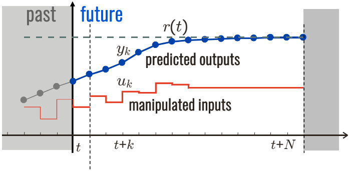
MODEL PREDICTIVE CONTROL

Alberto Bemporad

http://cse.lab.imtlucca.it/~bemporad/mpc_course.html

Academic year 2024-2025

IMT SCHOOL FOR ADVANCED STUDIES LUCCA



COURSE STRUCTURE

- **Basic concepts** of model predictive control (MPC) and **linear MPC**
- **Linear time-varying** and **nonlinear** MPC
- **Quadratic programming** (QP) and **explicit MPC**
- **Hybrid** MPC
- **Stochastic** MPC
- **Learning-based** MPC

- Numerical **examples**:
  - **MPC Toolbox** for MATLAB (linear/explicit/parameter-varying MPC)
  - **Hybrid Toolbox** for MATLAB (explicit MPC, hybrid systems)

COURSE STRUCTURE

- For additional background:
  - **Linear Systems**:

    `http://cse.lab.imtlucca.it/~bemporad/intro_control_course.html`

  - **Numerical Optimization**:

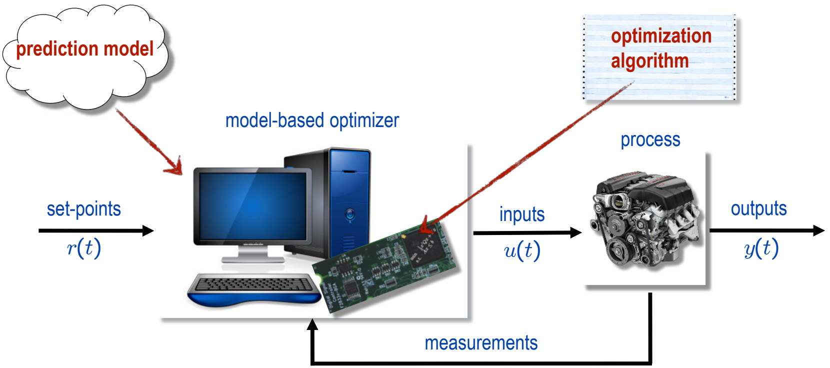
    `http://cse.lab.imtlucca.it/~bemporad/optimization_course.html`

  - **Machine Learning**:

    `http://cse.lab.imtlucca.it/~bemporad/ml.html`

MODEL PREDICTIVE CONTROL: BASIC CONCEPTS

MODEL PREDICTIVE CONTROL (MPC)



Use a simplified dynamical **model** of the process to **predict** its likely future evolution and choose ~~the "best"~~ a good **control** action

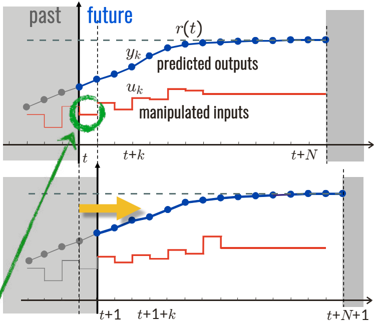
MODEL PREDICTIVE CONTROL

- **MPC problem:** find the best control sequence over a future horizon of N steps

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \|y_k - r(t)\|_2^2 + \rho\|u_k - u_r(t)\|_2^2$$

$$\text{s.t.}\quad x_{k+1} = f(x_k, u_k) \qquad \text{prediction model}$$
$$y_k = g(x_k)$$
$$u_{\min} \le u_k \le u_{\max} \qquad \text{constraints}$$
$$y_{\min} \le y_k \le y_{\max}$$
$$x_0 = x(t) \qquad \text{state feedback}$$

numerical optimization problem

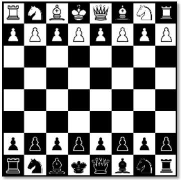


1. **estimate** current state $x(t)$
2. **optimize** wrt $\{u_0,\ldots,u_{N-1}\}$
3. only **apply** optimal u_0 as input $u(t)$

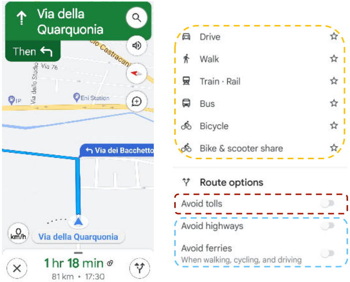
Repeat at all time steps t

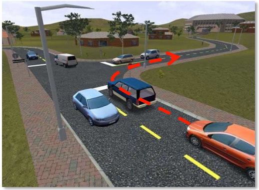
DAILY-LIFE EXAMPLES OF MPC

- MPC is like playing chess !



- Online (event-based) re-planning used in GPS navigation





- You use MPC too when you drive !

MPC IN INDUSTRY

- The MPC concept dates back to the 60's

Discrete Dynamic Optimization
Applied to On-Line Optimal Control

MARSHALL D. RAFAL and WILLIAM F. STEVENS

(Rafal, Stevens, AiChE Journal, 1968)

Том XXIV «АВТОМАТИКА И ТЕЛЕМЕХАНИКА» № 7
1963

УДК 62-50

ПРИМЕНЕНИЕ МЕТОДОВ ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ ДЛЯ СИНТЕЗА ИМПУЛЬСНЫХ АВТОМАТИЧЕСКИХ СИСТЕМ

А. И. ПРОПОЙ

USE OF LINEAR PROGRAMMING METHODS FOR SYNTHESIZING SAMPLED-DATA AUTOMATIC SYSTEMS

A. I. Propoi
(Moscow)
Translated from Avtomatika i Telemekhanika, Vol. 24, No. 7, pp. 912-920, July, 1963
Original article submitted September 24, 1962

(Propoi, 1963)

- MPC used in the process industries since the 80's

(Qin, Badgewell, 2003) (Bauer, Craig, 2008)

Today APC (advanced process control) = MPC

©SimulateLive.com

(Qin, Badgewell, 2003)

- Industrial survey of MPC applications conducted in mid 1999

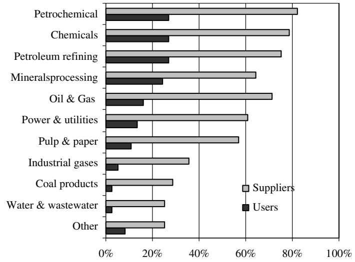
Area	Aspen Technology	Honeywell Hi-Spec	Adersa[b]	Invensys	SGS[c]	Total
Refining	1200	480	280	25		1985
Petrochemicals	450	80	—	20		550
Chemicals	100	20	3	21		144
Pulp and paper	18	50	—	—		68
Air & Gas	—	10	—	—		10
Utility	—	10	—	4		14
Mining/Metallurgy	8	6	7	16		37
Food Processing	—	—	41	10		51
Polymer	17	—	—	—		17
Furnaces	—	—	42	3		45
Aerospace/Defense	—	—	13	—		13
Automotive	—	—	7	—		7
Unclassified	40	40	1045	26	450	1601
Total	1833	696	1438	125	450	4542
First App.	DMC:1985 IDCOM-M:1987 OPC:1987	PCT:1984 RMPCT:1991	IDCOM:1973 HIECON:1986	1984	1985	
Largest App.	603×283	225×85	—	31×12	—	

Estimates based on vendor survey

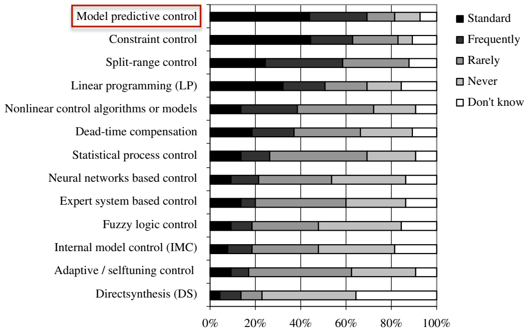
MPC IN INDUSTRY

(Bauer, Craig, 2008)

- Economic assessment of Advanced Process Control (APC)



participants of APC survey by industry (worldwide)



Industrial use of APC methods: survey results

MPC IN INDUSTRY

(Samad, IEEE CS Magazine, 2017)

- Impact of advanced control technologies in industry

TABLE 1 A list of the survey results in order of industry impact as perceived by the committee members.

Rank and Technology	High-Impact Ratings	Low- or No-Impact Ratings
PID control	100%	0%
$\boxed{\text{Model predictive control}}$	$\boxed{78\%}$	9%
System identification	61%	9%
Process data analytics	61%	17%
Soft sensing	52%	22%
Fault detection and identification	50%	18%
Decentralized and/or coordinated control	48%	30%
Intelligent control	35%	30%
Discrete-event systems	23%	32%
Nonlinear control	22%	35%
Adaptive control	17%	43%
Robust control	13%	43%
Hybrid dynamical systems	13%	43%

MPC IN INDUSTRY

Table 2
The percentage of survey respondents indicating whether a control technology had demonstrated ("Current Impact") or was likely to demonstrate over the next five years ("Future Impact") high impact in practice.

(Samad et al., 2020)

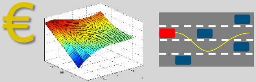
	Current Impact	Future Impact
Control Technology	%High	%High
PID control	91%	78%
System Identification	65%	72%
Estimation and filtering	64%	63%
Model-predictive control	62%	85%
Process data analytics	51%	70%
Fault detection and identification	48%	78%
Decentralized and/or coordinated control	29%	54%
Robust control	26%	42%
Intelligent control	24%	59%
Discrete-event systems	24%	39%
Nonlinear control	21%	42%
Adaptive control	18%	44%
Repetitive control	12%	17%
Hybrid dynamical systems	11%	33%
Other advanced control technology	11%	25%
Game theory	5%	17%

"As can be observed, MPC is clearly considered more impactful, and likely to be more impactful, vis-à-vis other control technologies, especially those that can be considered the "crown jewels" of control theory - robust control, adaptive control, and nonlinear control."

TYPICAL USE OF MPC

strategic planner

high-level optimizer



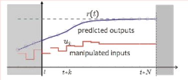
reference optimization

- Static optimizer of steady-state (static I/O models)
- Reference trajectory generator

reference-signal $r(t)$

tactical planner

dynamic optimizer

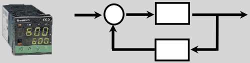


MPC

- optimize transient response
- ensure reference tracking
- handle constraints
- coordinate multiple inputs

measurements $y(t)$ $u(t)$ **actuator set-points**

regulators



low-level controllers

- fast sampling
- single loops

MPC OF AUTOMOTIVE SYSTEMS

(Bemporad, Bernardini, Borrelli, Cimini, Di Cairano, Esen, Giorgetti, Graf-Plessen, Hrovat, Kolmanovsky Levijoki, Livshiz, Long, Pattipati, Ripaccioli, Trimboli, Tseng, Verdejo, Yanakiev, ..., 2001-present)

Powertrain
engine control, magnetic actuators, robotized gearbox, power MGT in HEVs, cabin heat control, electrical motors

Vehicle dynamics
traction control, active steering, semiactive suspensions, autonomous driving

Ford Motor Company
Jaguar
DENSO Automotive
Fiat
General Motors
ODYS
Advanced Controls & Optimization

Most automotive OEMs are looking into MPC solutions today

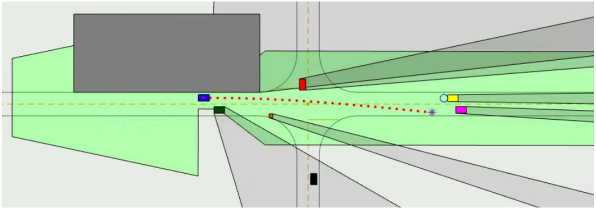
SYSTEMS

(Graf Plessen, Bernardini, Esen, Bemporad, 2018)

- Coordinate **torque request** and **steering** to achieve **safe and comfortable** autonomous driving with no collisions

- MPC combines **path planning**, **path tracking**, and **obstacle avoidance**

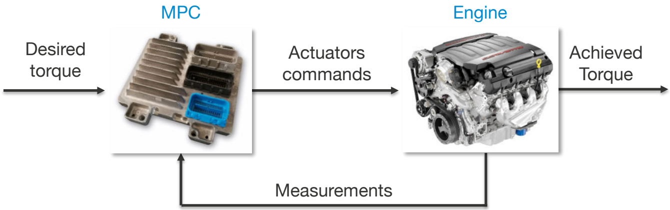
- **Stochastic prediction models** used to account for uncertainty (other vehicles/pedestrians, driver's requests)



ODYS
Advanced Controls & Optimization

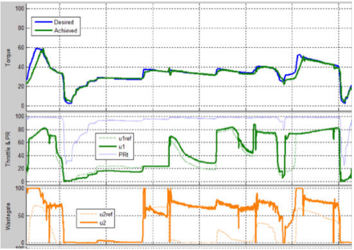
MPC OF GASOLINE TURBOCHARGED ENGINES

- Control **throttle, wastegate, intake & exhaust cams** to make **engine torque** track set-points, with max efficiency and satisfying **constraints**



numerical optimization problem solved in real-time on ECU

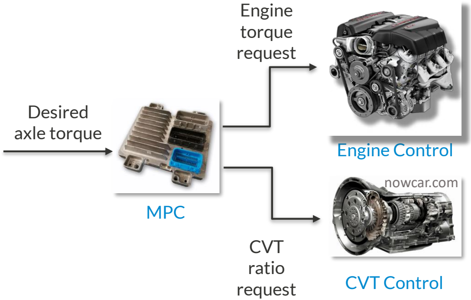
(Bemporad, Bernardini, Long, Verdejo, 2018)

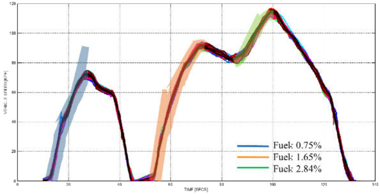


engine operating at low pressure (66 kPa)

SUPERVISORY MPC OF POWERTRAIN WITH CVT

- Coordinate **engine torque request** and continuously variable transmission (CVT) **ratio** to improve fuel economy and drivability
- Real-time MPC is able to take into account **coupled dynamics** and **constraints**, optimizing performance also during transients





US06 Double Hill driving cycle

(Bemporad, Bernardini, Livshiz, Pattipati, 2018)

MPC IN AUTOMOTIVE PRODUCTION

ODYS real-time embedded optimization and MPC software is currently running on **3+ million vehicles** worldwide

- Multivariable system, **4 inputs, 4 outputs**. QP solved **in real time on ECU**

  (Bemporad, Bernardini, Long, Verdejo, 2018)

- Supervisory **MPC for powertrain control** also in production since 2018

  (Bemporad, Bernardini, Livshiz, Pattipati, 2018)

First known mass production of MPC in the automotive industry

and more are underway...

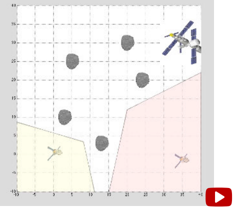
http://www.odys.it/odys-and-gm-bring-online-mpc-to-production

ODYS Advanced Controls & Optimization

AEROSPACE APPLICATIONS OF MPC

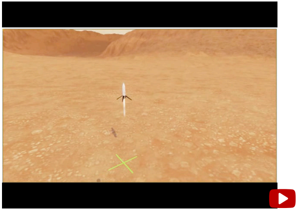
- MPC capabilities explored in space applications

cooperating UAVs



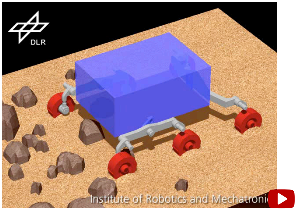
(Bemporad, Rocchi, 2011)

powered descent



(Pascucci, Bennani, Bemporad, 2016)

planetary rover



(Krenn et. al., 2012)

- Online convex optimization ($\approx$ MPC) is used by SpaceX to land reusable rockets

  (Blackmore, 2016)

MPC IN AERONAUTIC INDUSTRY

PRESS RELEASE

Pratt & Whitney's F135 Advanced Multi-Variable Control Team Receives UTC's Prestigious George Mead Award for Outstanding Engineering Accomplishment

EAST HARTFORD, CONN., THURSDAY, MAY 27, 2010

Pratt & Whitney engineers Louis Celiberti, Timothy Crowley, James Fuller and Cary Powell won the George Mead Award – United Technologies Corp.'s highest award for outstanding engineering achievement – for their pioneering work in developing the world's first advanced multi-variable control (AMVC) design for the only engine that powers the F-35 Lightning II flight test program. Pratt & Whitney is a United Technologies Corp. (NYSE:UTX) company.

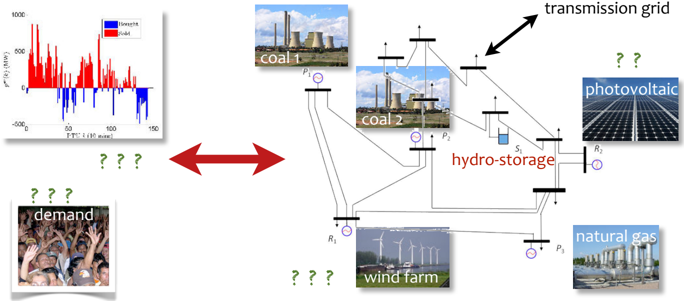
The AMVC, which uses a proprietary model predictive control methodology, is the most technically advanced propulsion system control ever produced by the aerospace industry, demonstrating the highest pilot rating for flight performance and providing independent control of vertical thrust and pitch from five sources. This innovative and industry-leading advanced design is protected with five broad patents for Pratt & Whitney and UTC, and is the new standard for propulsion system control for Pratt & Whitney military and commercial engines.

Pratt & Whitney
A United Technologies Company

http://www.pw.utc.com/Press/Story/20100527-0100/2010

MPC FOR SMART ELECTRICITY GRIDS

(Patrinos, Trimboli, Bemporad, 2011)



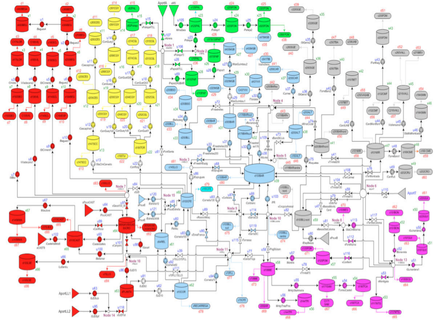
Dispatch power in smart distribution grids, trade energy on energy markets

Challenges: account for **dynamics**, network **topology**, physical **constraints**, and **stochasticity** (of renewable energy, demand, electricity prices)

FP7-ICT project "E-PRICE - Price-based Control of Electrical Power Systems" (2010-2013)

MPC OF DRINKING WATER NETWORKS

(Sampathirao, Sopasakis, Bemporad, Patrinos, 2017)



Drinking water network of Barcelona:

63 tanks

114 controlled flows

17 mixing nodes

- **≈5% savings** on energy costs w.r.t. current practice
- Demand and minimum pressure requirements met, smooth control actions
- Computation time: **≈20 s** on NVIDIA Tesla 2075 CUDA (sample time = 1 hr)

FP7-ICT project “EFFINET - Efficient Integrated Real-time Monitoring and Control of Drinking Water Networks” (2012-2015)

EFFINET

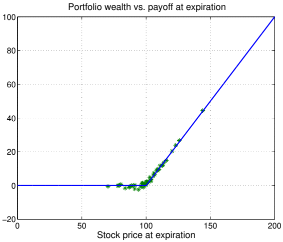
MPC FOR DYNAMIC HEDGING OF FINANCIAL OPTIONS

(Bemporad, Bellucci, Gabbriellini, *Quantitative Finance*, 2014)

- **Goal**: find a **dynamic hedging policy** of a portfolio replicating a **synthetic option**, so to **minimize risk** that payoff≠portfolio wealth at expiration date

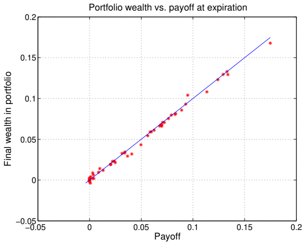
- A simple **linear stochastic model** describes the dynamics of **portfolio wealth**

- Stochastic MPC results:



$$p(T) = \max\{w(T) - K, 0\}$$

European call



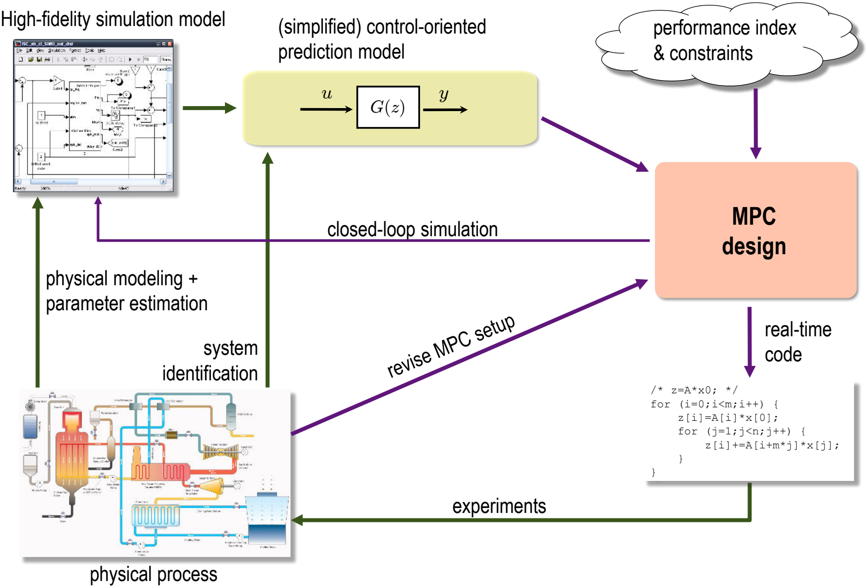
$$p(T) = \max\left\{0,\, C + \min_{i\in\{1,\ldots,N_{\rm fix}\}} \frac{x(t_i) - x(t_{i-1})}{x(t_{i-1})}\right\}$$

Napoleon cliquet

MPC RESEARCH IS DRIVEN BY APPLICATIONS

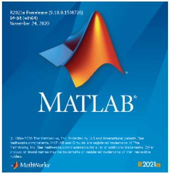
- Process control $\rightarrow$ **linear** MPC (some **nonlinear** too) — **1970-2000**
- Automotive control $\rightarrow$ **explicit**, **hybrid** MPC — **2001-2010**
- Aerospace systems and UAVs $\rightarrow$ **linear time-varying** MPC — **>2005**
- Information and Communication Technologies (ICT) (wireless nets, cloud) $\rightarrow$ **distributed/decentralized** MPC — **>2005**
- Energy, finance, automotive, water $\rightarrow$ **stochastic** MPC — **>2010**
- Industrial production $\rightarrow$ **embedded optimization** solvers for MPC — **>2010**
- Machine learning $\rightarrow$ **data-driven** MPC — **today**

MPC DESIGN FLOW



MPC TOOLBOXES

- **MPC Toolbox** (The Mathworks, Inc.): (Bemporad, Ricker, Morari, 1998-today)
  - Part of Mathworks' official toolbox distribution
  - All written in **MATLAB** code
  - Great for education and research



- **Hybrid Toolbox**: (Bemporad, 2003-today)
  - Free download: `http://cse.lab.imtlucca.it/~bemporad/hybrid/toolbox`
  - Great for research and education

11,500+ downloads

1.5 downloads/day

- **ODYS Embedded MPC Toolset**: (ODYS, 2013-today)
  - Very flexible MPC design and **seamless integration** in production
  - Real-time **MPC code** and **QP solver** written in **plain C**
  - Support for nonlinear models and deep learning
  - Designed and adopted for **industrial production**

`odys.it/embedded-mpc`

ODYS
Advanced Controls & Optimization

BENEFITS OF MPC

- Long history (decades) of success of MPC in industry

- MPC is a **universal control methodology**:
  - to coordinate **multiple inputs/outputs**, arbitrary **models** (linear, nonlinear, ...)
  - to **optimize performance** under **constraints**
  - intuitive to **design**, easy to **calibrate** and **reconfigure** = **short development time**

- MPC is a mature technology also in fast-sampling applications (e.g. **automotive**)
  - modern ECUs can solve MPC problems in **real-time**
  - advanced MPC **software tools** are available for design/calibration/deployment

Ready to learn how MPC works ?

BASICS OF CONSTRAINED OPTIMIZATION

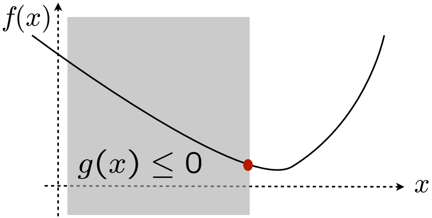
See more in "Numerical Optimization" course

http://cse.lab.imtlucca.it/~bemporad/optimization_course.html

MATHEMATICAL PROGRAMMING

$$\begin{array}{rl} \min_x & f(x) \\ \text{s.t.} & g(x) \leq 0 \end{array}$$

$$x \in \mathbb{R}^n,\ f : \mathbb{R}^n \to \mathbb{R},\ g : \mathbb{R}^n \to \mathbb{R}^m$$



$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad f(x) = f(x_1, x_2, \ldots, x_n), \quad g(x) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

In general, the problem is difficult to solve → **use software tools**

OPTIMIZATION SOFTWARE

- Comparison on benchmark problems:

  `http://plato.la.asu.edu/bench.html`

- Taxonomy of many solvers for different classes of optimization problems:

  `http://www.neos-guide.org`

- NEOS server for remotely solving optimization problems:

  `http://www.neos-server.org`

- Good open-source optimization software:

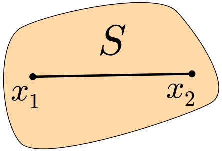
  `http://www.coin-or.org/`

- GitHub, MATLAB Central, Google, ...

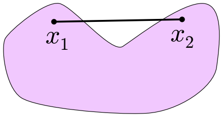
- A set $S \subseteq \mathbb{R}^n$ is **convex** if for all $x_1, x_2 \in S$

$$\boxed{\lambda x_1 + (1-\lambda)x_2 \in S,\ \forall \lambda \in [0,1]}$$

convex set



nonconvex set

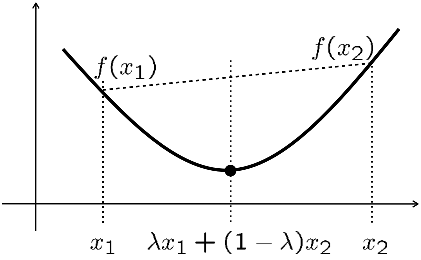


CONVEX FUNCTIONS

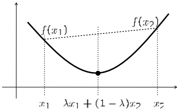
- A function $f : S \to \mathbb{R}$ is a **convex function** if S is convex and

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\forall x_1, x_2 \in S,\ \lambda \in [0,1]$$

Jensen's inequality

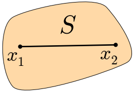


CONVEX OPTIMIZATION PROBLEM



The optimization problem

$$
\begin{array}{ll}
\min & f(x) \\
\text{s.t.} & x \in S
\end{array}
$$



is a **convex optimization problem** if S is a convex set and $f : S \to \mathbb{R}$ is a convex function

- Often S is defined by **linear equality constraints** $Ax = b$ and **convex inequality constraints** $g(x) \leq 0$, $g : \mathbb{R}^n \to \mathbb{R}^m$ convex

- Every local solution is also a global one (we will see this later)

- Efficient solution algorithms exist

- Often occurring in many problems in engineering, economics, and science

  Excellent textbook: "Convex Optimization" (Boyd, Vandenberghe, 2002)

POLYHEDRA

- Convex **polyhedron** = intersection of a finite set of half-spaces of $\mathbb{R}^n$
- Convex **polytope** = bounded convex polyhedron

- **Hyperplane (H-)representation**:

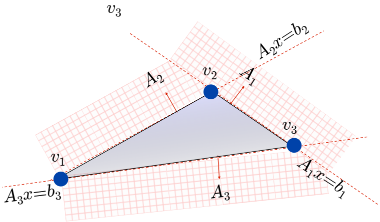
$$P = \{x \in \mathbb{R}^n : \ Ax \leq b\}$$

- **Vertex (V-)representation**:

$$P = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{q} \alpha_i v_i + \sum_{j=1}^{p} \beta_j r_j\}$$

$$\alpha_i, \beta_j \geq 0, \ \sum_{i=1}^{q} \alpha_i = 1, \ v_i, r_j \in \mathbb{R}^n$$

when $q = 0$ the polyhedron is a **cone**



Convex hull = transformation from V- to H-representation

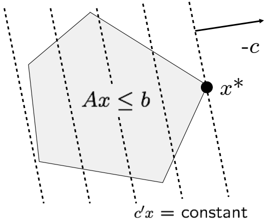
Vertex enumeration = transformation from H- to V-representation

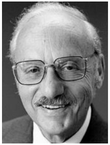
v_i = **vertex**, r_j = **extreme ray**

LINEAR PROGRAMMING

- Linear programming (LP) problem:

$$\begin{array}{rl} \min & c'x \\ \text{s.t.} & Ax \le b,\ x \in \mathbb{R}^n \\ & Ex = f \end{array}$$





George Dantzig
(1914–2005)

- LP in standard form:

$$\begin{array}{rl} \min & c'x \\ \text{s.t.} & Ax = b \\ & x \ge 0,\ x \in \mathbb{R}^n \end{array}$$

- Conversion to standard form:

  1. introduce **slack variables**

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i,\ s_i \ge 0$$

  2. split **positive and negative part** of x

$$\left\{ \begin{array}{l} \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \\ x_j \text{ free},\ s_i \ge 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \sum_{j=1}^{n} a_{ij}(x_j^+ - x_j^-) + s_i = b_i \\ x_j^+, x_j^-, s_i \ge 0 \end{array} \right.$$

LINEAR PROGRAMMING (LP)

- Converting maximization to minimization

$$\max_x c'x = -(\min_x -c'x) \qquad \text{(more generally: } \max_x f(x) = -\min_x\{-f(x)\}\text{)}$$

- Equalities to double inequalities (not recommended)

$$\sum_{j=1}^n a_{ij}x_j = b_i \Rightarrow \begin{cases} \sum_{j=1}^n a_{ij}x_j \leq b_i \\ \sum_{j=1}^n a_{ij}x_j \geq b_i \end{cases}$$

- Change direction of an inequality

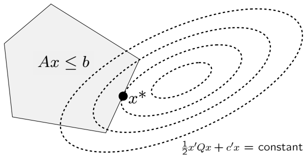
$$\sum_{j=1}^n a_{ij}x_j \geq b_i \Rightarrow \sum_{j=1}^n -a_{ij}x_j \leq -b_i$$

An LP can be always formulated using "$\min$" and "$\leq$"

QUADRATIC PROGRAMMING (QP)

- Quadratic programming (QP) problem:

$$\begin{array}{ll} \min & \frac{1}{2}x'Qx + c'x \\ \text{s.t.} & Ax \le b,\ x \in \mathbb{R}^n \\ & Ex = f \end{array}$$



- Convex optimization problem if $Q \succeq 0$ (Q = positive semidefinite matrix) [1]
- Without loss of generality, we can assume $Q = Q'$:

$$\begin{aligned} \frac{1}{2}x'Qx &= \frac{1}{2}x'\left(\frac{Q+Q'}{2} + \frac{Q-Q'}{2}\right)x = \frac{1}{2}x'\left(\frac{Q+Q'}{2}\right)x + \frac{1}{4}x'Qx - \frac{1}{4}(x'Q'x)' \\ &= \frac{1}{2}x'\left(\frac{Q+Q'}{2}\right)x \end{aligned}$$

- Hard problem if $Q \not\succeq 0$ (Q = indefinite matrix)

[1] A matrix $P \in \mathbb{R}^{n \times n}$ is **positive semidefinite** ($P \succeq 0$) if $x'Px \ge 0$ for all x.
It is **positive definite** ($P \succ 0$) if in addition $x'Px > 0$ for all $x \neq 0$.
It is **negative (semi)definite** ($P \prec 0$, $P \preceq 0$) if $-P$ is positive (semi)definite.
It is **indefinite** otherwise.

MIXED-INTEGER PROGRAMMING (MIP)

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax \le b,\ x = \left[\begin{smallmatrix} x_c \\ x_b \end{smallmatrix}\right] \\ & x_c \in \mathbb{R}^{n_c},\ x_b \in \{0,1\}^{n_b} \end{array}$$

mixed-integer linear program (MILP)

$$\begin{array}{ll} \min & \frac{1}{2}x'Qx + c'x \\ \text{s.t.} & Ax \le b,\ x = \left[\begin{smallmatrix} x_c \\ x_b \end{smallmatrix}\right] \\ & x_c \in \mathbb{R}^{n_c},\ x_b \in \{0,1\}^{n_b} \end{array}$$

mixed-integer quadratic program (MIQP)

- Some variables are real, some are binary (0/1)
- MILP and MIQP are $\mathcal{NP}$-hard problems, in general
- Many good solvers are available (CPLEX, Gurobi, GLPK, FICO Xpress, CBC, ...)
  For comparisons see `http://plato.la.asu.edu/bench.html`

MODELING LANGUAGES FOR OPTIMIZATION PROBLEMS

- **AMPL** (A Modeling Language for Mathematical Programming) most used modeling language, supports several solvers

- **GAMS** (General Algebraic Modeling System) is one of the first modeling languages

- **GNU MathProg** a subset of AMPL associated with the free package GLPK (GNU Linear Programming Kit)

- **YALMIP** MATLAB-based modeling language

- **CVX (CVXPY)** Modeling language for convex problems in MATLAB (python)

MODELING LANGUAGES FOR OPTIMIZATION PROBLEMS

- **CASADI + IPOPT** Nonlinear modeling + automatic differentiation, nonlinear programming solver (MATLAB, python, C++)
- **Optimization Toolbox**' modeling language (part of MATLAB since R2017b)
- **PYOMO** python-based modeling language
- **GEKKO** python-based mixed-integer nonlinear modeling language
- **PYTHON-MIP** python-based modeling language for mixed-integer linear programming
- **PuLP** A linear programming modeler for python
- **JuMP** A modeling language for linear, quadratic, and nonlinear constrained optimization problems embedded in julia

LINEAR MPC

LINEAR MPC - UNCONSTRAINED CASE

- Linear prediction model

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{cases}$$

$$x \in \mathbb{R}^n \quad u \in \mathbb{R}^m \quad y \in \mathbb{R}^p$$

Notation:

$x_0 = x(t)$

$x_k = x(t+k|t)$

$u_k = u(t+k|t)$

- Relation between input and states: $x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$
- Performance index

$$J(z, x_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$$

$$\begin{array}{rcl} R &=& R' \succ 0 \\ Q &=& Q' \succeq 0 \\ P &=& P' \succeq 0 \end{array} \qquad z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

- **Goal**: find the sequence z^* that minimizes $J(z, x_0)$, i.e., that steers the state x to the origin optimally

COMPUTATION OF COST FUNCTION

$$J(z,x_0) = x_0'Qx_0 + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}' \overbrace{\begin{bmatrix} Q & 0 & 0 & \dots & 0 \\ 0 & Q & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & Q & 0 \\ 0 & 0 & \dots & 0 & P \end{bmatrix}}^{\bar{Q}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}$$

$$+ \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}' \underbrace{\begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & R \end{bmatrix}}_{\bar{R}} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}_{\bar{S}} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\bar{T}} x_0$$

$$\begin{aligned} J(z,x_0) &= (\bar{S}z + \bar{T}x_0)'\bar{Q}(\bar{S}z + \bar{T}x_0) + z'\bar{R}z + x_0'Qx_0 \\ &= \frac{1}{2}z' \underbrace{2(\bar{R} + \bar{S}'\bar{Q}\bar{S})}_{H} z + x_0' \underbrace{2\bar{T}'\bar{Q}\bar{S}}_{F'} z + \frac{1}{2}x_0' \underbrace{2(Q + \bar{T}'\bar{Q}\bar{T})}_{Y} x_0 \end{aligned}$$

LINEAR MPC - UNCONSTRAINED CASE

$$J(z,x_0) = \frac{1}{2}z'Hz + x_0'F'z + \frac{1}{2}x_0'Yx_0$$

$$z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

condensed form of MPC

- The optimum is obtained by zeroing the gradient

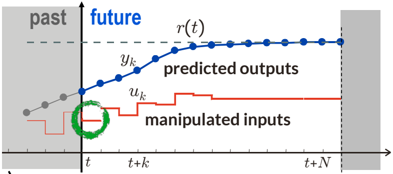
$$\nabla_z J(z,x_0) = Hz + Fx_0 = 0$$

and hence $z^* = \begin{bmatrix} u_0^* \\ u_1^* \\ \vdots \\ u_{N-1}^* \end{bmatrix} = -H^{-1}Fx_0$ ("batch" solution)

- **Alternative #1**: find z^* via **dynamic programming** (Riccati iterations)
- **Alternative #2**: keep also $x_1, \ldots, x_N$ as optimization variables and the equality constraints $x_{k+1} = Ax_k + Bu_k$ (**non-condensed form**, which is very **sparse**)

UNCONSTRAINED LINEAR MPC ALGORITHM

@ each sampling step t:



- Minimize quadratic function (no constraints)

$$\min_z f(z) = \frac{1}{2} z' H z + x'(t) F' z \qquad z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

- solution: $\nabla f(z) = Hz + Fx(t) = 0 \Rightarrow z^* = -H^{-1} F x(t)$

$$\Rightarrow \quad u(t) = -\begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix} H^{-1} F x(t) = K x(t)$$

unconstrained linear MPC = linear state-feedback!

CONSTRAINED LINEAR MPC

- Linear prediction model: $\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{cases}$

$$x \in \mathbb{R}^n$$
$$u \in \mathbb{R}^m$$
$$y \in \mathbb{R}^p$$

- Constraints to enforce:

$$\begin{cases} u_{\min} \leq u(t) \leq u_{\max} \\ y_{\min} \leq y(t) \leq y_{\max} \end{cases}$$

$$u_{\min}, u_{\max} \in \mathbb{R}^m$$
$$y_{\min}, y_{\max} \in \mathbb{R}^p$$

- Constrained optimal control problem (quadratic performance index):

$$\begin{aligned} \min_z \quad & x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \\ \text{s.t.} \quad & u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1 \\ & y_{\min} \leq y_k \leq y_{\max},\ k = 1, \ldots, N \end{aligned}$$

$$\begin{aligned} R &= R' \succ 0 \\ Q &= Q' \succeq 0 \\ P &= P' \succeq 0 \end{aligned} \qquad z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

CONSTRAINED LINEAR MPC

- Linear prediction model: $x_k = A^k x_0 + \sum_{i=0}^{k-1} A^i B u_{k-1-i}$
- Optimization problem (condensed form):

$$V(x_0) = \frac{1}{2}x_0' Y x_0 + \min_z \quad \frac{1}{2}z'Hz + x_0'F'z$$

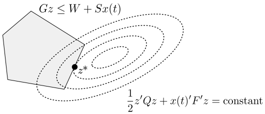
(quadratic objective)

$$\text{s.t.} \quad Gz \leq W + Sx_0$$

(linear constraints)

convex Quadratic Program (QP)

- $z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{Nm}$ is the optimization vector



- $H = H' \succ 0$, and H, F, Y, G, W, S depend on weights Q, R, P upper and lower bounds $u_{\min}, u_{\max}, y_{\min}, y_{\max}$ and model matrices A, B, C.

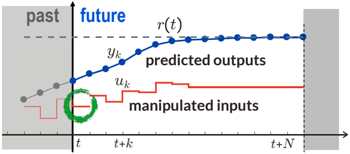
COMPUTATION OF CONSTRAINT MATRICES

- Input constraints $u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1$

$$\left\{\begin{array}{rcl} u_k & \leq & u_{\max} \\ -u_k & \leq & -u_{\min} \end{array}\right. \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \\ -1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & -1 \end{bmatrix} z \leq \begin{bmatrix} u_{\max} \\ u_{\max} \\ \vdots \\ u_{\max} \\ -u_{\min} \\ -u_{\min} \\ \vdots \\ -u_{\min} \end{bmatrix} \qquad z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

- Output constraints $y_k = CA^k x_0 + \sum_{i=0}^{k-1} CA^i B u_{k-1-i} \leq y_{\max},\ k = 1, \ldots, N$

$$\begin{bmatrix} CB & 0 & \ldots & 0 \\ CAB & CB & \ldots & 0 \\ \vdots & & & \vdots \\ CA^{N-1}B & \ldots & CAB & CB \end{bmatrix} z \leq \begin{bmatrix} y_{\max} \\ y_{\max} \\ \vdots \\ y_{\max} \end{bmatrix} - \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix} x_0$$



@ each sampling step t:

- Measure (or estimate) the current state $x(t)$
- Get the solution $z^* = \begin{bmatrix} u_0^* \\ u_1^* \\ \vdots \\ u_{N-1}^* \end{bmatrix}$ of the QP

$$\left\{ \begin{array}{ll} \min\limits_z & \frac{1}{2} z' H z + \overbrace{x'(t)}^{\text{feedback}} F' z \\ \text{s.t.} & Gz \leq W + S \underbrace{x(t)}_{\text{feedback}} \end{array} \right.$$

- Apply only $u(t) = u_0^*$, discarding the remaining optimal inputs $u_1^*, \dots, u_{N-1}^*$

DOUBLE INTEGRATOR EXAMPLE

- System: $x_2(t) = x_2(0) + \sum_{j=0}^{t-1} u(j), y(t) = x_1(0) + \sum_{j=0}^{t-1} x_2(j)$
- Sample time: $T_s = 1$ s
- State-space realization: $\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$
- Constraints: $-1 \leq u(t) \leq 1$
- Performance index: $\min \left(\sum_{k=0}^{1} y_k^2 + \frac{1}{10} u_k^2 \right) + x_N' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_N, \quad N = 2$
- QP matrices:

cost: $\frac{1}{2} z' H z + x'(t) F' z + \frac{1}{2} x'(t) Y x(t)$

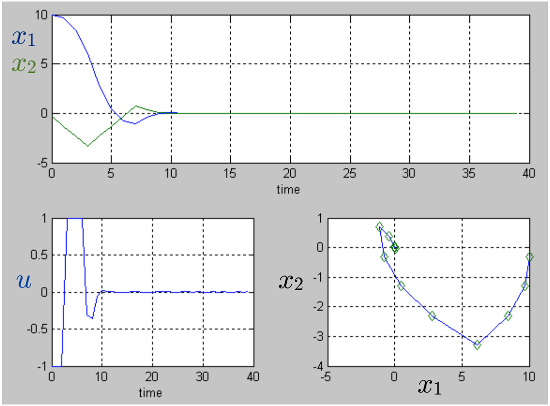
constraints: $Gz \leq W + Sx(t)$

$$H = \begin{bmatrix} 4.2 & 2 \\ 2 & 2.2 \end{bmatrix}, \; F = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$$Y = \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

DOUBLE INTEGRATOR EXAMPLE



go to demo `linear/doubleint.m` (Hybrid Toolbox)
(see also `mpcdoubleint.m` in MPC Toolbox)

DOUBLE INTEGRATOR EXAMPLE

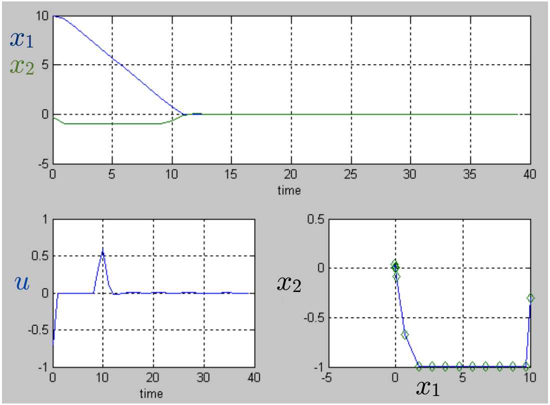
- Add constraint on second state at prediction time $t+1$:

$$x_{2,k} \geq -1,\ k=1$$

- New QP matrices:

$$H = \left[\begin{smallmatrix} 4.2 & 2 \\ 2 & 2.2 \end{smallmatrix}\right],\ F = \left[\begin{smallmatrix} 2 & 6 \\ 0 & 2 \end{smallmatrix}\right],\ Y = \left[\begin{smallmatrix} 4 & 6 \\ 6 & 12 \end{smallmatrix}\right]$$

$$G = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix},\ W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$



- State constraint $x_2(t) \geq -1$ is satisfied

LINEAR MPC - TRACKING

- Objective: make the output $y(t)$ track a reference signal $r(t)$
- Let us parameterize the problem using the **input increments**

$$\boxed{\Delta u(t) = u(t) - u(t-1)}$$

- As $u(t) = u(t-1) + \Delta u(t)$ we need to extend the system with a new state $x_u(t) = u(t-1)$

$$\left\{\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t-1) + B\Delta u(t) \\ x_u(t+1) & = & x_u(t) + \Delta u(t) \end{array}\right.$$

$$\boxed{\left\{\begin{array}{rcl} \begin{bmatrix} x(t+1) \\ x_u(t+1) \end{bmatrix} & = & \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ x_u(t) \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} \Delta u(t) \\ y(t) & = & \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_u(t) \end{bmatrix} \end{array}\right.}$$

- Again a linear system with states $x(t), x_u(t)$ and input $\Delta u(t)$

- Optimal control problem (quadratic performance index):

$$
\begin{array}{ll}
\min\limits_{z} & \displaystyle\sum_{k=0}^{N-1} \|W^y(y_{k+1}-r(t))\|_2^2 + \|W^{\Delta u}\Delta u_k\|_2^2 \\
& [\Delta u_k \triangleq u_k - u_{k-1}],\ u_{-1} = u(t-1) \\
\text{s.t.} & u_{\min} \le u_k \le u_{\max},\ k=0,\dots,N-1 \\
& y_{\min} \le y_k \le y_{\max},\ k=1,\dots,N \\
& \Delta u_{\min} \le \Delta u_k \le \Delta u_{\max},\ k=0,\dots,N-1
\end{array}
$$

$$
z = \left[\begin{array}{c} \Delta u_0 \\ \Delta u_1 \\ \vdots \\ \Delta u_{N-1} \end{array}\right] \text{ or } z = \left[\begin{array}{c} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{array}\right]
$$

weight $W^{(\cdot)}$ = diagonal matrix, or Cholesky factor of $Q^{(\cdot)} = (W^{(\cdot)})'W^{(\cdot)}$

$$
\begin{array}{ll}
\min\limits_{z} & J(z,x(t)) = \frac{1}{2}z'Hz + [x'(t)\, r'(t)\, u'(t-1)]F'z \\
\text{s.t.} & Gz \le W + S\left[\begin{array}{c} x(t) \\ r(t) \\ u(t-1) \end{array}\right]
\end{array}
$$

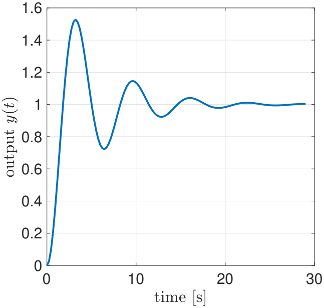
convex Quadratic Program

- Add the extra penalty $\|W^u(u_k - u_{\text{ref}}(t))\|_2^2$ to track **input references**
- Constraints may depend on $r(t)$, such as $e_{\min} \le y_k - r(t) \le e_{\max}$

LINEAR MPC TRACKING EXAMPLE

- System: $y(t) = \frac{1}{s^2+0.4s+1}u(t)$

  (or equivalently $\frac{d^2y}{dt^2} + 0.4\frac{dy}{dt} + y = u$)



- Sampling with period $T_s = 0.5$ s:

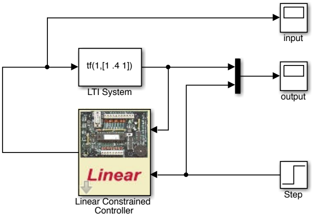
$$\begin{cases} x(t+1) &= \begin{bmatrix} 1.597 & -0.8187 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0.2294 & 0.2145 \end{bmatrix} x(t) \end{cases}$$

go to demo `linear/example3.m` (Hybrid Toolbox)

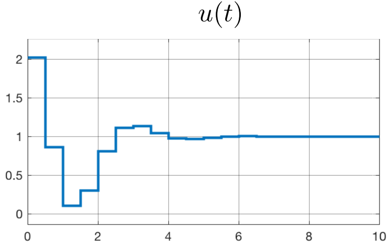
LINEAR MPC TRACKING EXAMPLE

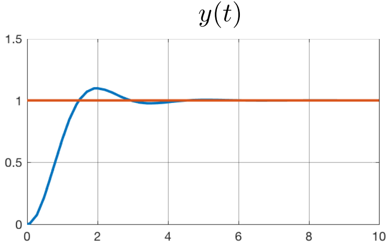
- Performance index:

$$\min \sum_{k=0}^{9} (y_{k+1} - r(t))^2 + 0.04\Delta u_k^2$$



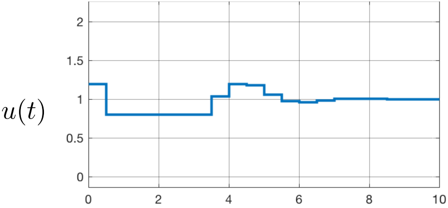
- Closed-loop MPC results:

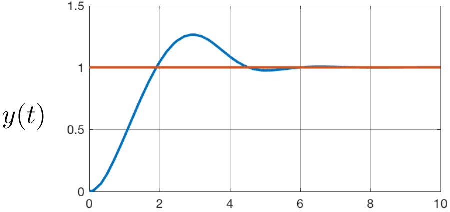




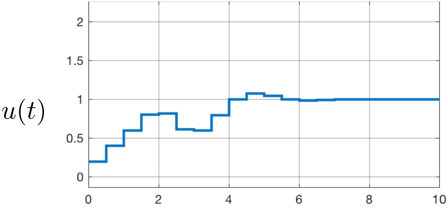
LINEAR MPC TRACKING EXAMPLE

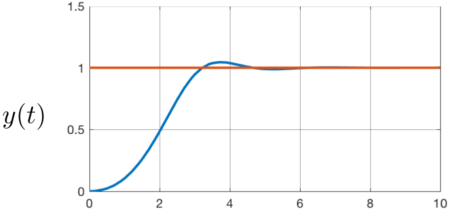
- Impose constraint $0.8 \leq u(t) \leq 1.2$ (amplitude)





- Impose instead constraint $-0.2 \leq \Delta u(t) \leq 0.2$ (slew-rate)



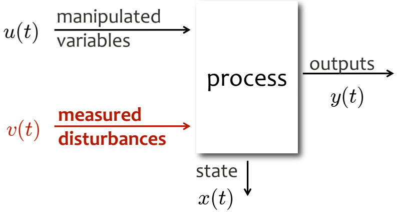


MEASURED DISTURBANCES

- **Measured disturbance** $v(t)$ = input that is measured but not manipulated

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + B_v v(t) \\ y_k = Cx_k + D_v v(t) \end{cases}$$

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} + A^j B_v v(t)$$



- Same performance index, same constraints. We still have a QP:

$$\begin{array}{ll} \min\limits_z & \frac{1}{2} z' H z + [x'(t)\; r'(t)\; u'(t-1)\; v'(t)] F' z \\ \text{s.t.} & Gz \le W + S \begin{bmatrix} x(t) \\ r(t) \\ u(t-1) \\ v(t) \end{bmatrix} \end{array}$$

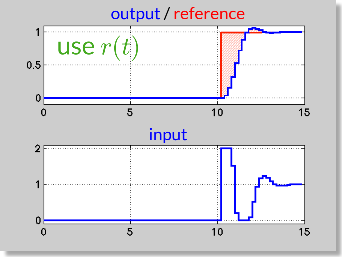
- Note that MPC naturally provides **feedforward action** on $v(t)$ and $r(t)$

ANTICIPATIVE ACTION (A.K.A. "PREVIEW")

$$\min_{z} \sum_{k=0}^{N-1} \|W^y(y_{k+1} - r(t+k))\|_2^2 + \|W^{\Delta u}\Delta u(k)\|_2^2$$

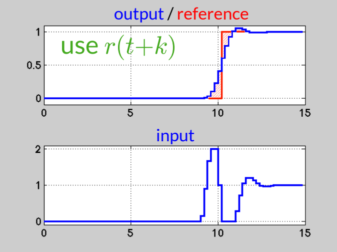
- Reference **not known** in advance (**causal**):

$$r_k \equiv r(t), \forall k = 0, \ldots, N-1$$



- Future refs (partially) **known** in advance (**anticipative action**):

$$r_k = r(t+k), \forall k = 0, \ldots, N-1$$

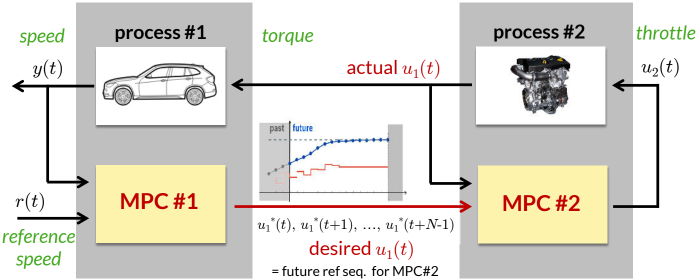


go to demo `mpcpreview.m` (MPC Toolbox)

- Same idea also applies for **preview of measured disturbances** $v(t+k)$

EXAMPLE: CASCADED MPC

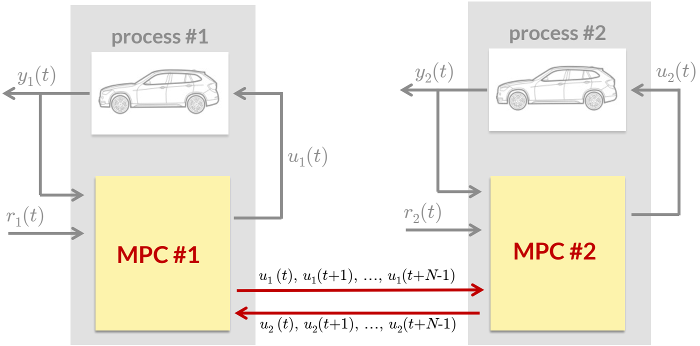
- We can use preview also to **coordinate** multiple MPC controllers
- Example: **cascaded MPC**



- MPC #1 sends current and future references to MPC #2

EXAMPLE: DECENTRALIZED MPC

- Example: decentralized MPC



- Commands generated by MPC #1 are measured dist. in MPC#2, and vice versa

SOFT CONSTRAINTS

- **Relax** output constraints to prevent **QP infeasibility**

$$
\begin{array}{ll}
\min\limits_{z} & \displaystyle\sum_{k=0}^{N-1} \|W^y(y_{k+1}-r(t))\|_2^2 + \|W^{\Delta u}\Delta u_k\|_2^2 + \|W^u(u_k - u_{\mathrm{ref}}(t))\|_2^2 + \rho_\epsilon \epsilon^2 \\
\text{s.t.} & x_{k+1} = Ax_k + Bu_k,\ k = 0,\ldots,N-1 \\
& u_{\min} \le u_k \le u_{\max},\ k = 0,\ldots,N-1 \\
& \Delta u_{\min} \le \Delta u_k \le \Delta u_{\max},\ k = 0,\ldots,N-1 \\
& y_{\min} - \epsilon V_{\min} \le y_k \le y_{\max} + \epsilon V_{\max},\ k = 1,\ldots,N
\end{array}
\qquad
z = \begin{bmatrix} \Delta u(0) \\ \vdots \\ \Delta u(N-1) \\ \epsilon \end{bmatrix}
$$

- ϵ = “panic” variable, with weight $\rho_\epsilon \gg W^y, W^{\Delta u}$

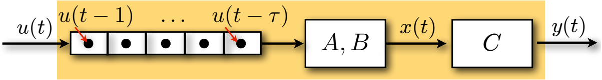
- $V_{\min}, V_{\max}$ = vectors with entries > 0. The larger the i-th entry of vector V, the relatively softer the corresponding i-th constraint

- Infeasibility can be due to:
  - modeling errors, disturbances
  - wrong MPC setup (e.g., prediction horizon is too short)

DELAYS - METHOD #1

- Linear model with **delays**

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t-\tau) \\
y(t) &= Cx(t)
\end{aligned}
$$



- Map delays to poles in $z = 0$:

$$
x_k(t) \triangleq u(t-k) \quad \Rightarrow \quad x_k(t+1) = x_{k-1}(t),\ k = 1, \dots, \tau
$$

$$
\begin{bmatrix} x \\ x_\tau \\ x_{\tau-1} \\ \vdots \\ x_1 \end{bmatrix}(t+1) = \begin{bmatrix} A & B & 0 & 0 & \dots & 0 \\ 0 & 0 & I_m & 0 & \dots & 0 \\ 0 & 0 & 0 & I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \\ x_{\tau-1} \\ \vdots \\ x_1 \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ I_m \end{bmatrix} u(t)
$$

- Apply MPC to the extended system

- **Note**: the prediction horizon N must be $\geq \tau$, otherwise no input $u_0, \dots, u_{N-1}$ has an effect on the output!

DELAYS - METHOD #2

- Linear model with **delays**: $\begin{cases} x(t+1) &= Ax(t) + Bu(t-\tau) \\ y(t) &= Cx(t) \end{cases}$
- **Delay-free** model: $\bar{x}(t) \triangleq x(t+\tau)$ $\Rightarrow$ $\begin{cases} \bar{x}(t+1) &= A\bar{x}(t) + Bu(t) \\ \bar{y}(t) &= C\bar{x}(t) \end{cases}$
- Design MPC for delay-free model, $u(t) = f_{\rm MPC}(\bar{x}(t))$
- Compute the predicted state

$$\bar{x}(t) = \hat{x}(t+\tau) = A^\tau x(t) + \sum_{j=0}^{\tau-1} A^j B \underbrace{u(t-1-j)}_{\text{past inputs!}}$$

- Compute the MPC control move $u(t) = f_{\rm MPC}(\hat{x}(t+\tau))$
- For better closed-loop performance and improved robustness, $\hat{x}(t+\tau)$ can be computed by a more complex model than (A, B, C)

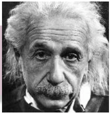
GOOD MODELS FOR (MPC) CONTROL

- Computation complexity depends on chosen prediction model

- Good models for MPC must be
  - **Descriptive** enough to capture the most significant dynamics of the system

  - **Simple** enough for solving the optimization problem

> “Things should be made as simple as possible, but not any simpler.”



Albert Einstein
(1879–1955)

- **After** the industrial success of MPC, a lot of research done:

  - **linear** MPC $\Rightarrow$ linear prediction model
  - **nonlinear** MPC $\Rightarrow$ nonlinear prediction model
  - **robust** MPC $\Rightarrow$ uncertain (linear) prediction model
  - **stochastic** MPC $\Rightarrow$ stochastic prediction model
  - **distributed/decentralized** MPC$\Rightarrow$ multiple MPCs cooperating together
  - **economic** MPC $\Rightarrow$ MPC based on arbitrary (economic) performance indices
  - **hybrid** MPC $\Rightarrow$ prediction model integrating logic and dynamics
  - **explicit** MPC $\Rightarrow$ offline (exact/approximate) computation of MPC
  - **solvers** for MPC $\Rightarrow$ online numerical algorithms for solving MPC problems
  - **data-driven** MPC $\Rightarrow$ machine learning methods tailored to MPC design

- Main theoretical issues: **feasibility**, **stability**, solution **algorithms** (Mayne, 2014)

FEASIBILITY

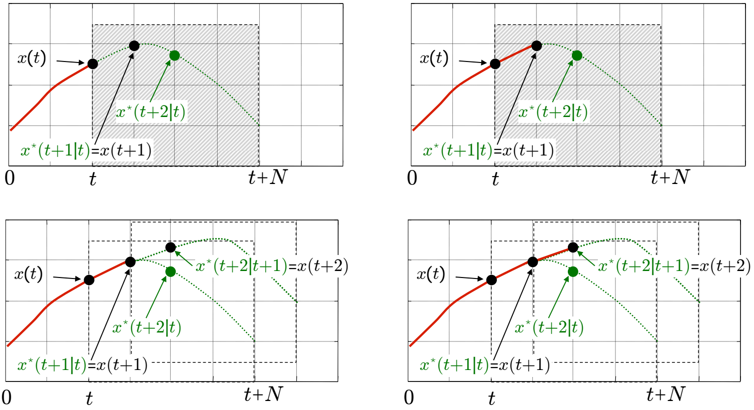
$$
\begin{array}{ll}
\min\limits_{z} & \sum\limits_{k=0}^{N-1} \|W^y(y_{k+1}-r(t))\|^2 + \|W^{\Delta u}\Delta u_k\|^2 \\
\text{subj. to} & u_{\min} \le u_k \le u_{\max},\ k=0,\dots,N-1 \\
& y_{\min} \le y_k \le y_{\max},\ k=1,\dots,N \\
& \Delta u_{\min} \le \Delta u_k \le \Delta u_{\max},\ k=0,\dots,N-1
\end{array}
$$

QP problem

- **Feasibility**: Will the QP problem be feasible at all sampling instants t?
- **Input constraints** only: always feasible if $u/\Delta u$ constraints are consistent
- **Hard output constraints**:
  - When $N < \infty$ there is no guarantee that the QP problem will remain feasible at all t, even in the nominal case
  - $N = \infty$ ok in the nominal case, but we have an infinite number of constraints!
  - **Maximum output admissible set** theory: $N < \infty$ is enough (Gutman, Ckwikel, 1987) (Gilbert, Tan, 1991) (Chmielewski, Manousiouthakis, 1996) (Kerrigan, Maciejowski, 2000)

PREDICTED AND ACTUAL TRAJECTORIES

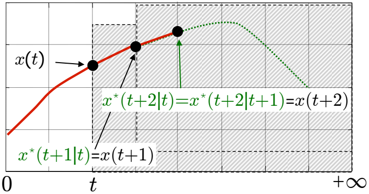
- Consider predicted and actual trajectories

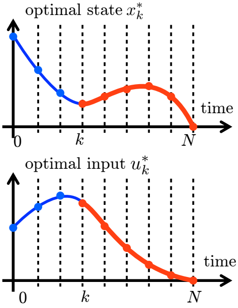


- Even assuming a perfect model and no disturbances, **predicted** open-loop trajectories and **actual** closed-loop trajectories can be different

PRINCIPLE

- Special case: when the horizon is **infinite**, open-loop and closed-loop trajectories coincide





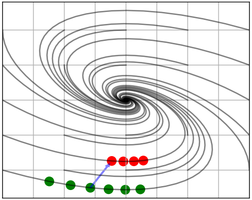
- This follows by **Bellman's principle of optimality**:

> "Given the optimal sequence $u_0^*, \ldots, u_{N-1}^*$ and the corresponding optimal trajectory $x_0^*, \ldots, x_N^*$, the **subsequence** $u_k^*, \ldots, u_{N-1}^*$ **is optimal** for the sub-problem on the horizon $[k, N]$, starting **from the optimal state** x_k^*"

Richard Bellman
(1920–1984)

PRINCIPLE

- Let us focus on the subproblem from k to N. Clearly, the solution $u_k^*, \ldots, u_{N-1}^*$ only depends on the initial state x_k^*



- In particular, the first sample u_k^* only depends on x_k^*!

- Hence, an optimal control policy can always be computed in **state-feedback form** $u_k^* = g_k(x_k^*)$, $k = 0, \ldots, N-1$, which is a **more robust** form

- Bellman's principle holds since the dynamics are causal, i.e., $x_{k+1} = f_k(x_k, u_k)$, and the cost function is separable:

$$\min_{u_0,\ldots,u_{N-1}} \left\{ \sum_{j=0}^{N-1} \ell_j(x_j, u_j) \right\} = \min_{u_0,\ldots,u_{k-1}} \left\{ \sum_{j=0}^{k-1} \ell_j(x_j, u_j) + \underbrace{\min_{u_k,\ldots,u_{N-1}} \left\{ \sum_{j=k}^{N-1} \ell_j(x_j, u_j) \right\}}_{V_k^*(x_k)} \right\}$$

This is the basis of **dynamic programming** ...

CONVERGENCE AND STABILITY

$$\begin{array}{ll} \min\limits_{z} & x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \\ \text{s.t.} & u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1 \\ & y_{\min} \leq C x_k \leq y_{\max},\ k = 1, \ldots, N \end{array}$$

QP problem

$$Q = Q' \succeq 0, R = R' \succ 0, P = P' \succeq 0$$

- Stability is a complex function of the model (A, B, C) and the MPC parameters $N, Q, R, P, u_{\min}, u_{\max}, y_{\min}, y_{\max}$
- **Stability constraints** and weights on the terminal state x_N can be imposed over the prediction horizon to ensure stability of MPC

BASIC CONVERGENCE PROPERTIES

(Keerthi, Gilbert, 1988) (Bemporad, Chisci, Mosca, 1994)

- **Theorem**: Let the MPC law be based on

$$\begin{array}{rl} V^*(x(t)) = \min & \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \\ \text{s.t.} & x_{k+1} = A x_k + B u_k \\ & u_{\min} \leq u_k \leq u_{\max} \\ & y_{\min} \leq C x_k \leq y_{\max} \\ & x_N = 0 \quad \leftarrow \text{“terminal constraint”} \end{array}$$

  with $R, Q \succ 0, u_{\min} < 0 < u_{\max}, y_{\min} < 0 < y_{\max}$.
  If the **optimization problem is feasible at time** $t = 0$ then

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} u(t) = 0$$

  and the constraints are satisfied at all time $t \geq 0$, for all $R, Q \succ 0$.

- Many more convergence and stability results exist (Mayne, 2014)

CONVERGENCE PROOF

Proof: Main idea = use the **value function** $V^*(x(t))$ as a **Lyapunov-like function**

- Let $z_t = [u_0^t \ \ldots \ u_{N-1}^t]'$ be the optimal control sequence at time t
- By construction $\bar{z}_{t+1} = [u_1^t \ \ldots \ u_{N-1}^t \ 0]'$ is a feasible sequence at time $t+1$
- The cost of $\bar{z}_{t+1}$ is $V^*(x(t)) - x'(t)Qx(t) - u'(t)Ru(t) \geq V^*(x(t+1))$
- $V^*(x(t))$ is monotonically decreasing and ≥ 0, so $\exists \lim_{t\to\infty} V^*(x(t)) \triangleq V_\infty$
- Hence
  $0 \leq x'(t)Qx(t) + u'(t)Ru(t) \leq V^*(x(t)) - V^*(x(t+1)) \to 0$ for $t \to \infty$
- Since $R, Q \succ 0, \lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} u(t) = 0$ ■

Reaching the global optimum exactly is not needed to prove convergence

MORE GENERAL CONVERGENCE RESULT - OUTPUT WEIGHTS

(Keerthi, Gilbert, 1988) (Bemporad, Chisci, Mosca, 1994)

- **Theorem**: Consider the MPC control law based on optimizing

$$
\begin{aligned}
V^*(x(t)) = \min \quad & \sum_{k=0}^{N-1} y_k' Q_y y_k + u_k' R u_k \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k \\
& y_k = Cx_k \\
& u_{\min} \leq u_k \leq u_{\max} \\
& y_{\min} \leq y_k \leq y_{\max} \\
& \text{either} \quad N = \infty \quad \text{or} \quad x_N = 0
\end{aligned}
$$

If the optimization problem is feasible at time $t = 0$ then for all $R = R' \succ 0$, $Q_y = Q_y' \succ 0$

$$\lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} u(t) = 0$$

and the constraints are satisfied at all time $t \geq 0$. Moreover, if (C, A) is a detectable pair then $\lim_{t\to\infty} x(t) = 0$.

CONVERGENCE PROOF

Proof:

- The shifted optimal sequence $\bar{z}_{t+1} = [u_1^t \ \ldots \ u_{N-1}^t \ 0]'$ (or $\bar{z}_{t+1} = [u_1^t \ u_2^t \ \ldots]'$ in case $N = \infty$) is feasible sequence at time $t+1$ and has cost

$$V^*(x(t)) - y'(t)Q_y y(t) - u'(t)Ru(t) \geq V^*(x(t+1))$$

- Therefore, by convergence of $V^*(x(t))$, we have that

$$0 \leq y'(t)Q_y y(t) + u'(t)Ru(t) \leq V^*(x(t)) - V^*(x(t+1)) \to 0$$

  for $t \to \infty$

- Since $R, Q_y \succ 0$, also $\lim_{t\to\infty} y(t) = 0, \ \lim_{t\to\infty} u(t) = 0$
- For all $k = 0, \ldots, n-1$ we have that

$$0 = \lim_{t\to\infty} y'(t+k)Q_y y(t+k) = \lim_{t\to\infty} \|LC(A^k x(t) + \sum_{j=0}^{k-1} A^j Bu(t+k-1-j))\|_2^2$$

  where $Q_y = L'L$ (Cholesky factorization) and L is nonsingular

CONVERGENCE PROOF (CONT'D)

- As $u(t) \to 0$, also $LCA^k x(t) \to 0$, and since L is nonsingular $CA^k x(t) \to 0$ too, for all $k = 0, \ldots, n-1$
- Hence $\Theta x(t) \to 0$, where Θ is the observability matrix of (C, A)
- If (C, A) is observable then Θ is nonsingular and hence $\lim_{t\to\infty} x(t) = 0$
- If (C, A) is only detectable, we can make a canonical observability decomposition and show that the observable states converge to zero
- As also $u(t) \to 0$ and the unobservable subsystem is asymptotically stable, the unobservable states must converge to zero asymptotically too $\blacksquare$

EXTENSION TO REFERENCE TRACKING

- We want to track a constant reference r. Assume x_r and u_r exist such that

$$\begin{array}{rcl} x_r &=& Ax_r + Bu_r \\ r &=& Cx_r \end{array} \qquad \text{(equilibrium state/input)}$$

- Formulate the MPC problem (assume $u_{\min} < u_r < u_{\max}, y_{\min} < r < y_{\max}$)

$$\begin{array}{ll} \min & \displaystyle\sum_{k=0}^{N-1} (y_k - r)' Q_y (y_k - r) + (u_k - u_r)' R (u_k - u_r) \\ \text{s.t.} & x_{k+1} = Ax_k + Bu_k \\ & u_{\min} \leq u_k \leq u_{\max}, \quad y_{\min} \leq Cx_k \leq y_{\max} \\ & x_N = x_r \end{array}$$

- We can repeat the convergence proofs in the **shifted-coordinates** $x_{k+1} - x_r = A(x_k - x_r) + B(u_k - u_r), y_k - r = C(x_k - x_r)$

- **Drawback**: the approach only works well in nominal conditions, as the input reference u_r depends on A, B, C (inverse static model)

STABILITY CONSTRAINTS

1. No stability constraint, infinite prediction horizon $N \to \infty$

   (Keerthi, Gilbert, 1988) (Rawlings, Muske, 1993) (Bemporad, Chisci, Mosca, 1994)

2. End-point constraint $x_N = 0$

   (Kwon, Pearson, 1977) (Keerthi, Gilbert, 1988) (Bemporad, Chisci, Mosca, 1994)

3. Relaxed terminal constraint $x_N \in \Omega$

   (Scokaert, Rawlings, 1996)

4. Contraction constraint $\|x_{k+1}\| \leq \alpha \|x(t)\|,\ \alpha < 1$

   (Polak, Yang, 1993) (Bemporad, 1998)

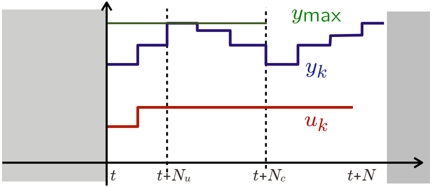
   All the proofs in (1,2,3) use the value function $V^*(x(t)) = \min_z J(z, x(t))$ as a Lyapunov function.

CONTROL AND CONSTRAINT HORIZONS

$$
\begin{array}{ll}
\min\limits_{z} & \displaystyle\sum_{k=0}^{N-1} \|W^y(y_k - r(t))\|_2^2 + \|W^{\Delta u}\Delta u_k\|_2^2 + \rho_\epsilon \epsilon^2 \\
\text{subj. to} & u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1 \\
& \Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max},\ k = 0, \ldots, N-1 \\
& \Delta u_k = 0,\ k = N_u, \ldots, N-1 \\
& y_{\min} - \epsilon V_{\min} \leq y_k \leq y_{\max} + \epsilon V_{\max},\ k = 1, \ldots, N_c
\end{array}
$$

- The **input horizon** N_u limits the number of free variables
  - Reduced performance
  - Reduced computation time

  typically $N_u = 1 \div 5$



- The **constraint horizon** N_c limits the number of constraints
  - Higher chance of violating output constraints but reduced computation time

- Other variable-reduction methods exist (Bemporad, Cimini, 2020)

MPC AND LINEAR QUADRATIC REGULATION (LQR)

- Special case: $J(z,x_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$, $N_u = N$, with matrix P solving the Discrete Algebraic Riccati Equation

$$P = A'PA - A'PB(B'PB+R)^{-1}B'PA + Q$$

Jacopo Francesco Riccati (1676–1754)

(unconstrained) **MPC = LQR** for any choice of the prediction horizon N

Proof: Easily follows from Bellman's principle of optimality (dynamic programming): $x_N' P x_N$ = optimal "cost-to-go" from time N to ∞.

MPC AND CONSTRAINED LQR

- Consider again the constrained MPC law based on minimizing

$$\begin{array}{ll} \min_{z} & x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \\ \text{s.t.} & u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1 \\ & y_{\min} \leq y_k \leq y_{\max},\ k = 1, \ldots, N \\ & u_k = K x_k,\ k = N_u, \ldots, N-1 \end{array}$$

- Choose matrix P and terminal gain K by solving the LQR problem

$$\begin{aligned} K &= -(R + B'PB)^{-1} B'PA \\ P &= (A+BK)'P(A+BK) + K'RK + Q \end{aligned}$$

- In a polyhedral region around the origin, **constrained MPC = constrained LQR** for any choice of the prediction and control horizons N, N_u

(Sznaier, Damborg, 1987) (Chmielewski, Manousiouthakis, 1996) (Scokaert, Rawlings, 1998)
(Bemporad, Morari, Dua Pistikopoulos, 2002)

- The larger the horizon N, the larger the region where MPC $\equiv$ constrained LQR

MPC AND CONSTRAINED LQR

- Some MPC formulations also include the **terminal constraint** $x_N \in \mathcal{X}_\infty$

$$\mathcal{X}_\infty = \left\{x : \begin{bmatrix} u_{\min} \\ y_{\min} \end{bmatrix} \leq \begin{bmatrix} K \\ C \end{bmatrix} (A+BK)^k x \leq \begin{bmatrix} u_{\max} \\ y_{\max} \end{bmatrix}, \forall k \geq 0\right\}$$

  that is the **maximum output admissible set** for the closed-loop system $(A+BK, B, C)$ and constraints $u_{\min} \leq Kx \leq u_{\max}, y_{\min} \leq Cx \leq y_{\max}$

- This makes MPC $\equiv$ constrained LQR **where the MPC problem is feasible**

- Recursive feasibility in ensured by the terminal constraint for all $x(0)$ such that the MPC problem is feasible @$t = 0$

- The domain of feasibility may be reduced because of the additional constraint

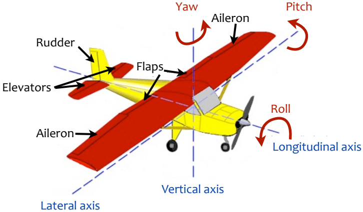
Linearized model:

$$\begin{cases} \dot{x} = \begin{bmatrix} -0.0151 & -60.5651 & 0 & -32.174 \\ -0.0001 & -1.3411 & 0.9929 & 0 \\ 0.00018 & 43.2541 & -0.86939 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} -2.516 & -13.136 \\ -0.1689 & -0.2514 \\ -17.251 & -1.5766 \\ 0 & 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \end{cases}$$

(Kapasouris, Athans, Stein, 1988)

- Inputs: elevator and flaperon (=flap+aileron) angles
- Outputs: attack and pitch angles
- Sampling time: $T_s = 0.05$ sec (+ ZOH)
- Constraints: $\pm 25^\circ$ on both angles
- Open-loop unstable, poles are
  $-7.6636, -0.0075 \pm 0.0556j, 5.4530$

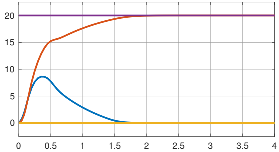
go to demo `linear/afti16.m` (Hybrid Toolbox)
see also `mpcaircraft.m` (MPC Toolbox)



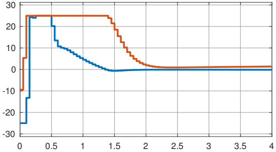
EXAMPLE: AFTI-F16 AIRCRAFT

- Prediction horizon $N = 10$, control horizon $N_u = 2$
- Input weights $W^{\Delta u} = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]$, $W^u = 0$
- Input constraints $u_{\min} = -u_{\max} = \left[\begin{smallmatrix} 25^\circ \\ 25^\circ \end{smallmatrix}\right]$

$y_1(t)$
$y_2(t)$

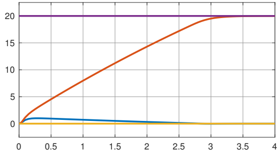


$u(t)$

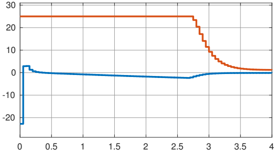


$W^y = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$

$y_1(t)$
$y_2(t)$



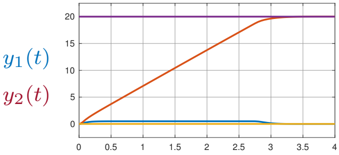
$u(t)$

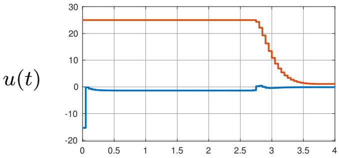


$W^y = \left[\begin{smallmatrix} 100 & 0 \\ 0 & 1 \end{smallmatrix}\right]$

EXAMPLE: AFTI-F16 AIRCRAFT

- Add output constraints $y_{1,\min} = -y_{1,\max} = 0.5°$

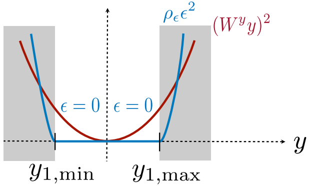




$$W^y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Soft output constraint = convex penalty with dead zone

$$\begin{array}{rl} \min & \rho_\epsilon \epsilon^2 \\ \text{s.t.} & y_{1,\min} - \epsilon V_{1,\min} \leq y_1 \leq y_{1,\max} - \epsilon V_{1,\max} \end{array}$$

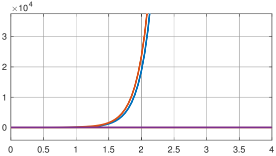


EXAMPLE: AFTI-F16 AIRCRAFT

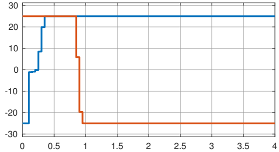
- Linear control (=unconstrained MPC) + input clipping

$y_1(t)$

$y_2(t)$



$u(t)$



$$W^y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

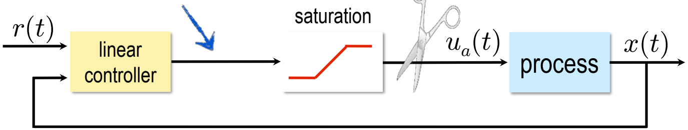
unstable!

Saturation needs to be considered in the control design!

SATURATION

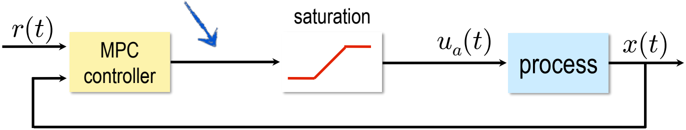
- Saturation is dangerous because it can **break the control loop**

$$u(t) = Kx(t) + Hr(t) \neq u_a(t)$$



- MPC takes saturation into account and handles it automatically (and optimally)

$$u(t) = f_{\mathrm{MPC}}(x(t), r(t)) \equiv u_a(t)$$



TUNING GUIDELINES

$$
\begin{array}{ll}
\min\limits_{\Delta U} & \sum\limits_{k=0}^{N-1} \|W^y(y_{k+1} - r(t))\|_2^2 + \|W^{\Delta u}\Delta u_k\|_2^2 + \rho_\epsilon \epsilon^2 \\
\text{subj. to} & \Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max},\ k = 0, \ldots, N_u - 1 \\
& \Delta u_k = 0,\ k = N_u, \ldots, N - 1 \\
& u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N_u - 1 \\
& y_{\min} - \epsilon V_{\min} \leq y_k \leq y_{\max} + \epsilon V_{\max},\ k = 1, \ldots, N_c
\end{array}
$$

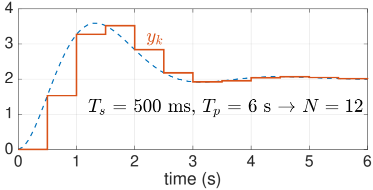
- **weights**: the larger the ratio $W^y/W^{\Delta u}$ the more aggressive the controller
- **input horizon**: the larger N_u, the more "optimal" but more complex the controller
- **prediction horizon**: the smaller N, the more aggressive the controller
- **constraints horizon**: the smaller N_c, the simpler the controller
- **limits**: controller less aggressive if $\Delta u_{\min}, \Delta u_{\max}$ are small
- **penalty ρ_ϵ**: pick up smallest ρ_ϵ that keeps soft constraints reasonably satisfied

Always try to set N_u as small as possible!

- Let T_s = **sampling time** used in MPC predictions

- For predicting T_p time units in the future we must set

$$N = \left\lceil \frac{T_p}{T_s} \right\rceil$$



- Slew-rate constraints $\dot{u}_{\min} \leq \frac{du}{dt} \leq \dot{u}_{\max}$ on actuators are related to T_s by

$$\dot{u} \triangleq \frac{du}{dt} \approx \frac{u_k - u_{k-1}}{T_s} \quad \Rightarrow \quad \overbrace{T_s \dot{u}_{\min}}^{\Delta u_{\min}} \leq \Delta u_k \leq \overbrace{T_s \dot{u}_{\max}}^{\Delta u_{\max}}$$

TUNING GUIDELINES - EFFECT OF SAMPLING TIME

- The MPC cost to minimize can be thought in continuous-time as

$$J = \int_0^{T_p} \left(\|W^y(y(\tau)-r)\|_2^2 + \|W^u(u(\tau)-u_r)\|_2^2 + \|W^{\dot{u}}\dot{u}(\tau)\|_2^2 \right) d\tau + \rho_\epsilon \epsilon^2$$

$$\approx T_s \left(\sum_{k=0}^{N-1} \|W^y(y_{k+1}-r)\|_2^2 + \|W^u(u_k-u_r)\|_2^2 + \|\underbrace{W^{\dot{u}}T_s^{-1}}_{W^{\Delta u}}\Delta u_k\|_2^2 + \rho_\epsilon T_s^{-1}\epsilon^2 \right)$$

- Hence, when changing the sampling time from T_1 to T_2, can can keep the same MPC cost function by leaving W^y, W^u unchanged and simply rescaling

$$W_2^{\Delta u} = \frac{W^{\dot{u}}}{T_2} = \frac{T_1}{T_2}\frac{W^{\dot{u}}}{T_1} = \frac{T_1}{T_2}W_1^{\Delta u} \qquad \rho_{\epsilon 2} = \frac{\rho_\epsilon}{T_2} = \frac{T_1}{T_2}\frac{\rho_\epsilon}{T_1} = \frac{T_1}{T_2}\rho_{\epsilon 1}$$

- **Note**: T_s used for controller execution can be $\neq$ than T_s used in prediction

- Small controller $T_s \Rightarrow$ fast **reaction** to set-point changes / disturbances

SCALING

- Humans think infinite precision ...computers do not !

- Numerical difficulties may arise if variables assume very small/large values

- Example:

$$\begin{array}{ll} y_1 \in [-1e-4, 1e-4] & \text{[V]} \\ y_2 \in [-1e4, 1e4] & \text{[Pa]} \end{array} \quad \Longrightarrow \quad \begin{array}{ll} y_1 \in [-0.1, 0.1] & \text{[mV]} \\ y_2 \in [-10, 10] & \text{[kPa]} \end{array}$$

- Ideally all variables should be in $[-1, 1]$. For example, one can replace y with $y/y_{\rm max}$

- Scaling also possible after formulating the QP problem (=**preconditioning**)

PRE-STABILIZATION OF OPEN-LOOP UNSTABLE MODELS

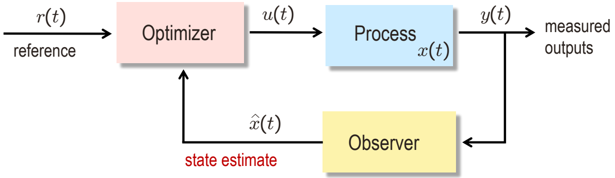
- **Numerical issues** may arise with **open-loop unstable** models
- When condensing the QP we substitute $x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$ that may lead to a badly conditioned Hessian matrix H

$$H = \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}' \begin{bmatrix} Q & 0 & 0 & \dots & 0 \\ 0 & Q & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & Q & 0 \\ 0 & 0 & \dots & 0 & P \end{bmatrix} \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix} + \begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & R \end{bmatrix}$$

- **Pre-stabilizing** the system with $u_k = Kx_k + v_k$ leads to $x_{k+1} = A_K x_k + B v_k$, $A_K \triangleq A + BK$, and therefore $x_k = A_K^k x_0 + \sum_{j=0}^{k-1} A_K^j B v_{k-1-j}$
- Input constraints become mixed constraints $u_{\min} \leq K x_k + v_k \leq u_{\max}$
- K can be chosen for example by pole-placement or LQR

OBSERVER DESIGN FOR MPC

STATE OBSERVER FOR MPC

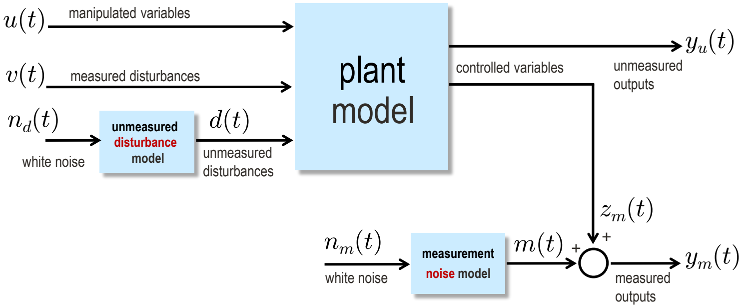


- Full state $x(t)$ of process may not be available, only outputs $y(t)$
- Even if $x(t)$ is available, noise should be filtered out
- Prediction and process models may be quite different
- The state $x(t)$ may not have any physical meaning
  (e.g., in case of model reduction or subspace identification)

We need to use a state observer

- Example: **Luenberger observer** $\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$

EXTENDED MODEL FOR OBSERVER DESIGN



unmeasured disturbance model

$$\begin{cases} x_d(t+1) &= \bar{A}x_d(t) + \bar{B}n_d(t) \\ d(t) &= \bar{C}x_d(t) + \bar{D}n_d(t) \end{cases}$$

measurement noise model

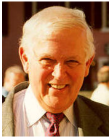
$$\begin{cases} x_m(t+1) &= \tilde{A}x_m(t) + \tilde{B}n_m(t) \\ m(t) &= \tilde{C}x_m(t) + \tilde{D}n_m(t) \end{cases}$$

- Note: the measurement noise model is not used for optimization, as we want z_m to go to its reference, not y_m

KALMAN FILTER DESIGN

- Plant model

$$\begin{cases} x(t+1) &= Ax(t) + B_u u(t) + B_v v(t) + B_d d(t) \\ y(t) &= Cx(t) + D_v v(t) + D_d d(t) \end{cases}$$



Rudolf Emil Kalman
(1930–2016)

- Full model for designing Kalman filter

$$\begin{aligned} \begin{bmatrix} x(t+1) \\ x_d(t+1) \\ x_m(t+1) \end{bmatrix} &= \begin{bmatrix} A & B_d\bar{C} & 0 \\ 0 & \bar{A} & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \\ x_m(t) \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} B_v \\ 0 \\ 0 \end{bmatrix} v(t) + \\ & \begin{bmatrix} B_d\bar{D} \\ \bar{B} \\ 0 \end{bmatrix} n_d(t) + \begin{bmatrix} 0 \\ 0 \\ \tilde{B} \end{bmatrix} n_m(t) + \begin{bmatrix} B_u \\ 0 \\ 0 \end{bmatrix} n_u(t) \\ y_m(t) &= \begin{bmatrix} C_m & D_{dm}\bar{C} & \tilde{C} \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \\ x_m(t) \end{bmatrix} + D_{vm} v(k) + \bar{D}_m n_d(t) + \tilde{D}_m n_m(t) \end{aligned}$$

- $n_d(k)$ = source of **modeling errors**
- $n_m(k)$ = source of **measurement noise**
- $n_u(k)$ = white noise on input u (added to compute the Kalman gain)

OBSERVER IMPLEMENTATION

- Measurement update

$$
\begin{aligned}
\hat{y}_m(t|t-1) &= C_m\hat{x}(t|t-1)\\
\hat{x}(t|t) &= \hat{x}(t|t-1) + M(y_m(t) - \hat{y}_m(t|t-1))
\end{aligned}
$$

- Time update

$$
\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + L(y_m(t) - \hat{y}_m(t|t-1))
$$

- Note that if $L = AM$ then $\hat{x}(t+1|t) = A\hat{x}(t|t) + Bu(t)$

- The **separation principle** holds (under certain assumptions)

  (Muske, Meadows, Rawlings, 1994)

I/O FEEDTHROUGH

- We always assumed **no feedthrough from u to measured y**

$$y_k = Cx_k + Du_k + D_v v_k + D_d d_k, \quad D_m = 0$$

- This avoids **static loops** between state observer, as $\hat{x}(t|t)$ depends on $u(t)$ via $\hat{y}_m(t|t-1)$, and MPC ($u(t)$ depends on $\hat{x}(t|t)$)

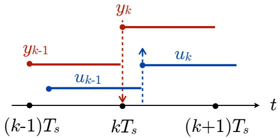
- Often $D = 0$ is not a limiting assumption as
  - often **actuator dynamics** must be considered (u is the set-point to a low-level controller of the actuators)
  - most physical models described by ordinary differential equations are **strictly causal**, and so is the discrete-time version of the model

- In case $D \neq 0$, we can assume a **delay** in executing the commanded u

$$y_k = C_m x_k + Du_{k-1}$$

and treat $u(t-1)$ as an extra state

- Not an issue for **unmeasured** outputs



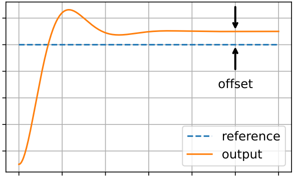
INTEGRAL ACTION IN MPC

CONSTANT DISTURBANCE MODEL

- **Tracking errors** in steady state can be due to **model mismatch** / **disturbances**

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{cases}$$

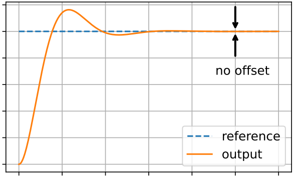
prediction model $\neq$ plant dynamics



- **Possible remedy**: introduce a **constant disturbance model**

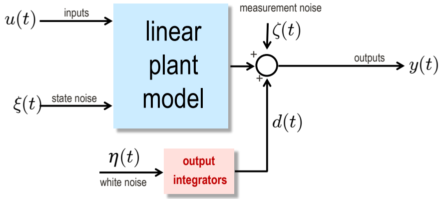
$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ d_{k+1} &= d_k \\ y_k &= Cx_k + d_k \end{cases}$$

augmented prediction model



and estimate x_k, d_k by a **state observer**

OUTPUT INTEGRATORS AND OFFSET-FREE TRACKING



$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \xi_k \\ d_{k+1} = d_k + \eta_k \\ y_k = Cx_k + d_k + \zeta_k \end{cases}$$

ξ_k, η_k, ζ_k are noise terms

- The state observer (e.g., a **Kalman filter**) estimates both $\hat{x}(t)$ and $\hat{d}(t)$ from $y(t)$
- Intuitively, we get offset-free tracking in steady-state because:
  - the observer makes $C\hat{x}(t) + \hat{d}(t) \to y(t)$ **(estimation error)**
  - the MPC controller makes $C\hat{x}(t) + \hat{d}(t) \to r(t)$ **(predicted tracking error)**
  - the combination of the two makes $y(t) \to r(t)$ **(actual tracking error)**
- In steady state, the term $\hat{d}(t)$ compensates for model mismatch

OUTPUT INTEGRATORS AND OFFSET-FREE TRACKING

- **More general result**: consider the disturbance model

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + B_d d_k \\ d_{k+1} = d_k \\ y_k = Cx_k + D_d d_k \end{cases}$$

special case: **output integrators**

$B_d = 0, D_d = I$

- **Theorem:** (Pannocchia, Rawlings, 2003)

  If **# disturbances = # measured outputs**, then all pairs (B_d, D_d) satisfying

$$\text{rank}\begin{bmatrix} I - A & -B_d \\ C & D_d \end{bmatrix} = n_x + n_y$$

  guarantee **zero offset in steady-state** $(y = r)$, provided that constraints are not active in steady-state and the closed-loop system is asymptotically stable

- See more on survey paper (Pannocchia, Gabiccini, Artoni, 2015)

DISTURBANCE MODEL EXAMPLE

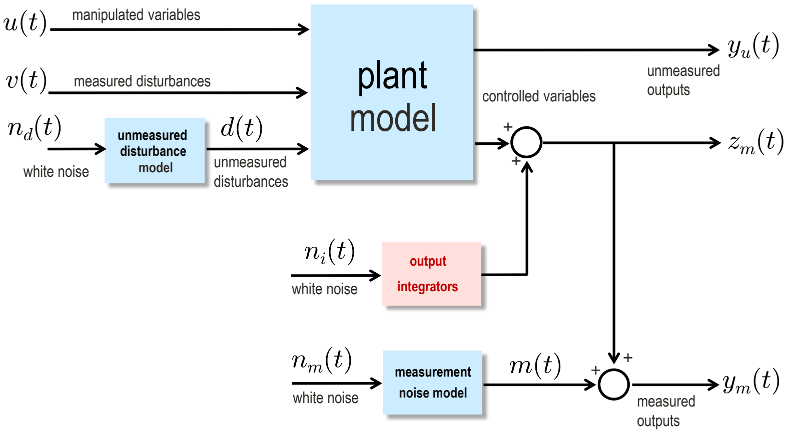
- Open-loop system: $A = \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & -2 & 3 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 2 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, C_m = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$

- We cannot add an **output integrator** since the pair $\left(\left[\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} C & 1 \end{smallmatrix}\right]\right)$ is not observable

- Add an **input integrator**: $B_d = B, D_d = 0, \text{rank} \left[\begin{smallmatrix} A & B_d \\ C_m & D_d \end{smallmatrix}\right] = 5 = n_x + n_y$

$$\begin{cases} x(t+1) & = & Ax(t) + B(u(t) + d(t)) \\ d(t+1) & = & d(t) \\ y(t) & = & Cx(t) + 0 \cdot d(t) \end{cases}$$

OVERALL OBSERVER MODEL FOR OFFSET-FREE MPC



(observer model structure used in the MPC Toolbox for MATLAB)

ERROR FEEDBACK

(Kwakernaak, Sivan, 1972)

- **Idea**: add the integral of the tracking error as an additional state (original idea developed for integral action in state-feedback control)

- Extended prediction model:

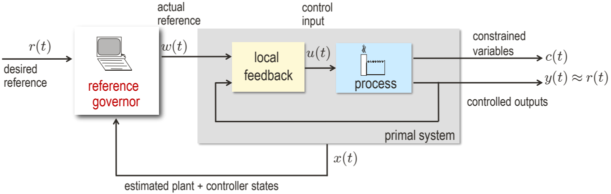
$$\left\{\begin{array}{rcll} x(t+1) & = & Ax(t)+B_u u(t)+0\cdot r(t) & \leftarrow \text{r(t) is seen as a meas. disturbance} \\ q(t+1) & = & q(t)+\underbrace{Cx(t)-r(t)}_{\text{tracking error}} & \leftarrow \text{integral action} \\ y(t) & = & Cx(t) & \end{array}\right.$$

- $\|W^i q\|_2^2$ is penalized in the cost function, otherwise it is useless. W^i is a new tuning knob

- Intuitively, if the MPC closed-loop is asymptotically stable then $q(t)$ converges to a constant, and hence $y(t)-r(t)$ converges to zero.

REFERENCE / COMMAND GOVERNOR

(Bemporad, Mosca, 1994) (Gilbert, Kolmanovsky, Tan, 1994) (Garone, Di Cairano, Kolmanovsky, 2016)



- MPC manipulates set-points to a linear feedback loop to enforce constraints
- Separation of problems:
  - Local feedback guarantees offset-free tracking $y(t) - w(t) \to 0$ in steady-state **in the absence of constraints**
  - Actual reference $w(t)$ generated by MPC to **take care of constraints**
- **Advantages**: **small-signal properties** preserved, **fewer variables** to optimize

$$\begin{aligned} w(t) = \arg\min_w \quad & \|w - r(t)\|_2^2 \\ \text{s.t.} \quad & c_{t+k} \in \mathcal{C} \end{aligned}$$

INTEGRAL ACTION AND Δu-FORMULATION

- In control systems, **integral action** occurs if the controller has a transfer-function from the output to the input of the form

$$u(t) = \frac{B(z)}{(z-1)A(z)}y(t), \qquad B(1) \neq 0$$

- One may think that the Δu-formulation of MPC provides integral action ...

  ... is it true ?

- **Example**: we want to regulate the output $y(t)$ to zero of the scalar system

$$\begin{array}{rcl} x(t+1) & = & \alpha x(t) + \beta u(t) \\ y(t) & = & x(t) \end{array}$$

INTEGRAL ACTION AND Δu-FORMULATION

- Design an unconstrained MPC controller with horizon $N = 1$

$$\begin{array}{rl} \Delta u(t) = & \arg\min_{\Delta u_0} \Delta u_0^2 + \rho y_1^2 \\ \text{s.t.} & u_0 = u(t-1) + \Delta u_0 \\ & y_1 = x_1 = \alpha x(t) + \beta(\Delta u_0 + u(t-1)) \end{array}$$

- By substitution, we get

$$\begin{array}{rcl} \Delta u(t) & = & \arg\min_{\Delta u_0} \Delta u_0^2 + \rho(\alpha x(t) + \beta u(t-1) + \beta \Delta u_0)^2 \\ & = & \arg\min_{\Delta u_0} (1+\rho\beta^2)\Delta u_0^2 + 2\beta\rho(\alpha x(t) + \beta u(t-1))\Delta u_0 \\ & = & -\frac{\beta\rho\alpha}{1+\rho\beta^2} x(t) - \frac{\rho\beta^2}{1+\rho\beta^2} u(t-1) \end{array}$$

- Since $x(t) = y(t)$ and $u(t) = u(t-1) + \Delta u(t)$ we get the linear controller

$$u(t) = -\frac{\frac{\rho\beta\alpha}{1+\rho\beta^2} z}{z - \frac{1}{1+\rho\beta^2}} y(t)$$

No pole in $z = 1$

- Reason: MPC gives a feedback gain on both $x(t)$ and $u(t-1)$, not just on $x(t)$

INTEGRAL ACTION AND Δu-FORMULATION

- Numerical test (with MPC Toolbox)

```
alpha=.5;
beta=3;
sys=ss(alpha,beta,1,0);sys.ts=1;
rho=1;p=1;m=1;
weights=struct('OV',rho,'MVRate',1);
mpc1=mpc(sys,1,p,m,weights);
setoutdist(mpc1,'remove',1); % no output disturbance model
mpc1tf=tf(mpc1);
sum(mpc1tf.den{1})

ans = 0.9000
```

- Now add an output integrator as disturbance model

```
setoutdist(mpc1,'integrators'); % add output disturbance model
mpc1tf=tf(mpc1);
sum(mpc1tf.den{1})

ans = -6.4185e-16
```


MPC AND INTERNAL MODEL PRINCIPLE

- Assume we have an **internal model** of the reference signal $r(t)$

$$\begin{array}{rcl} x^r(t+1) & = & A_r x^r(t) \\ r(t) & = & C_r x^r(t) \end{array}$$

- Consider the augmented prediction model

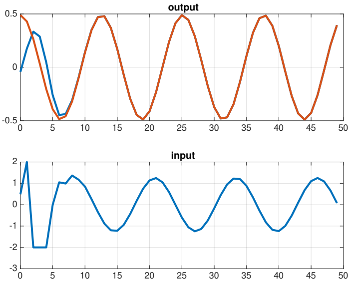
$$\begin{bmatrix} x_{k+1} \\ x^r_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x_k \\ x^r_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k$$

- Design a state observer (e.g., via pole-placement) to estimate $x(t), x^r(t)$ from $\begin{bmatrix} y(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C_r \end{bmatrix} \begin{bmatrix} x(t) \\ x^r(t) \end{bmatrix}$

- Design an MPC controller with output $e_k = y_k - r_k = \begin{bmatrix} C & -C_r \end{bmatrix} \begin{bmatrix} x_k \\ x^r_k \end{bmatrix}$

MPC AND INTERNAL MODEL PRINCIPLE

- Penalize $\sum_{k=1}^{N} e_k^2$ (+small penalty on u^2 or Δu^2)
- Set control horizon $N_u = N$



- **Example**: $r(t) = r_0 \sin(\omega t + \phi_0), \omega = 1$ rad/s

$$A = \begin{bmatrix} 1.6416 & -0.7668 & 0.2416 \\ 1 & 0 & 0 \\ 0 & 0.125 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.1349 & 0.0159 & -0.1933 \end{bmatrix}$$

$$A_r = \begin{bmatrix} 1.7552 & -1 \\ 1 & 0 \end{bmatrix}$$
$$C_r = \begin{bmatrix} .2448 & .2448 \end{bmatrix}$$

observer poles placed in $\{0.3, 0.4, 0.5, 0.6, 0.7\}$

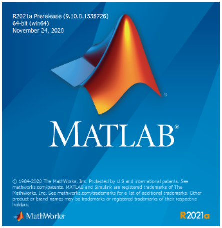
sample time $T_s = 0.5$ s

input saturation $-2 \leq u(t) \leq 2$

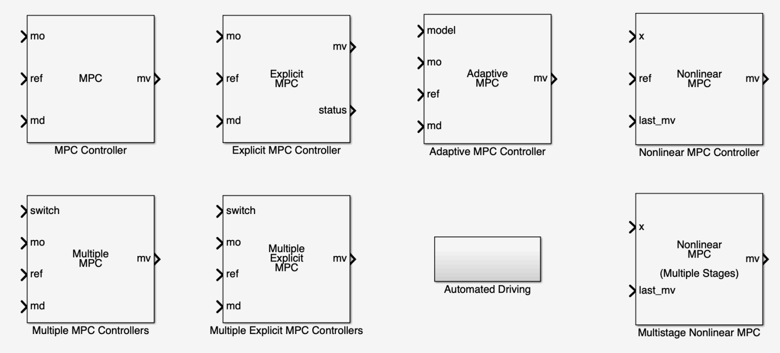
MODEL PREDICTIVE CONTROL TOOLBOX

(Bemporad, Ricker, Morari, 1998-present)

- Several MPC design features available:
  - explicit MPC
  - time-varying/adaptive models, nonlinear models, weights, constraints
  - stability/frequency analysis of closed-loop (inactive constraints)
  - ...
- Prediction models can be generated by the Identification Toolbox or automatically linearized from Simulink
- Fast command-line MPC functions (compiled EML-code)
- Graphical User Interface
- Simulink library (compiled EML-code)

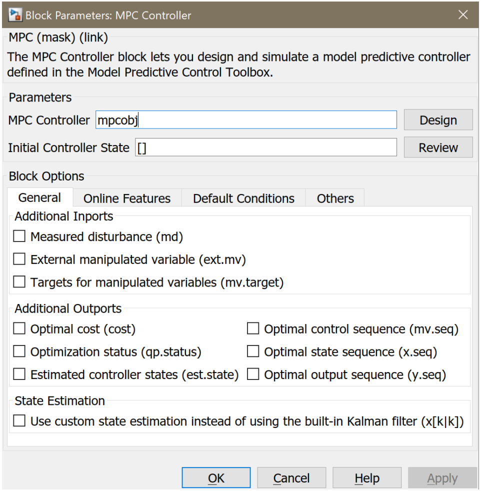


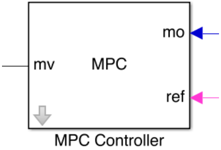
MPC SIMULINK LIBRARY



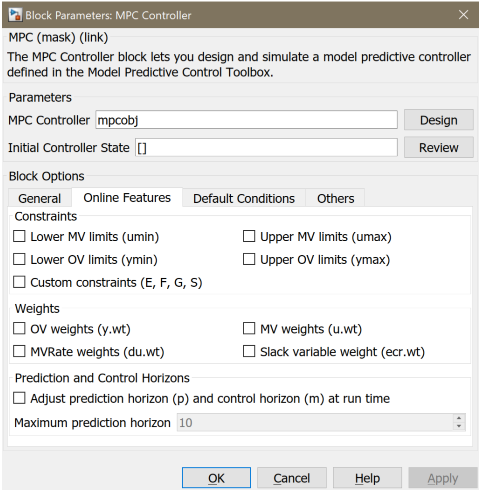

```
>> mpclib
```

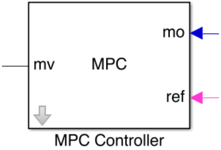
MPC SIMULINK BLOCK



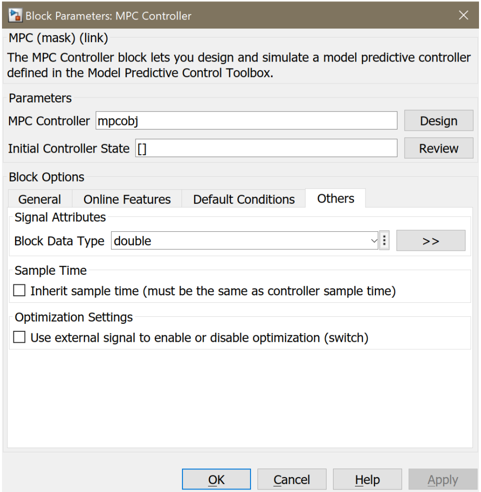


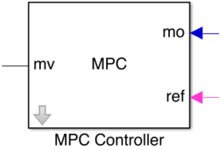
MPC SIMULINK BLOCK



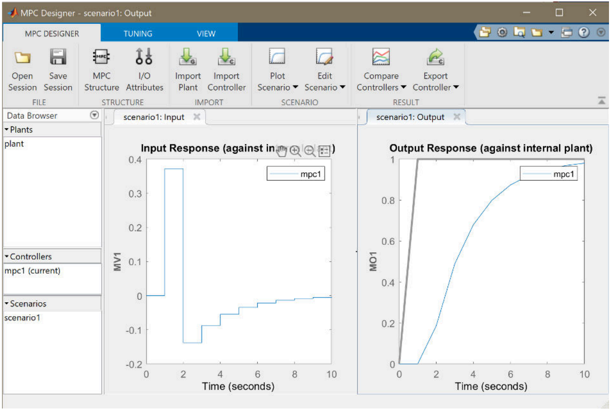


MPC SIMULINK BLOCK





MPC GRAPHICAL USER INTERFACE

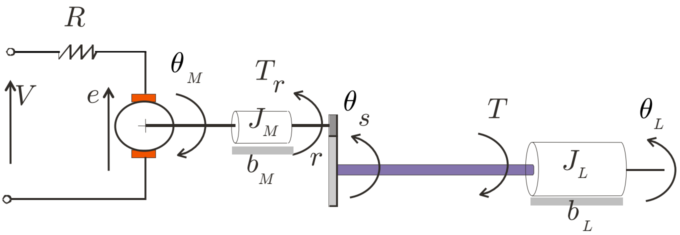


>> mpcDesigner

(old version: >> mpctool)

See video on Mathworks' web site (link)

EXAMPLE: MPC OF A DC SERVOMOTOR



Symbol	Value (MKS)	Meaning
L_S	1.0	shaft length
d_S	0.02	shaft diameter
J_S	negligible	shaft inertia
J_M	0.5	motor inertia
β_M	0.1	motor viscous friction coefficient
R	20	resistance of armature
k_T	10	motor constant
ρ	20	gear ratio
k_θ	1280.2	torsional rigidity
J_L	$50J_M$	nominal load inertia
β_L	25	load viscous friction coefficient
T_s	0.1	sampling time

```
>> openExample mpcmotor
```

see also

`linear/dcmotor.m`

(Hybrid Toolbox)

DC SERVOMOTOR MODEL

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_\theta}{J_L} & -\frac{\beta_L}{J_L} & \frac{k_\theta}{\rho J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_\theta}{\rho J_M} & 0 & -\frac{k_\theta}{\rho^2 J_M} & -\frac{\beta_M + k_T^2/R}{J_M} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_T}{R J_M} \end{bmatrix} V \\
\theta_L &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x \\
T &= \begin{bmatrix} k_\theta & 0 & -\frac{k_\theta}{\rho} & 0 \end{bmatrix} x
\end{aligned}
$$

$$x = \begin{bmatrix} \theta_L \\ \dot{\theta}_L \\ \theta_M \\ \dot{\theta}_M \end{bmatrix}$$

$$y = \begin{bmatrix} \theta_L \\ T \end{bmatrix}$$

```
>> [plant, tau] = mpcmotormodel;
>> plant = setmpcsignals(plant,'MV',1,'MO',1,'UO',2);
```

DC SERVOMOTOR CONSTRAINTS

- The input DC voltage V is bounded withing the range

$$|V| \leq 220\,V$$

- Finite shear strength $\tau_{adm} = 50N/mm^2$ requires that the torsional torque T satisfies the constraint

$$|T| \leq 78.5398\,Nm$$

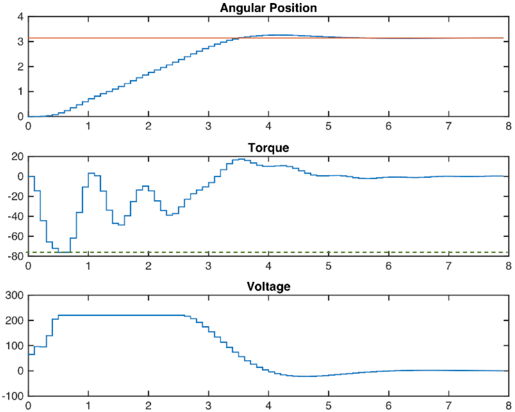
- Sampling time of model/controller: $T_s = 0.1s$

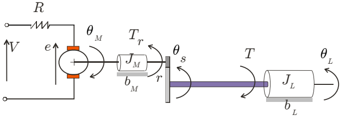
```
>> MV = struct('Min',-220,'Max',220);
>> OV = struct('Min',{-Inf,-78.5398},'Max',{Inf,78.5398});
>> Ts = 0.1;
```

DC SERVOMOTOR - MPC SETUP

$$
\begin{array}{ll}
\min\limits_{\Delta U} & \displaystyle\sum_{k=0}^{p-1} \|W^y(y_{k+1} - r(t))\|^2 + \|W^{\Delta u}\Delta u_k\|^2 + \rho_\epsilon \epsilon^2 \\
\text{subj. to} & \Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max},\ k = 0, \ldots, m-1 \\
& \Delta u_k = 0,\ k = m, \ldots, p-1 \\
& u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, m-1 \\
& y_{\min} - \epsilon V_{\min} \leq y_k \leq y_{\max} + \epsilon V_{\max},\ k = 1, \ldots, p
\end{array}
$$

```
>> Weights = struct('MV',0,'MVRate',0.1,'OV',[0.1 0]);
>> p = 10;
>> m = 2;
>> mpcobj = mpc(plant,Ts,p,m,Weights,MV,OV);
```

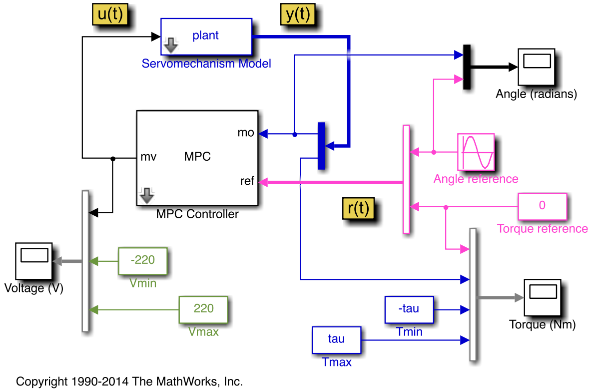




Closed-loop simulation using the `sim` command

```
>> Tstop = 8; % seconds
>> Tf = round(Tstop/Ts); % simulation iterations
>> r = [pi*ones(Tf,1) zeros(Tf,1)];
>> [y1,t1,u1] = sim(mpcobj,Tf,r);
```

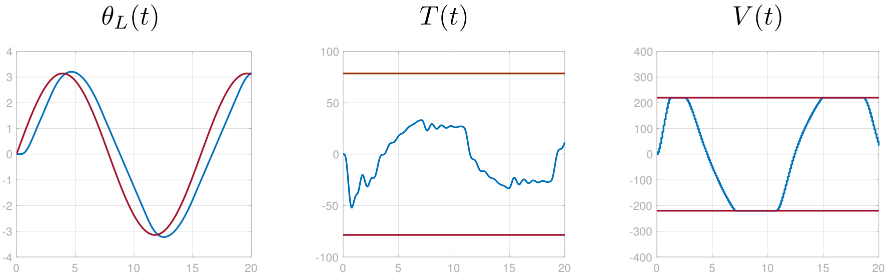

DC SERVOMOTOR - CLOSED-LOOP SIMULATION



closed-loop simulation in Simulink

```
>> mdl = 'mpc_motor';
>> open_system(mdl)
>> sim(mdl)
```

DC SERVOMOTOR - CLOSED-LOOP SIMULATION

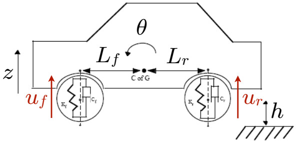


- same MPC tuning parameters

- setpoint $r(t) = \pi \sin(0.4t)$ deg

EXAMPLE: MPC OF ACTIVE SUSPENSIONS

- Take simulation model `sldemo_suspn` in MATLAB



$$F_{front} = 2K_f(L_f\theta - (z+h)) + 2C_f(L_f\dot{\theta} - \dot{z}) + u_f$$

$$\begin{aligned}
F_{front}, F_{rear} &= \text{upward force on body from front/rear suspension}\\
K_f, K_r &= \text{front and rear suspension spring constant}\\
C_f, C_r &= \text{front and rear suspension damping rate}\\
L_f, L_r &= \text{horizontal distance from c.g. to front/rear suspension}\\
\theta, \dot{\theta} &= \text{pitch (rotational) angle and its rate of change}\\
z, \dot{z} &= \text{bounce (vertical) distance and its rate of change}
\end{aligned}$$

- For control purposes we add external forces u_f, u_r as manipulated variables
- The system has 4 states: $\theta, \dot{\theta}, z, \dot{z}$
- Measured disturbances: road height h, pitch moment from vehicle acceleration

EXAMPLE: MPC OF ACTIVE SUSPENSIONS

- Step #1: get a linear discrete-time model (4 inputs, 3 outputs, 4 states)

```
>> plant_mdl = 'suspension_model';
>> op = operspec(plant_mdl);
>> [op_point, op_report] = findop(plant_mdl,op);
>> sys = linearize(plant_mdl, op_point);
>> Ts = 0.025; % sample time (s)
>> plant = c2d(sys,Ts);

>> plant = setmpcsignals(plant,'MV',[1 2],'MD',...
[3 4],'MO',[1 2],'UO',3);
```

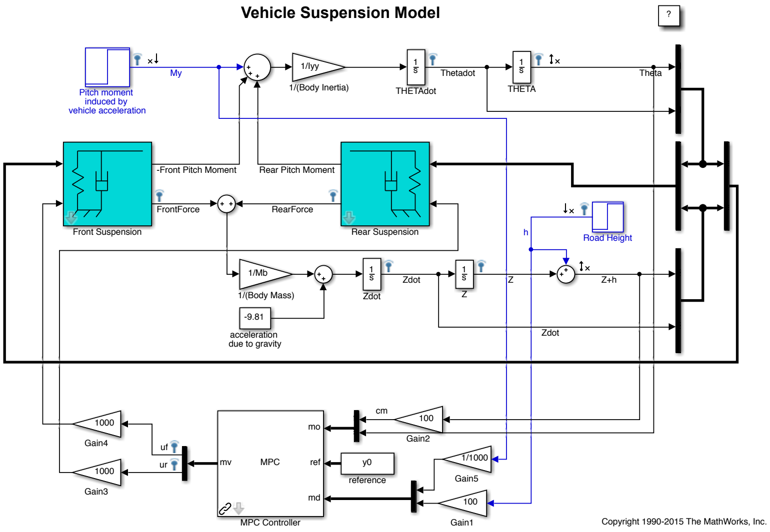
EXAMPLE: MPC OF ACTIVE SUSPENSIONS

- Step #2: design the MPC controller

```
>> dfmax = .1/.1; % [kN/s]
>> MV = struct('RateMin',-Ts*dfmax, Ts*dfmax,...
'RateMax',Ts*dfmax,Ts*dfmax);
>> OV = [];
>> Weights = struct('MV',[0 0], 'MVRate',[.01 .01],...
'OV',[.01 0 10]);

>> p = 50; % Prediction horizon
>> m = 5; % Control horizon

>> mpcobj = mpc(plant,Ts,p,m,Weights,MV,OV);
```



Vehicle Suspension Model
?
My
Pitch moment induced by vehicle acceleration
1/Iyy
1/(Body Inertia)
THETAdot
Thetadot
THETA
Theta
-Front Pitch Moment
Rear Pitch Moment
FrontForce
RearForce
Front Suspension
Rear Suspension
h
Road Height
1/Mb
1/(Body Mass)
-9.81
acceleration due to gravity
Zdot
Z
Z+h
1000
Gain4
1000
Gain3
uf
ur
mv
MPC
mo
ref
md
MPC Controller
cm
100
Gain2
y0
reference
1/1000
Gain5
100
Gain1
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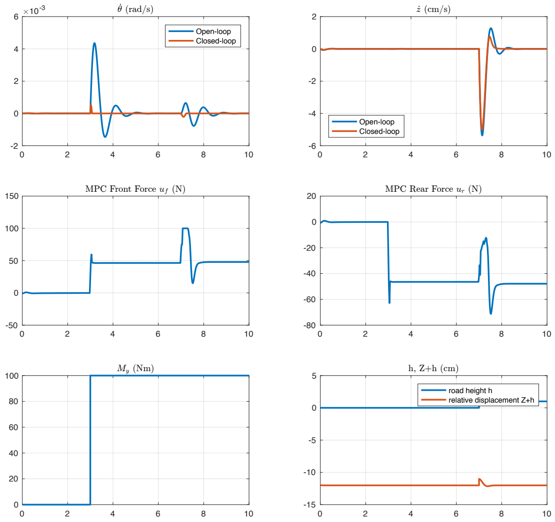
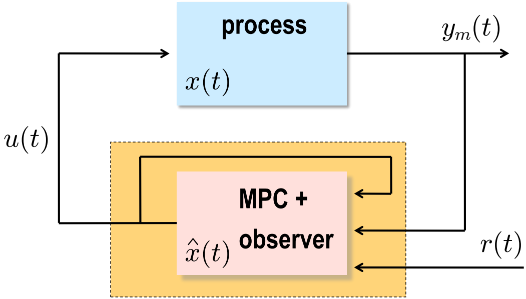
EXAMPLE: MPC OF ACTIVE SUSPENSIONS

- Closed-loop MPC results



FREQUENCY ANALYSIS OF MPC (FOR SMALL SIGNALS)

- **Unconstrained MPC** gain + linear observer = linear dynamical system
- Closed-loop MPC analysis can be performed using standard frequency-domain tools (e.g., Bode plots for sensitivity analysis)




```
>> sys=ss(mpcobj)
>> sys=tf(mpcobj)
```

returns the LTI object of the MPC controller (when constraints are inactive)

- Given a desired linear controller $u = K_d x$, find a set of weights Q, R, P defining an MPC problem such that

$$-\begin{bmatrix} I \; 0 \; \dots \; 0 \end{bmatrix} H^{-1} F = K_d$$

i.e., the **MPC law coincides with K_d** when the constraints are **inactive**

- Recall that the QP matrices are $H = 2(\bar{R} + \bar{S}'\bar{Q}\bar{S})$, $F = 2\bar{S}'\bar{Q}\bar{T}$, where

$$\bar{Q} = \begin{bmatrix} Q & 0 & 0 & \dots & 0 \\ 0 & Q & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & Q & 0 \\ 0 & 0 & \dots & 0 & P \end{bmatrix}, \; \bar{R} = \begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & R \end{bmatrix}, \; \bar{S} = \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}, \; \bar{T} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}$$

- The above **inverse optimality problem** can be cast to a convex problem

(Di Cairano, Bemporad, 2010)

- Result extended to match any linear controller/observer by LQR/Kalman filter

(Zanon, Bemporad, 2022)

CONTROLLER MATCHING - EXAMPLE

(Di Cairano, Bemporad, 2010)

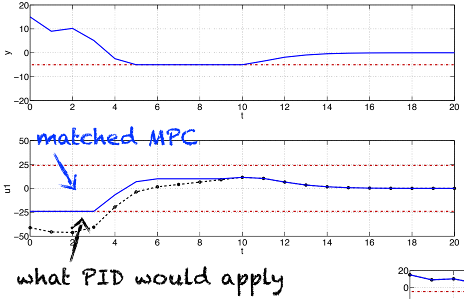
- Open-loop process: $y(t) = 1.8y(t-1) + 1.2y(t-2) + u(t-1)$
- Constraints: $-24 \leq u(t) \leq 24, y(t) \geq -5$
- Desired controller = PID with gains $K_I = 0.248, K_P = 0.752, K_D = 2.237$

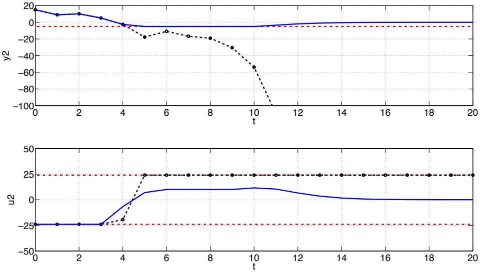
$$\begin{array}{rcl} u(t) & = & -\left(K_I\mathcal{I}(t) + K_P y(t) + \frac{K_D}{T_s}(y(t) - y(t-1))\right) \\ \mathcal{I}(t) & = & \mathcal{I}(t-1) + T_s y(t) \end{array} \qquad x(t) = \begin{bmatrix} y(t-1) \\ y(t-2) \\ \mathcal{I}(t-1) \\ u(t-1) \end{bmatrix}$$

- Matching result (using inverse LQR):

$$Q^* = \begin{bmatrix} 6.401 & 0.064 & -0.001 & 0.020 \\ 0.064 & 6.605 & 0.006 & 0.080 \\ -0.001 & 0.006 & 6.647 & -0.020 \\ 0.019 & 0.080 & -0.020 & 6.378 \end{bmatrix},\ R^* = 1,\ P^* = \begin{bmatrix} 422.7 & 241.7 & 50.39 & 201.4 \\ 241.7 & 151.0 & 32.13 & 120.4 \\ 50.39 & 32.13 & 19.85 & 26.75 \\ 201.4 & 120.4 & 26.75 & 106.6 \end{bmatrix}$$

CONTROLLER MATCHING - EXAMPLE





- **Note**: This is not trivially a saturation of a PID controller. In this case saturating the PID output leads to closed-loop instability!

LINEAR MPC BASED ON LINEAR PROGRAMMING

LINEAR MPC BASED ON LP

(Propoi, 1963) (Bemporad, Borrelli, Morari, 2003)

- Linear prediction model: $\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases}$ $\quad x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

- Constraints to enforce:

$$\begin{cases} u_{\min} \le u(t) \le u_{\max} \\ y_{\min} \le y(t) \le y_{\max} \end{cases}$$

- Constrained optimal control problem (∞-norms): $\|v\|_\infty \triangleq \max_{i=1,\ldots,n} |v_i|$

$$\begin{array}{ll} \min\limits_z & \|Px_N\|_\infty + \sum_{k=0}^{N-1} \|Qx_k\|_\infty + \|Ru_k\|_\infty \\ \text{s.t.} & u_{\min} \le u_k \le u_{\max},\ k = 0,\ldots,N-1 \\ & y_{\min} \le y_k \le y_{\max},\ k = 1,\ldots,N \end{array}$$

R, Q, P full rank $\quad z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$

LINEAR MPC BASED ON LP

- **Basic trick**: introduce slack variables (Q^i = i-th row of Q)

$$\epsilon_k^x \geq \|Qx_k\|_\infty \quad \Rightarrow \quad \begin{array}{lll} \epsilon_k^x & \geq & Q^i x_k \qquad i = 1, \ldots, \text{\#rows of } Q \\ \epsilon_k^x & \geq & -Q^i x_k \qquad k = 0, \ldots, N-1 \end{array}$$

$$\epsilon_k^u \geq \|Ru_k\|_\infty \quad \Rightarrow \quad \begin{array}{lll} \epsilon_k^u & \geq & R^i u_k \qquad i = 1, \ldots, \text{\#rows of } Q \\ \epsilon_k^u & \geq & -R^i u_k \qquad k = 0, \ldots, N-1 \end{array}$$

$$\epsilon_N^x \geq \|Px_k\|_\infty \quad \Rightarrow \quad \begin{array}{lll} \epsilon_N^x & \geq & P^i x_N \qquad i = 1, \ldots, \text{\#rows of } P \\ \epsilon_N^x & \geq & -P^i x_N \end{array}$$

and minimize $\epsilon_N^x + \sum_{k=0}^{N-1} \epsilon_k^x + \epsilon_k^u$ w.r.t. $u_0, \ldots, u_{N-1}, \epsilon_0^x, \ldots, \epsilon_N^x, \epsilon_0^u, \ldots, \epsilon_N^u$

(not that ϵ_0^x can be eliminated as x_0 is given)

- Example: $$\min_x |x| \;\rightarrow\; \begin{array}{ll} \min_{x,\epsilon} & \epsilon \\ \text{s.t.} & \epsilon \geq x,\ \epsilon \geq -x \end{array}$$

At optimility, $\epsilon^* = |x^*|$

LINEAR MPC BASED ON LP

- Linear prediction model: $x_k = A^k x_0 + \sum_{i=0}^{k-1} A^i B u_{k-1-i}$
- Optimization problem:

$$\begin{array}{rcll} V(x_0) = & \min\limits_z & [1 \;\ldots\; 1\, 0 \;\ldots\; 0]z & \text{(linear objective)} \\ & \text{s.t.} & Gz \leq W + Sx_0 & \text{(linear constraints)} \end{array}$$

Linear Program (LP)

- optimization vector: $z \triangleq [\epsilon_0^u \;\ldots\; \epsilon_{N-1}^u \; \epsilon_1^x \;\ldots\; \epsilon_N^x \; u_0', \ldots, u_{N-1}']' \in \mathbb{R}^{N(n_u+2)}$
- G, W, S are obtained from weights Q, R, P, and model matrices A, B, C
- Q, R, P can be selected to guarantee closed-loop stability

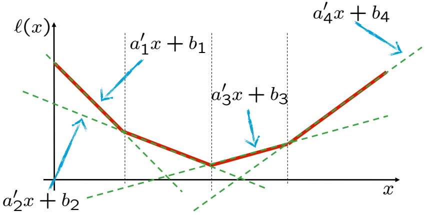
(Bemporad, Borrelli, Morari, 2003)

EXTENSION TO ARBITRARY CONVEX PWA FUNCTIONS

Result

Every convex piecewise affine function $\ell:\mathbb{R}^n\to\mathbb{R}$ can be represented as the max of affine functions, and vice versa

(Schechter, 1987)



$$\ell(x)=\max\left\{a_1'x+b_1,\ldots,a_4'x+b_4\right\}$$

Example: $\ell(x)=|x|=\max\{x,-x\}$

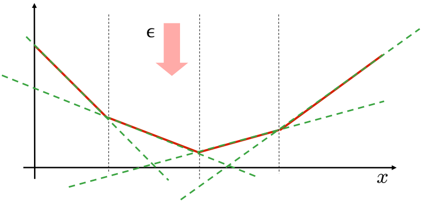
- Constrained optimal control problem

$$\begin{array}{ll}\min\limits_{U} & \ell_N(x_N)+\sum\limits_{k=0}^{N-1}\ell_k(x_k,u_k)\\[2ex] \text{s.t.} & g_k(x_k,u_k)\leq 0,\ k=0,\ldots,N-1\\ & g_N(x_N)\leq 0\end{array}$$

ℓ_k, ℓ_N, g_k, g_N are arbitrary convex piecewise affine (PWA) functions

CONVEX PWA OPTIMIZATION PROBLEMS AND LP

- Minimization of a **convex PWA function** $\ell(x)$:



$$\min_{\epsilon,x} \quad \epsilon$$
$$\text{s.t.} \quad \begin{cases} \epsilon \geq a_1'x + b_1 \\ \epsilon \geq a_2'x + b_2 \\ \epsilon \geq a_3'x + b_3 \\ \epsilon \geq a_4'x + b_4 \end{cases}$$

- By construction $\epsilon \geq \max\{a_1'x + b_1, a_2'x + b_2, a_3'x + b_3, a_4'x + b_4\}$

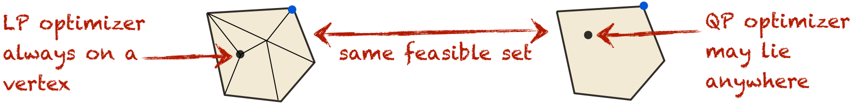
- By contradiction it is easy to show that at the optimum we have that

$$\epsilon = \max\{a_1'x + b_1, a_2'x + b_2, a_3'x + b_3, a_4'x + b_4\}$$

- **Convex PWA constraints** $\ell(x) \leq 0$ can be handled similarly by imposing $a_i'x + b_i \leq 0, \forall i = 1, 2, 3, 4$

LP-BASED VS QP-BASED MPC

- QP- and LP-based share the same set of feasible inputs ($Gz \leq W + Sx$)
- When constraints dominate over performance there is little difference between them (e.g., during transients)
- Small-signal response, however, is usually less smooth with LP than with QP, because in LP an optimal point is always on a vertex



$$\left\{ z : \begin{array}{l} Gz \leq W + Sx \\ \bar{G}z + \bar{E}\epsilon \leq \bar{W} + \bar{S}x \end{array} \right\} \qquad \{z : \ Gz \leq W + Sx\}$$

ϵ = additional slack variables introduced to represent convex PWA costs