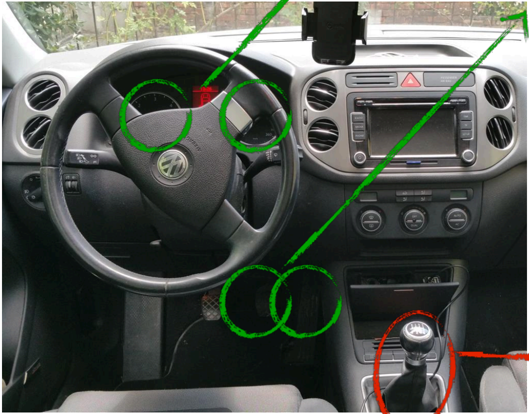
cyber-physical systems

>1995 >2005 >2010

AN EXAMPLE OF "INTRINSICALLY HYBRID" SYSTEM

- Vehicle

continuous dynamical variables
(speed, torque, ...)



continuous commands
(brake & gas pedal)

+

discrete command
(R,N,1,2,3,4,5)

KEY REQUIREMENTS FOR HYBRID MODELS

- **Descriptive** enough to capture the behavior of the system
  - **continuous** dynamics (physical systems)
  - **logic** components (switches, automata)
  - **interconnection** between logic and dynamics

- **Simple** enough for solving analysis and synthesis problems

$$\begin{cases} x' &= Ax + Bu \\ y &= Cx + Du \end{cases} \quad \longleftrightarrow \quad \begin{cases} x' &= f(x,u,t) \\ y &= g(x,u,t) \end{cases}$$

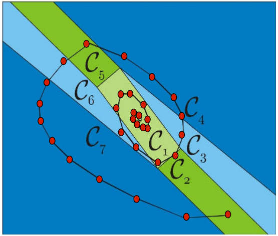
linear hybrid systems

> "Perfection is achieved not when there is nothing more to add, but when there is nothing left to take away."

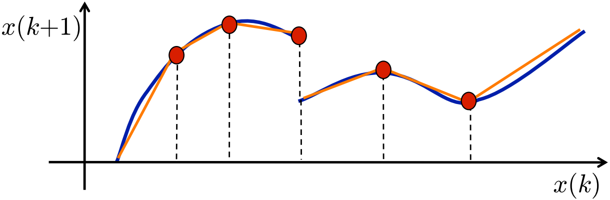
A. de Saint-Exupéry
(1900–1944)

PIECEWISE AFFINE SYSTEMS

$$
\begin{aligned}
x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\
y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\
i(k) \quad \text{s.t.} \quad & H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}
\end{aligned}
$$



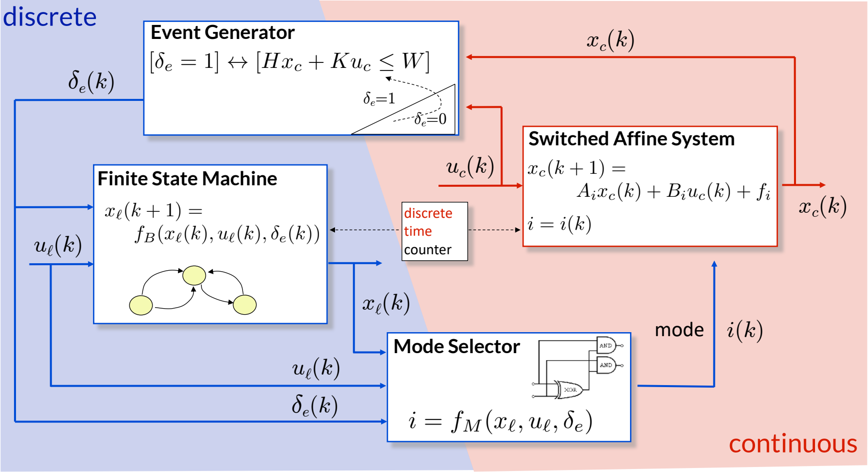
- PWA systems can approximate nonlinear dynamics arbitrarily well (even discontinuous ones)



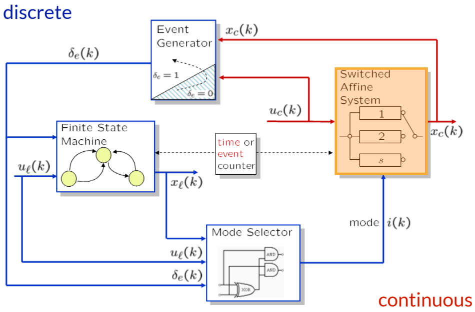
- Model **descriptiveness** vs model **complexity tradeoff**

DISCRETE HYBRID AUTOMATON (DHA)

(Torrisi, Bemporad, 2004)



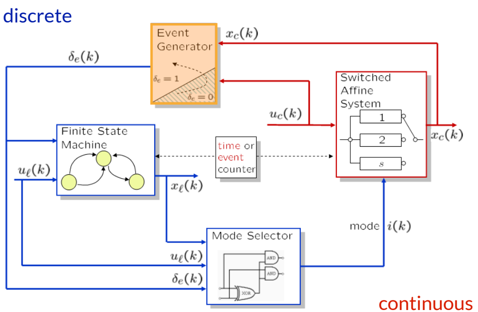
$x_\ell \in \{0,1\}^{n_\ell}$	$=$	binary state	$x_c \in \mathbb{R}^{n_c}$	$=$	real-valued state
$u_\ell \in \{0,1\}^{m_\ell}$	$=$	binary input	$u_c \in \mathbb{R}^{m_c}$	$=$	real-valued input
$\delta_e \in \{0,1\}^{n_e}$	$=$	event variable	$i \in \{1,\dots,s\}$	$=$	current mode



- The **affine dynamics** depend on the current mode $i(k)$:

$$x_c(k+1) = A_{i(k)}x_c(k) + B_{i(k)}u_c(k) + f_{i(k)}$$

$x_c \in \mathbb{R}^{n_c}, u_c \in \mathbb{R}^{m_c}$

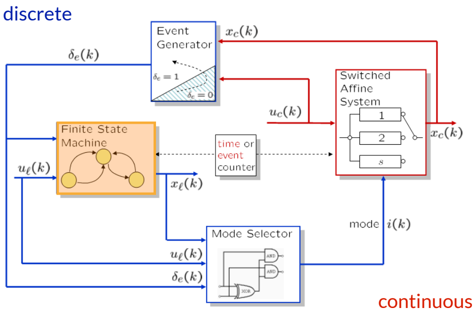


- **Event variables** are generated by **linear threshold conditions** over continuous states, continuous inputs, and time:

$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) + K^i u_c(k) \leq W^i]$$

$$x_c \in \mathbb{R}^{n_c}, \quad u_c \in \mathbb{R}^{m_c}$$
$$\delta_e \in \{0, 1\}^{n_e}$$

- Example: $[\delta_e(k) = 1] \leftrightarrow [x_c(k) \geq 0]$



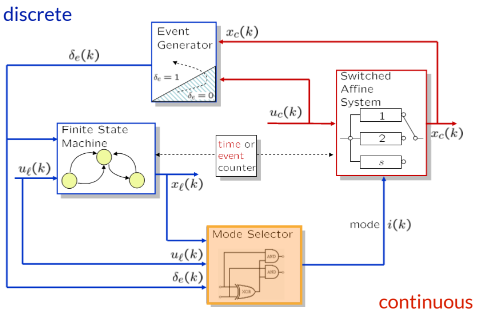
- The binary state of the **finite state machine** evolves according to a Boolean state update function $f_B : \{0,1\}^{n_\ell+m_\ell+n_e} \to \{0,1\}^{n_\ell}$:

$$x_\ell(k+1) = f_B(x_\ell(k), u_\ell(k), \delta_e(k))$$

$$x_\ell \in \{0,1\}^{n_\ell}, \quad u_\ell \in \{0,1\}^{m_\ell}$$
$$\delta_e \in \{0,1\}^{n_e}$$

- Example: $x_\ell(k+1) = \neg\delta_e(k) \vee (x_\ell(k) \wedge u_\ell(k))$

MODE SELECTOR



The mode selector can be seen as the output function of the discrete dynamics

- The active **mode** $i(k)$ is selected by a Boolean function of the current binary states, binary inputs, and event variables:

$$i(k) = f_M(x_\ell(k), u_\ell(k), \delta_e(k))$$

$$x_\ell \in \{0,1\}^{n_\ell}, \quad u_\ell \in \{0,1\}^{m_\ell}$$
$$\delta_e \in \{0,1\}^{n_e}$$

- Example:

$$i(k) = \begin{bmatrix} \neg u_\ell(k) \vee x_\ell(k) \\ u_\ell(k) \wedge x_\ell(k) \end{bmatrix}$$

u_ℓ / x_ℓ	0	1
0	$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
1	$i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

the system has 3 modes

CONVERSION OF LOGIC FORMULAS TO LINEAR INEQUALITIES

(Glover, 1975) (Williams, 1977) (Hooker, 2000)

- Key observation: $X_1 \vee X_2 = \texttt{true}$ → $\delta_1 + \delta_2 \geq 1, \delta_1, \delta_2 \in \{0, 1\}$
- We want to impose the Boolean statement

$$F(X_1, \ldots, X_n) = \texttt{true}$$

- Convert the formula to **Conjunctive Normal Form (CNF)**

$$\bigwedge_{j=1}^{m} \left(\bigvee_{i \in P_j} X_i \bigvee_{i \in N_j} \bar{X}_i \right) = \texttt{true}, \quad P_j \cup N_j \subseteq \{1, \ldots, n\}$$

- Transform the CNF into the equivalent linear inequalities

$$\left\{ \begin{array}{rcl} \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) & \geq & 1 \\ \vdots & & \vdots \\ \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) & \geq & 1 \end{array} \right. \quad \rightarrow \quad A\delta \leq b,\ \delta \in \{0, 1\}^n$$

polyhedron

Any logic proposition can be translated into integer linear inequalities

LOGIC $\longrightarrow$ INEQUALITIES: SYMBOLIC APPROACH

- Example: we want to impose the following condition on X_1, X_2, X_3

$$F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2] = \texttt{true}$$

- Convert F to Conjunctive Normal Form (CNF):

  (see e.g. `https://www.wolframalpha.com` or just google "CNF + converter" ...)

$$(X_3 \vee \neg X_1 \vee \neg X_2) \wedge (X_1 \vee \neg X_3) \wedge (X_2 \vee \neg X_3)$$

- Transform into inequalities:

$$\left\{\begin{array}{rcl} \delta_3 + (1-\delta_1) + (1-\delta_2) & \geq & 1 \\ \delta_1 + (1-\delta_3) & \geq & 1 \\ \delta_2 + (1-\delta_3) & \geq & 1 \end{array}\right.$$

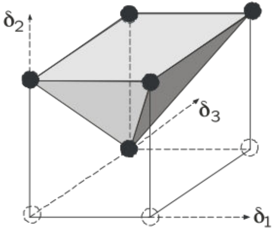
- Consider the Boolean statement $F(X_1, \ldots, X_n) =$ `true` and collect the rows of the **truth table** $T(F)$ of F

> The **convex hull** $P = \{\delta \in \mathbb{R}^n : A\delta \leq b\}$ of the **rows in** $T(F)$ is the smallest polytope equivalent to the Boolean statement F

(Mignone, Bemporad, Morari, 1999)

$T(F)$:

X_1	X_2	...	X_n
0	0	...	1
0	1	...	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	...	0

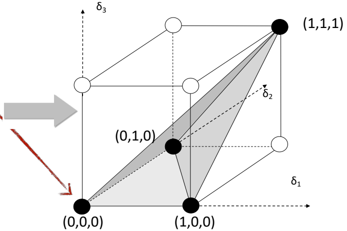


- Convex hull packages: `cdd`, `lrs`, `qhull`, `chD`, `Hull`, `Porto`
  `CDDMEX` package by K. Fukuda included in the Hybrid Toolbox

- Example: $F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$ (logic **and**)

$T(F)$:

X_1	X_2	X_3
0	0	0
0	1	0
1	0	0
1	1	1



- **Key idea**: white points cannot be inside the convex hull of black points

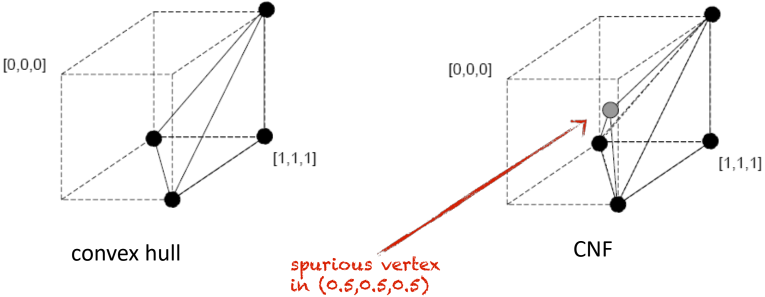
$$\text{conv}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \left\{\delta \in \mathbb{R}^3 : \begin{array}{rcl} -\delta_1 + \delta_3 & \leq & 0 \\ -\delta_2 + \delta_3 & \leq & 0 \\ \delta_1 + \delta_2 - \delta_3 & \leq & 1 \end{array}\right\}$$

```
>> V=struct('V',[0 0 0;0 1 0;1 0 0;1 1 1]);
>> H=cddmex('hull',V);A=H.A,b=H.B
```


GEOMETRIC VS SYMBOLIC APPROACH

- The polyhedron obtained via convex hull is the smallest one
- The one obtained via CNF may be larger. Example:

$$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) = \texttt{true}$$



- **Note**: no other example with 3 vars but
$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge (\neg X_1 \vee \neg X_2 \vee \neg X_3)$ = $\texttt{true}$

BIG-M TECHNIQUE (IFF)

- Consider the **if-and-only-if** condition

$$[\delta = 1] \leftrightarrow [a'x_c - b \leq 0]$$

$$x_c \in \mathcal{X}$$
$$\delta \in \{0, 1\}$$

- Assume $\mathcal{X} \subset \mathbb{R}^{n_c}$ bounded. Let M and m such that $\forall x_c \in \mathcal{X}$

$$\begin{array}{rcl} M & > & a'x_c - b \\ m & < & a'x_c - b \end{array}$$

- The if-and-only-if condition is equivalent to

$$\left\{ \begin{array}{rcl} a'x_c - b & \leq & M(1 - \delta) \\ a'x_c - b & > & m\delta \end{array} \right.$$

- We can replace the second constraint with $a'x_c - b \geq \epsilon + (m - \epsilon)\delta$ to avoid strict inequalities, where $\epsilon > 0$ is a small number (e.g., the machine precision)

COMPUTING THE BIG-M

- If $\mathcal{X}$ is a polyhedron, we can use linear programming

$$\begin{aligned} m &< \min_{x_c \in \mathcal{X}} \{a' x_c\} - b \\ M &> -\min_{x_c \in \mathcal{X}} \{-a' x_c\} - b \end{aligned}$$

- If $\mathcal{X}$ is a box, $\mathcal{X} = [\ell, u]$, we can use the following simpler method

  (Lee, Kouvaritakis, 2000) (Bemporad, 2022)

$$\begin{aligned} a_+ &= \max\{a, 0\} \\ a_- &= a_+ - a = \max\{-a, 0\} \\ m &< a_+' \ell - a_-' u - b \\ M &> a_+' u - a_-' \ell - b \end{aligned}$$

BIG-M TECHNIQUE (IF-THEN-ELSE)

- Consider the **if-then-else** condition

$$z = \begin{cases} a_1' x_c - b_1 & \text{if } \delta = 1 \\ a_2' x_c - b_2 & \text{otherwise} \end{cases}$$

$$x_c \in \mathcal{X}$$
$$\delta \in \{0, 1\}$$
$$z \in \mathbb{R}$$

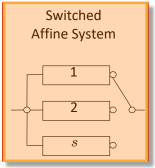
- Assume $\mathcal{X} \subset \mathbb{R}^{n_c}$ bounded. Let M_1, M_2 and m_1, m_2 such that $\forall x_c \in \mathcal{X}$

$$\begin{array}{ccccc} M_1 & > & a_1' x_c - b_1 & > & m_1 \\ M_2 & > & a_2' x_c - b_2 & > & m_2 \end{array}$$

- The if-then-else condition is equivalent to

$$\begin{cases} z & \leq & a_1' x_c - b_1 - (m_1 - M_2)(1 - \delta) \\ z & \geq & a_1' x_c + b_1 + (m_2 - M_1)(1 - \delta) \\ z & \leq & a_2' x_c - b_2 - (m_2 - M_1)\delta \\ z & \geq & a_2' x_c - b_2 + (m_1 - M_2)\delta \end{cases}$$

SWITCHED AFFINE SYSTEM



- The state-update equation of a SAS can be rewritten as

$$\boxed{x_c(k+1) = \sum_{i=1}^{s} z_i(k)} \quad z_i(k) \in \mathbb{R}^{n_c}$$

with

$$z_1(k) = \begin{cases} A_1 x_c(k) + B_1 u_c(k) + f_1 & \text{if } \delta_1(k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\vdots$$

$$z_s(k) = \begin{cases} A_s x_c(k) + B_s u_c(k) + f_s & \text{if } \delta_s(k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and with $\delta_i(k) \in \{0, 1\}$ subject to the **exclusive or** condition

$$\sum_{i=1}^{s} \delta_i(k) = 1 \text{ or equivalently } \begin{cases} \sum_{i=1}^{s} \delta_i(k) & \geq & 1 \\ \sum_{i=1}^{s} \delta_i(k) & \leq & 1 \end{cases}$$

- Output eq. $y_c(k) = C_i x_c(k) + D_i u_c(k) + g_i$ admits a similar transformation

TRANSFORMATION OF A DHA INTO LINEAR (IN)EQUALITIES

$X_1 \vee X_2 =$ TRUE $\quad \longrightarrow \quad \delta_1 + \delta_2 \geq 1, \qquad \delta_1, \delta_2 \in \{0, 1\}$

Any logic statement

$f(X) =$ TRUE

$$\bigwedge_{j=1}^{m} \left(\vee_{i \in P_j} X_i \vee_{i \in N_j} \neg X_i \right) \quad \text{(CNF)}$$

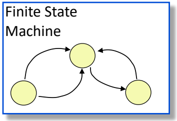
$N_j, P_j \subseteq \{1, \ldots, n\}$

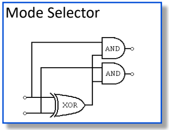
$$\longrightarrow \quad \left\{ \begin{array}{l} 1 \leq \sum\limits_{i \in P_1} \delta_i + \sum\limits_{i \in N_1} (1 - \delta_i) \\ \vdots \\ 1 \leq \sum\limits_{i \in P_m} \delta_i + \sum\limits_{i \in N_m} (1 - \delta_i) \end{array} \right.$$

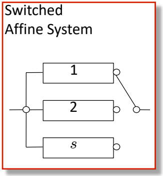
$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) \leq W^i] \quad \longrightarrow \quad \left\{ \begin{array}{l} H^i x_c(k) - W^i \leq M^i (1 - \delta_e^i(k)) \\ H^i x_c(k) - W^i > m^i \delta_e^i(k) \end{array} \right.$$

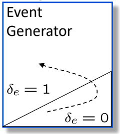
IF $[\delta = 1]$ THEN $z = a_1^T x + b_1^T u + f_1$
ELSE $z = a_2^T x + b_2^T u + f_2$

$$\longrightarrow \quad \left\{ \begin{array}{rcl} (m_1 - M_2)(1 - \delta) + z & \leq & a_1 x + b_1 u + f_1 \\ (m_2 - M_1)(1 - \delta) - z & \leq & -a_1 x - b_1 u - f_1 \\ (m_2 - M_1)\delta + z & \leq & a_2 x + b_2 u + f_2 \\ (m_1 - M_2)\delta - z & \leq & -a_2 x - b_2 u - f_2 \end{array} \right.$$





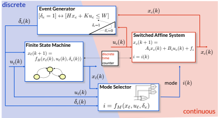




MIXED LOGICAL DYNAMICAL (MLD) SYSTEMS

(Bemporad, Morari, 1999)

- By converting logic relations into mixed-integer linear inequalities a DHA can be rewritten as the **Mixed Logical Dynamical (MLD)** system



$$\begin{cases} x(k+1) = Ax(k) + B_1 u(k) + B_2\delta(k) + B_3 z(k) + B_5 \\ y(k) = Cx(k) + D_1 u(k) + D_2\delta(k) + D_3 z(k) + D_5 \\ E_2\delta(k) + E_3 z(k) \le E_4 x(k) + E_1 u(k) + E_5 \end{cases}$$

$$x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b},\ u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$$
$$y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_b},\ \delta \in \{0,1\}^{r_b},\ z \in \mathbb{R}^{r_c}$$

- The translation from DHA to MLD can be automatized, see e.g. the language **HYSDEL** (HYbrid Systems DEscription Language) (Torrisi, Bemporad, 2004)

- MLD models allow solving MPC, verification, state estimation, and fault detection problems via **mixed-integer programming**

A SIMPLE EXAMPLE OF MLD SYSTEM

- PWA system[1]: $$x(k+1) = \begin{cases} 0.8x(k)+u(k) & \text{if} \quad x(k)\geq 0 \\ -0.8x(k)+u(k) & \text{if} \quad x(k)<0 \end{cases}$$

- Introduce event variable $[\delta(k)=1] \leftrightarrow [x(k)\geq 0]$ and use big-M technique:

$$\Rightarrow \quad \begin{array}{l} x(k)\geq m(1-\delta(k)) \\ x(k)\leq -\epsilon+(M+\epsilon)\delta(k) \end{array} \qquad \begin{array}{l} M=-m=10 \\ \epsilon>0 \text{ “small”} \end{array}$$

- Since $x(k+1)=1.6\delta(k)x(k)-0.8x(k)+u(k)$, introduce the aux variable

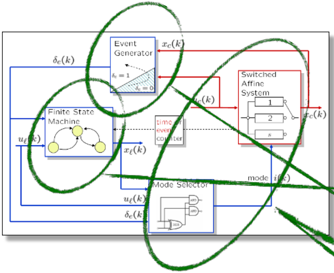
$$z(k)=\delta(k)x(k) \quad \Rightarrow \quad \begin{array}{rcl} z(k) & \leq & M\delta(k) \\ z(k) & \geq & m\delta(k) \\ z(k) & \leq & x(k)-m(1-\delta(k)) \\ z(k) & \geq & x(k)-M(1-\delta(k)) \end{array}$$

$$\delta(k)\in\{0,1\}$$

- Linear state update: $$x(k+1)=-0.8x(k)+1.6z(k)+u(k)$$

[1] This is the nonlinear system $x(k+1)=0.8|x(k)|+u(k)$

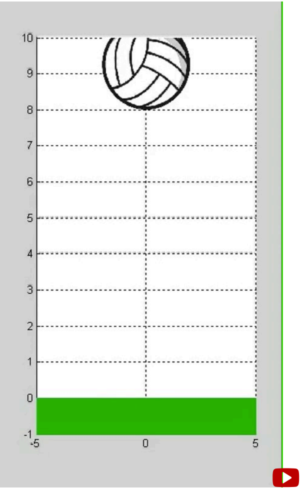
DHA AND HYSDEL MODELS




```
SYSTEM name {
   INTERFACE {
      STATE {
         REAL xc [xmin,xmax];
         BOOL xl; }
      INPUT {
         REAL uc [umin,umax];
         BOOL ul; }
      PARAMETER {
         REAL param1 = 1;}
   } /* end of interface */

   IMPLEMENTATION {
      AUX { BOOL d;
            REAL z; }

      AUTOMATA { xl = xl & ~ul; }

      AD { d = xc - 1 <= 0; }

      DA { z = { IF d THEN 2*xc ELSE -xc };  }

      CONTINUOUS {
         xc = z; }

      MUST {
         xc + uc <= 2;
         ~(xl & ul); }
   } /* end of implementation */
} /* end of system */
```

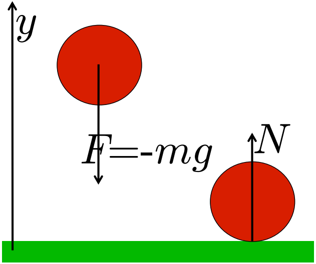
Additional relations constraining system's variables

BOUNCING BALL EXAMPLE



$$
\begin{aligned}
&\ddot{y} = -g \\
&y \leq 0 \;\Rightarrow\; \dot{y}(t^+) = -(1-\alpha)\dot{y}(t^-)
\end{aligned}
$$

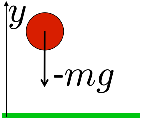
$$\alpha \in [0, 1]$$



How to model the bouncing ball as a discrete-time hybrid system ?

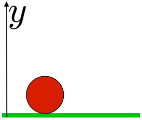
BOUNCING BALL – TIME DISCRETIZATION

- Case $y(k) > 0$ (ball falling):
$$\begin{array}{rcl} v(k) & \approx & \frac{y(k)-y(k-1)}{T_s} \\ -g & \approx & \frac{v(k)-v(k-1)}{T_s} \end{array}$$



$$\left\{\begin{array}{rcl} v(k+1) & = & v(k) - T_s g \\ y(k+1) & = & y(k) + T_s v(k+1) \\ & = & y(k) + T_s v(k) - T_s^2 g \end{array}\right.$$

- Case $y(k) \leq 0$ (ground level):
$$\begin{array}{rcl} v(k+1) & = & -(1-\alpha)v(k) \\ y(k+1) & = & y(k-1) = y(k) - T_s v(k) \end{array}$$



$$\left\{\begin{array}{rcl} v(k+1) & = & -(1-\alpha)v(k) \\ y(k+1) & = & y(k) - T_s v(k) \end{array}\right.$$

- We need a binary variable $[\delta(k) = 1] \leftrightarrow [y(k) \leq 0]$

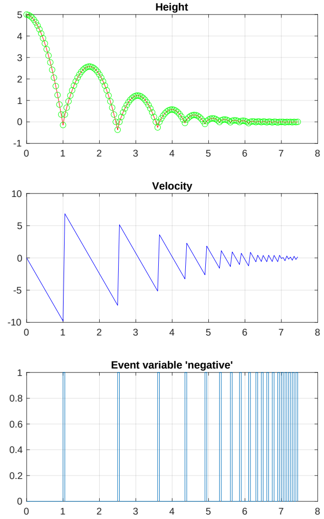
BOUNCING BALL - HYSDEL MODEL

```
SYSTEM bouncing_ball {
INTERFACE {
/* Description of variables and constants */
        STATE { REAL height [-10,10];
                REAL velocity [-100,100];    }

        PARAMETER {
                REAL g;
                REAL alpha;  /* 0=elastic, 1=completely anelastic */
                REAL Ts; }
}
IMPLEMENTATION {
        AUX {   BOOL negative;
                REAL hnext;
                REAL vnext;    }

        AD {    negative = height <= 0; }

        DA {    hnext = {  IF negative THEN height-Ts*velocity
                        ELSE height+Ts*velocity-Ts*Ts*g};
                vnext = {  IF negative THEN -(1-alpha)*velocity
                        ELSE velocity-Ts*g};  }
        CONTINUOUS {
                height   = hnext;
                velocity = vnext;}
}}
```

go to demo `demos/hybrid/bball.m`

BOUNCING BALL - SIMULATION

```
>> Ts=0.05;
>> g=9.8;
>> alpha=0.3;

>> S=mld('bouncing_ball',Ts);

>> N=150;
>> U=zeros(N,0);
>> x0=[5 0]';

>> [X,T,D]=sim(S,x0,U);
```

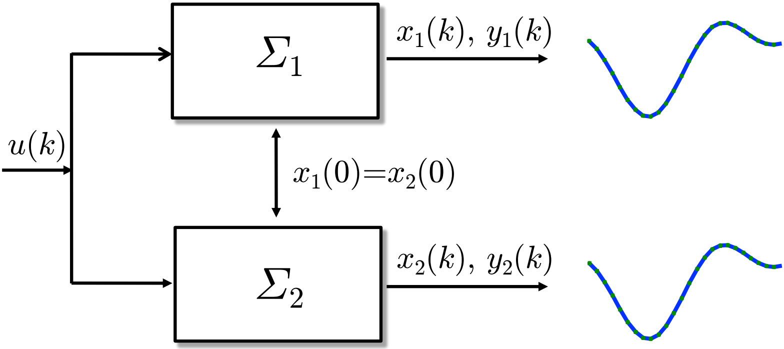


- **Note**: no **Zeno effect** in discrete time !

EQUIVALENCE OF HYBRID MODELS

EQUIVALENCE OF HYBRID MODELS

- Two hybrid models Σ_1, Σ_2 are **equivalent** if for all initial states $x_1(0) = x_2(0)$ and input excitations $u_1(k) \equiv u_2(k)$, the corresponding trajectories $x_1(k) \equiv x_2(k)$ and $y_1(k) \equiv y_2(k), \forall k = 0, 1, \ldots$



EQUIVALENCE OF HYBRID MODELS

- **MLD** and **PWA** systems are equivalent (Bemporad, Ferrari-Trecate, Morari, 2000)

  Proof: For a given combination (x_ℓ, u_ℓ, δ) of an MLD model, the state and output equation are linear and valid in a polyhedron.

  Conversely, a PWA system can be modeled as MLD system (see next slide)

- Efficient **conversion algorithms** from MLD to PWA form exist

  (Bemporad, 2004) (Geyer, Torrisi, Morari, 2003)

- Further equivalences exist with other classes of hybrid dynamical systems, such as **Linear Complementarity (LC)** systems (Heemels, De Schutter, Bemporad, 2001)

MODELING PWA SYSTEMS IN MLD FORM

- PWA system with bounded states $x \in \mathcal{X}$ and inputs $u \in \mathcal{U}$ with s regions

$$
\begin{array}{rcl}
x(k+1) & = & A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\
y(k) & = & C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\
i(k) & = & \text{such that } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{C}_{i(k)}
\end{array}
\qquad i(k) \in \{1, 2, \ldots, s\}
$$

- The sets $\mathcal{C}_i = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : H_i x + J_i u \leq K_i\}$ make a **polyhedral partition** of $\mathcal{X} \times \mathcal{U}$ $(\cup_{i=1}^{s} \mathcal{C}_i = \mathcal{X} \times \mathcal{U}$ and $\mathring{\mathcal{C}}_i \cap \mathring{\mathcal{C}}_j = \emptyset, \forall i \neq j, i, j = 1, \ldots, s)$

- Introduce s binary variables $\delta_i, i = 1, \ldots, s$ and set the logic constraints

$$
\begin{array}{l}
[\delta_i = 1] \to [H_i x + J_i u \leq K_i] \\
\displaystyle\bigoplus_{i=1}^{s} [\delta_i = 1] = \texttt{true}
\end{array}
\quad \Longrightarrow \quad
\begin{array}{l}
H_i x + J_i u \leq K_i + M_i(1 - \delta_i) \\
\displaystyle\sum_{i=1}^{s} \delta_i = 1
\end{array}
$$

where the vector M_i of upper-bounds can be computed, e.g., via LP

MODELING PWA SYSTEMS IN MLD FORM

- Introduce auxiliary real vectors z_i, w_i defined by if-then-else rules

$$z_i = \begin{cases} A_i x + B_i u + f_i & \text{if } \delta_i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad w_i = \begin{cases} C_i x + D_i u + g_i & \text{if } \delta_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and convert into mixed-integer inequalities (**big-M technique**):

$$\begin{cases} z_i & \leq & A_i x + B_i u + f_i - m_i(1-\delta_i) \\ z_i & \geq & A_i x + B_i u + f_i - M_i(1-\delta_i) \\ z_i & \leq & M_i \delta \\ z_i & \geq & m_i \delta \end{cases}$$

$$m \leq A_i x + B_i u + f_i \leq M_i$$

- Finally, write the state update and output equations

$$x(k+1) = \sum_{i=1}^{s} z_i(k) \qquad y(k) = \sum_{i=1}^{s} w_i(k)$$

MODELING PWA SYSTEMS VIA DISJUNCTIVE PROGRAMMING

(Balas, 1985)

- A PWA system with bounded states and inputs is equivalent to the **disjunction**

$$\bigvee_{i=1}^{s}\left[\begin{array}{c} H_i x(k) + J_i u(k) \le K_i \\ x(k+1) = A_i x(k) + B_i u(k) + f_i \end{array}\right] \qquad \begin{array}{c} x_{\ell b} \le x(k) \le x_{ub} \\ u_{\ell b} \le u(k) \le u_{ub} \end{array}$$

- Introduce s binary variables $\delta_1(k),\ldots,\delta_s(k)$ subject to $\sum_{i=1}^{s}\delta_i(k) = 1$
- Introduce the **convex hull relaxation** of the disjunction

$$x(k) = \sum_{i=1}^{s} v_i(k), \quad x_{\ell b}\delta_i(k) \le v_i(k) \le x_{ub}\delta_i(k)$$

$$u(k) = \sum_{i=1}^{s} w_i(k), \quad u_{\ell b}\delta_i(k) \le w_i(k) \le u_{ub}\delta_i(k)$$

and set

$$x(k+1) = \sum_{i=1}^{s} A_i v_i(k) + B_i w_i(k) + f_i\delta_i(k), \quad H_i v_i(k) + J_i w_i(k) \le K_i\delta_i(k)$$

VARIABLES

(Bemporad, 2022)

- Only introduce s binary variables $\delta_i, i = 1, \ldots, s$ and set:

$$\underbrace{m_i^x(1-\delta_i(k)) \leq x(k+1) - A_i x(k) - B_i u(k) - f_i \leq M_i^x(1-\delta_i(k))}_{[\delta_i(k)=1] \to [x(k+1) = A_i x(k) + B_i u(k) + f_i]}$$

$$\underbrace{m_i^y(1-\delta_i(k)) \leq y(k) - C_i x(k) - D_i u(k) - g_i \leq M_i^y(1-\delta_i(k))}_{[\delta_i(k)=1] \to [y(k) = C_i x(k) + D_i u(k) + g_i]}$$

$$H_i x(k) + J_i u(k) \leq K_i + M_i(1-\delta_i(k)) \qquad [\delta_i(k)=1] \to [H_i x(k) + J_i u(k) \leq K_i]$$

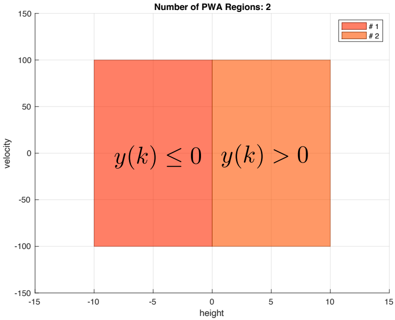
$$\sum_{i=1}^{s} \delta_i(k) = 1 \qquad \bigoplus_{i=1}^{s} [\delta_i(k)=1] = \texttt{true}$$

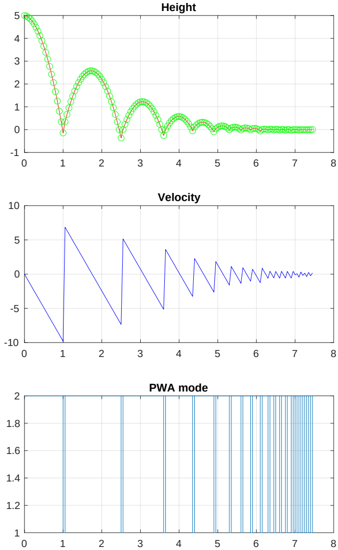
where $m_i^x, M_i^x, m_i^y, M_i^y$ are suitably defined upper and lower bounds

BOUNCING BALL - PWA EQUIVALENT

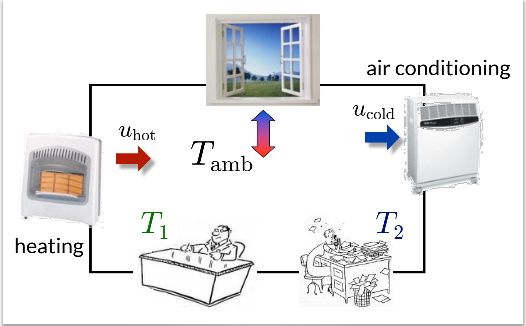
```
>> P=pwa(S);
>> plot(P)

>> [X,T,I]=sim(P,x0,U);
```

(Bemporad, 2004)





EXAMPLE: ROOM TEMPERATURE CONTROL



discrete dynamics

- #1 = cold $\rightarrow$ heater = on
- #2 = cold $\rightarrow$ heater = on **unless** #1 hot
- A/C activation has similar rules

continuous dynamics

$$\frac{dT_i}{dt} = -\alpha_i(T_i - T_{\text{amb}}) + k_i(u_{\text{hot}} - u_{\text{cold}})$$

$i = 1, 2$

go to demo `demos/hybrid/heatcool.m`

EXAMPLE: ROOM TEMPERATURE CONTROL

```
SYSTEM heatcool {

INTERFACE {
    STATE { REAL T1 [-10,50];
            REAL T2 [-10,50]; }
    INPUT { REAL Tamb [-50,50]; }
    PARAMETER {
        REAL Ts, alpha1, alpha2, k1, k2;
        REAL Thot1, Tcold1, Thot2, Tcold2, Uc, Uh; }
    }

IMPLEMENTATION {
    AUX { REAL uhot, ucold;
          BOOL hot1, hot2, cold1, cold2; }

    AD { hot1 = T1>=Thot1;
         hot2 = T2>=Thot2;
         cold1 = T1<=Tcold1;
         cold2 = T2<=Tcold2; }

    DA { uhot = { IF cold1 | (cold2 & ~hot1) THEN Uh ELSE 0 };
         ucold = { IF hot1 | (hot2 & ~cold1) THEN Uc ELSE 0 }; }

    CONTINUOUS { T1 = T1+Ts*(-alpha1*(T1-Tamb)+k1*(uhot-ucold));
                 T2 = T2+Ts*(-alpha2*(T2-Tamb)+k2*(uhot-ucold)); }
    }
}
```

```
>> S=mld('heatcoolmodel',Ts);
```

get the MLD model in MATLAB

```
>> [XX,TT]=sim(S,x0,U);
```

simulate the MLD model

EXAMPLE: ROOM TEMPERATURE CONTROL

- MLD model of the room temperature system

$$\begin{cases} x(k+1) &= & Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) &= & Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\ E_2\delta(k) &+ & E_3z(k) \leq E_4x(k) + E_1u(k) + E_5 \end{cases}$$

  - 2 continuous states (temperature T_1, T_2)
  - 1 continuous input (room temperature $T_{\rm amb}$)
  - 2 auxiliary continuous vars (power flows $u_{\rm hot}, u_{\rm cold}$)
  - 6 auxiliary binary vars (4 threshold events + 2 for the OR condition)
  - 20 mixed-integer inequalities

- In principle, we have $2^6 = 64$ possible combinations of binary variables

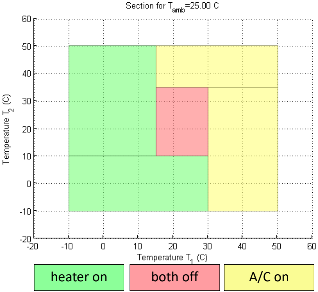
EXAMPLE: ROOM TEMPERATURE CONTROL

- PWA model of the room temperature system

$$\begin{aligned} x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \end{aligned}$$

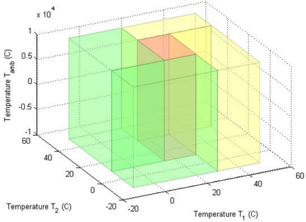
```
>> P=pwa(S);
```

$$i(k) \quad \text{s.t.} \quad H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}$$



5 polyhedral regions

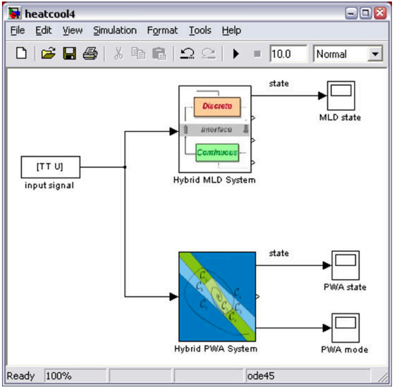
(partition does not depend on input)

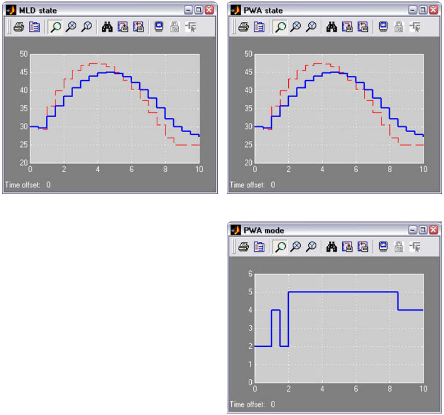


2 continuous states (T_1, T_2)

1 continuous input (T_{amb})

EXAMPLE: ROOM TEMPERATURE CONTROL





- MLD and PWA models are equivalent, hence simulated states are the same

USING PWA EQUIVALENCE FOR MODEL ANALYSIS

- Assume plant + controller can be modeled as DHA:
  - **plant** = approximated as PWA system (e.g.: nonlinear switched model)
  - **controller** = switched linear controller (e.g: combination of threshold conditions, logic, linear feedback laws, …)
- Convert DHA to MLD form, then to PWA form
- The resulting closed-loop PWA model reveals how the closed-loop system behaves in different regions of the state-space
- Can analyze **closed-loop stability** analysis using **piecewise quadratic Lyapunov functions** (Johansson, Rantzer, 1998) (Mignone, Ferrari-Trecate, Morari, 2000)

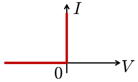
OTHER EXISTING HYBRID MODELS

(Heemels, De Schutter, Bemporad, 2001)

- **Linear complementarity (LC)** systems (Heemels, 1999)

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2w(k) \\ y(k) &= Cx(k) + D_1u(k) + D_2w(k) \\ v(k) &= E_1x(k) + E_2u(k) + E_3w(k) + E_4 \\ 0 &\leq v(k) \perp w(k) \geq 0 \end{aligned}$$

Examples:
mechanical systems,
electrical circuits



- **Min-max-plus-scaling (MMPS)** systems (De Schutter, Van den Boom, 2000)

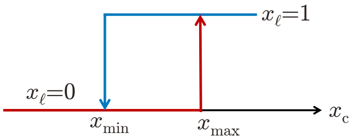
$$\begin{aligned} x(k+1) &= M_x(x(k), u(k), w(k)) \\ y(k) &= M_y(x(k), u(k), w(k)) \\ 0 &\geq M_c(x(k), u(k), w(k)) \end{aligned}$$

Example:
discrete-event system
k = event counter

where $M_{()}$ are MMPS functions defined by the grammar
$M := x_i|\alpha|\max(M_1, M_2)|\min(M_1, M_2)|M_1 + M_2|\beta M_1$

Example: $x(k+1) = 2\max(x(k), 0) + \min(\frac{1}{2}u(k), 1)$

MODELING HYSTERESIS



- Hysteresis between $x_{\min} \leq x_c(k) \leq x_{\max}$

- Introduce two binary variables

$$\begin{array}{rcl} [\delta_{\min}(k)=1] & \leftrightarrow & [x_c(k) \leq x_{\min}] \\ [\delta_{\max}(k)=1] & \leftrightarrow & [x_c(k) \geq x_{\max}] \end{array}$$

- Introduce logic state $x_\ell \in \{0,1\}$ with dynamics

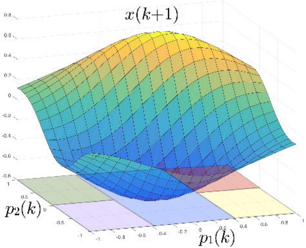
$$x_\ell(k+1) = (x_\ell(k) \wedge \neg\delta_{\min}(k)) \vee (\neg x_\ell(k) \wedge \delta_{\max}(k))$$

IDENTIFICATION OF HYBRID SYSTEMS

HYBRID SYSTEM IDENTIFICATION

- A hybrid model of the process may not be available from physical principles
- Therefore, a model must be either
  - **estimated from data** (model is unknown)
  - or **hybridized** (model is known but nonlinear)
- If one linear model is enough: easy problem (SYS-ID TBX) (Ljung, 1999)
- If switching sequence known: easy, just identify one linear model per mode
- If modes & dynamics must be identified simultaneously, we need **hybrid system identification** (or **piecewise affine regression**)

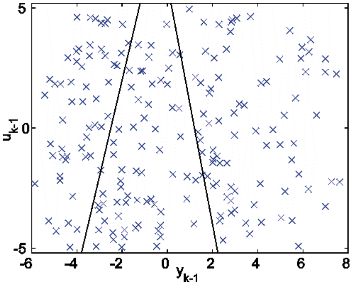
> In industrial MPC most effort is spent in **identifying (multiple) linear prediction models** from data



Problem: Estimate from data **both** the **parameters** of the affine submodels and the **partition** of the PWA map

Example: Let the data be generated by the PWARX system

$$y_k = \begin{cases} \begin{bmatrix} -0.4 & 1 & 1.5 \end{bmatrix} \phi_k + \epsilon_k \\ \text{if} \begin{bmatrix} 4 & -1 & 10 \end{bmatrix} \phi_k < 0 \\ \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} \phi_k + \epsilon_k \\ \text{if} \begin{bmatrix} -4 & 1 & 10 \\ 5 & 1 & -6 \end{bmatrix} \phi_k \leq 0 \\ \begin{bmatrix} -0.3 & 0.5 & -1.7 \end{bmatrix} \phi_k + \epsilon_k \\ \text{if} \begin{bmatrix} -5 & -1 & 6 \end{bmatrix} \phi_k < 0 \end{cases}$$



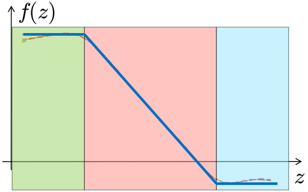
with $\phi_k = [y_{k-1}\ u_{k-1}\ 1]'$, $|u_k| \leq 5$, and $|\epsilon_k| \leq 0.1$

PWA REGRESSION PROBLEM

- **Problem**: Given input/output pairs $\{x(k), y(k)\}$, $k = 1, \ldots, N$ and number s of models, compute a **piecewise affine** (PWA) approximation $y \approx f(x)$

$$v(k) = \begin{cases} F_1 z(k) + g_1 & \text{if } H_1 z(k) \leq K_1 \\ \vdots & \\ F_s z(k) + g_s & \text{if } H_s z(k) \leq K_s \end{cases}$$

$$v(k) = \begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix}, \quad z(k) = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$



- Need to learn **both** the parameters $\{F_i, g_i\}$ of the affine submodels **and** the partition $\{H_i, K_i\}$ of the PWA map from data (**offline learning**)

- Possibly update model+partition as new data are available (**online learning**)

APPROACHES TO PWA IDENTIFICATION

- Mixed-integer linear or quadratic programming (Roll, Bemporad, Ljung, 2004)
- Partition of infeasible set of inequalities (Bemporad, Garulli, Paoletti, Vicino, 2005)
- K-means clustering in a feature space (Ferrari-Trecate, Muselli, Liberati, Morari, 2003)
- Bayesian approach (Juloski, Wieland, Heemels, 2004)
- Kernel-based approaches (Pillonetto, 2016)
- Hyperplane clustering in data space (Münz, Krebs, 2002)
- **Recursive multiple least squares & PWL separation** (Breschi, Piga, Bemporad, 2016)
- Jump models (Bemporad, Breschi, Piga, Boyd, 2018)
- **Piecewise affine regression and classification** (PARC) (Bemporad, 2022)
- Any **machine learning** technique producing a PWA model, such as (leaky)ReLU-NNs, decision trees, softmax regression, K-nearest neighbors, ...

PWA REGRESSION ALGORITHM

(Breschi, Piga, Bemporad, 2016)

1. Estimate models $\{F_i,\ g_i\}$ **recursively**. Let $e_i(k) = v(k) - F_i z(k) - g_i$ and only update model $i(k)$ such that

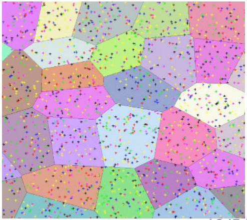
$$i(k) \leftarrow \arg\min_{i=1,\ldots,s} \quad \underbrace{e_i(k)'\Lambda_e^{-1}e_i(k)}_{\text{one-step prediction error of model \#i}} \quad + \underbrace{(z(k)-c_i)'\,R_i^{-1}(z(k)-c_i)}_{\text{proximity to centroid of cluster \#i}}$$

using **recursive LS** and **inverse QR decomposition** (Alexander, Ghirnikar, 1993)

This also splits the data points $z(k)$ in **clusters** $C_i = \{z(k)\ :\ i(k) = i\}$

2. Compute a polyhedral partition $\{H_i,\ K_i\}$ of the regressor space via **multi-category linear separation**

$$\phi(z) = \max_{i=1,\ldots,s}\{w_i'z - \gamma_i\} \qquad i(z) = \arg\max_i\{w_i'z - \gamma_i\}$$



3. Repeat the entire procedure M times

PWA REGRESSION EXAMPLES

(Breschi, Piga, Bemporad, 2016)

- Identification of **piecewise-affine ARX** model

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -0.83 & 0.20 \\ 0.30 & -0.52 \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} + \begin{bmatrix} -0.34 & 0.45 \\ -0.30 & 0.24 \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix}$$
$$+ \begin{bmatrix} 0.20 \\ 0.15 \end{bmatrix} + \max \left\{ \begin{bmatrix} 0.20 & -0.90 \\ 0.10 & -0.42 \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 0.42 & 0.20 \\ 0.50 & 0.64 \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + \begin{bmatrix} 0.40 \\ 0.30 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} + e(k),$$

- **Quality of fit**: best fit rate (BFR) = $\max\left\{1 - \frac{\|y_i - \hat{y}_i\|_2}{\|y_i - \bar{y}_i\|_2}, 0\right\}, i = 1, 2$

y = measured, $\hat{y}$ = open-loop simulated, $\bar{y}$ = sample mean of y

		$N = 4000$	$N = 20000$	$N = 100000$
y_1	(offline) RLP	96.0 %	96.5 %	99.0 %
	(Offline) RPSN	96.2 %	96.4 %	98.9 %
	(Online) ASGD	86.7 %	95.0 %	96.7 %
y_2	(offline) RLP	96.2 %	96.9 %	99.0 %
	(offline) RPSN	96.3 %	96.8 %	99.0 %
	(online) ASGD	87.4 %	95.2 %	96.4 %

BFR on validation data, open-loop validation

RLP = Robust linear programming
(Bennett, Mangasarian, 1994)

RPSN = Piecewise-smooth Newton method
(Bemporad, Bernardini, Patrinos, 2015)

ASGD = Averaged stochastic gradient descent
(Bottou, 2012)

- **CPU time for computing the partition**:
(i7 2.40-GHz Intel core)

	$N = 4000$	$N = 20000$	$N = 100000$
(Offline) RLP	0.308 s	3.227 s	112.435 s
(Offline) RPSN	0.016 s	0.086 s	0.365 s
(Online) ASGD	0.013 s	0.023 s	0.067 s

PWA REGRESSION EXAMPLES

(Breschi, Piga, Bemporad, 2016)

- Identification of **linear parameter varying ARX** model

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} \bar{a}_{1,1}(p(k)) & \bar{a}_{1,2}(p(k)) \\ \bar{a}_{2,1}(p(k)) & \bar{a}_{2,2}(p(k)) \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} + \begin{bmatrix} \bar{b}_{1,1}(p(k)) & \bar{b}_{1,2}(p(k)) \\ \bar{b}_{2,1}(p(k)) & \bar{b}_{2,2}(p(k)) \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + e_o(k)$$

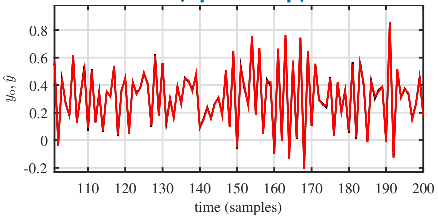
$\bar{a}(p)$ = PWA function of p
$\bar{b}(p)$ has **quadratic** and **sin** terms

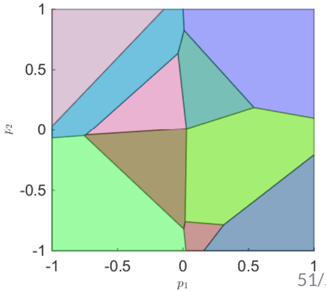
- **Quality of fit (BFR)**:

	y_1	y_2
PWA regression	87 %	84 %
parametric LPV*	80 %	70 %

* (Bamieh, Giarré, 2002)

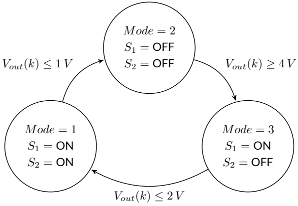
- **Validation data (open-loop)**:



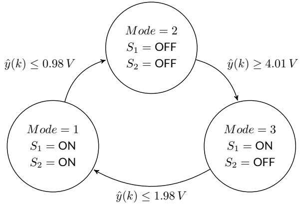


(Breschi, Piga, Bemporad, 2016)

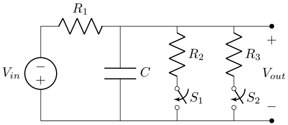
- Identification of a **hybrid model** with logic states

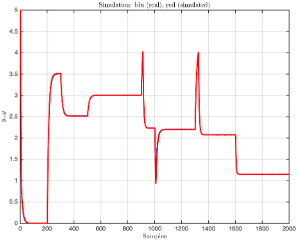


true system



identified system





Quality of fit: BFR=96.64 % (validation)

CPU time for identification: 78 ms

(2000 samples, MacBook Pro 2.8 GHz)

PARC - PIECEWISE AFFINE REGRESSION AND CLASSIFICATION

(Bemporad, 2022)

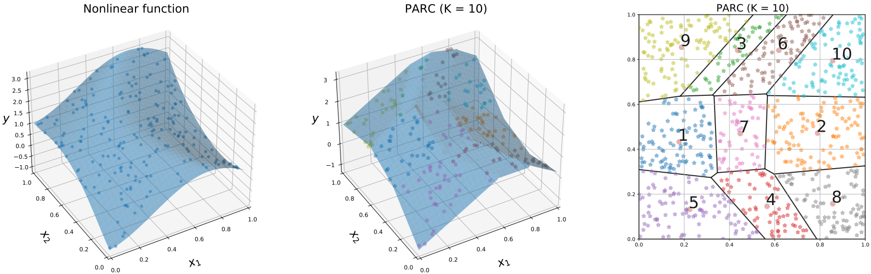
- **Piecewise Affine Regression and Classification** (**PARC**) algorithm
- Training dataset:
  - **feature vector** $z \in \mathbb{R}^n$ (categorical features **one-hot encoded** in $\{0,1\}$)
  - **target vector** $v_c \in \mathbb{R}^{m_c}$ (numeric), $v_{di} \in \{w_{di}^1, \ldots, w_{di}^{m_i}\}$ (categorical)
- PARC iteratively **clusters** training data in K sets and **fits** linear predictors:
  1. fit $v_c = a_j z + b_j$ by **ridge regression** (=ℓ_2-regularized least squares)
  2. fit $v_{di} = w_{di}^{h_*}, h_* = \arg\max\{a_{dih}^h z + b_{di}^h\}$ by **softmax regression**
  3. fit a convex **PWL separation function** by **softmax regression**

$$\Phi(z) = \omega^{j(z)} z + \gamma^{j(z)}, \qquad j(z) = \min\left\{\arg\max_{j=1,\ldots,K}\{\omega^j z + \gamma^j\}\right\}$$

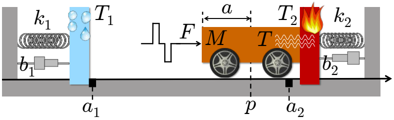
- Data reassigned to clusters based on weighted fit/PWL separation criterion
- PARC is a **block-coordinate descent** algorithm $\Rightarrow$ (local) convergence ensured

PARC - PIECEWISE AFFINE REGRESSION AND CLASSIFICATION

- Simple PWA regression example:
  - 1000 samples of $y = \sin(4x_1 - 5(x_2 - 0.5)^2) + 2x_2$ (use 80% for training)
  - PWA approximation over $K = 10$ polyhedral regions



- Code: `pip install pyparc` **github.com/bemporad/PyPARC**



- **Example**: moving cart and bumpers + heat transfer during bumps.

  Spring and viscous forces are **nonlinear**.

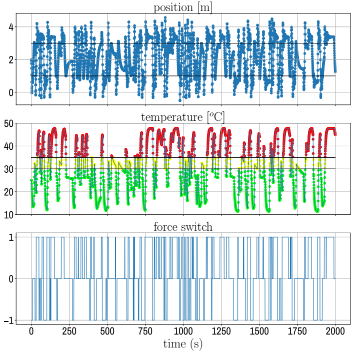
- Categorical input $F \in \{-\bar{F}, 0, \bar{F}\}$ and categorical output $c \in \{green, yellow, red\}$

- 4000 training samples

- Feature vector $z_k = [y_k, \dot{y}_k, T_k, F_k]$

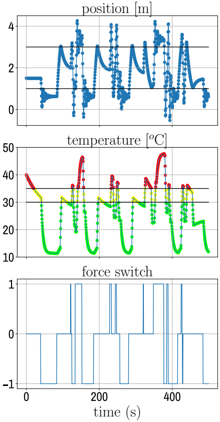
- Target vector $v_k = [y_{k+1}, \dot{y}_{k+1}, T_{k+1}, c_k]$

- Hybrid model learned by **PARC** ($K = 5$ regions)

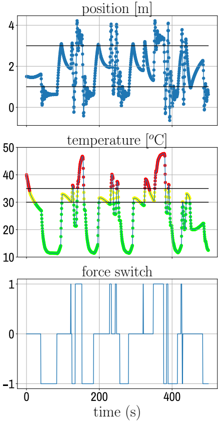


PARC - CART & BUMPERS EXAMPLE

- **Open-loop** simulation on 500 s **test** data:



continuous-time system



discrete-time PWA model

- Model fit is good enough for MPC design purposes (see later ...)