IDENTIFICATION, ANALYSIS AND CONTROL OF DYNAMICAL SYSTEMS

PART 2: SYSTEMS IDENTIFICATION

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IMT SCHOOL FOR ADVANCED STUDIES LUCCA

System identification: introduction

Building mathematical models

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- Different kinds of models:
  - mental or intuitive models. For example:
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  - graphical models. For example:
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- We will focus on mathematical models of dynamical systems, described, in general, by differential or difference equations.
- Mathematical models can be derived from:
  - first principle laws of physics, chemistry, biology, etc.
    (physical modeling approach)
  - observed data generated by the system
    (system identification approach)

System identification procedure

The system identification procedure involves three basic entities:

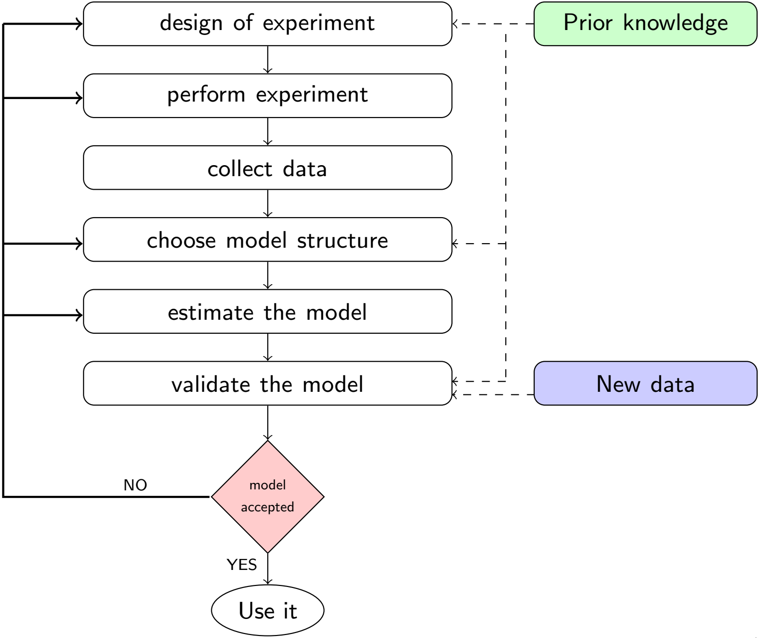
1. Data, which can be either recorded from specifically designed experiments or from normal operations of the system.
2. Set of candidate models, obtained by specifying within which set of models we are going to look for a suitable one. Different kinds of models may be specified (e.g., linear vs nonlinear; continuous time vs discrete time; deterministic vs stochastic, etc.). Two types of model sets:
   - gray boxes. A model with some unknown parameters is derived from physical laws. The parameters are then estimated from data.
   - black boxes. A model structure is chosen (e.g., linear models). The parameters of the model do not reflect any physical consideration.
3. Rule to assess candidate models using data. This is the identification method, used to determine the "best" model in the set, guided by data.

Model validation

Test whether the estimated model is an “appropriate” representation of the system. Assess how the model relates to:

- prior knowledge. Does the model adequately describes prior known physical behaviour of the system?
- experimental data (not used for training). Compare the simulated outputs of the model with the observed outputs.

System identification loop



References and educational material

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- T. Söderstrom and P. Stoica, *System identification*, Prentice Hall International, 1989. Available online at: `http://user.it.uu.se/~ps/ps.html`
- Parametric System Identification - theory and tools, R. De Callafon, University of California San Diego, `http://mechatronics.ucsd.edu/mae283a_10/index.html`
- IEEE CSS Technical Commettee on *System Identification and Adaptive Control*, `http://system-identification.ieeecss.org`

LTI systems

Input/Output representation

Given a discrete-time signal $u(k)$, $k = 0, 1, \ldots$, we define the (unilater) z-transform of u as

$$\mathcal{Z}\{u(k)\} = U(z) = \sum_{k=0}^{\infty} u(k) z^{-k}$$

- $\mathcal{Z}\{u(k-d)\} = \mathcal{Z}\{q^{-d}u(k)\} = z^{-d}U(z), \quad d \in \mathbb{Z}$
- $\mathcal{Z}\{g(k) * u(k)\} = \mathcal{Z}\left\{\sum_{\ell=0}^{\infty} g(\ell)u(k-\ell)\right\} = \mathcal{Z}\{G(q)u(k)\} = G(z)U(z)$

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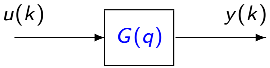

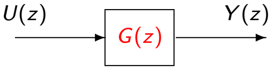

Analogy between the time-domain operator $G(q)$ and the DT transfer function $G(z)$

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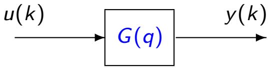

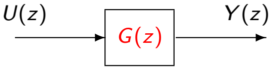

Analogy between the time-domain operator $G(q)$ and the DT transfer function $G(z)$

Thanks to this analogy, we can treat $G(q)$ as polynomials in q. Product and ratio between $G_1(q)$ and $G_2(q)$ have a meaning!

Example: $y(k) = \frac{b_1 q^{-1}}{1+a_1 q^{-1}} u(k) \rightarrow (1 + a_1 q^{-1})y(k) = b_1 q^{-1} u(k)$

Linear regression representation

- Linear regression representation of the system:

$$y(k) = \varphi^\top(k)\theta$$

θ: parameter vector, $\varphi(k)$: regressor vector, typically containing past values of inputs and outputs.

$$\varphi(k) = [-y(k-1) \quad \ldots \quad -y(k-n_\mathrm{a}) \quad u(k) \quad \ldots \quad u(k-n_\mathrm{b})]^\top$$
$$\theta = [a_1 \quad \ldots \quad a_{n_\mathrm{a}} \quad b_0 \quad \ldots \quad b_{n_\mathrm{b}}]^\top$$

Writing out the product gives:

$$y(k) = G(q)u(k), \qquad G(q) = \frac{b_0 + b_1 q^{-1} + \cdots + b_{n_\mathrm{b}} q^{-n_\mathrm{b}}}{1 + a_1 q^{-1} + \cdots + a_{n_\mathrm{a}} q^{-n_\mathrm{a}}}$$

Non-linear systems can be easily represented in a linear regression form.
Just include nonlinear terms (e.g., $y^2(k-1)$; $u(k)y(k-1)$) in the regressor!

Least-squares estimation

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- Least-squares (LS) estimate:

$$\hat{\theta}_{\mathrm{LS}} = \arg\min_{\theta} \sum_{k=1}^{N} \varepsilon^2(k, \theta) = \arg\min_{\theta} \sum_{k=1}^{N} \left(y(k) - \varphi^\top(k)\theta \right)^2$$

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Linear least-squares: Cholesky factorization

$$\hat{\theta}_{\mathrm{LS}} = \left(\Phi^\top \Phi\right)^{-1} \Phi^\top Y$$

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$$\Phi^\top \Phi = L L^\top \qquad L : \text{Lower Triangular Matrix}$$

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$$Lz = \Phi^\top Y, \qquad L^\top \hat{\theta}_{\mathrm{LS}} = z$$

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Linear least-squares: QR factorization

$$\hat{\theta}_{\mathrm{LS}} = \left(\Phi^\top \Phi\right)^{-1} \Phi^\top Y$$

- Compute a QR factorization of the (full-column rank) matrix $\Phi \in \mathbb{R}^{N,n}$, i.e.,

$$\Phi = \underbrace{\left[[Q_1]_{N,n} \quad [Q_2]_{N,(N-n)}\right]}_{Q} \underbrace{\begin{bmatrix} [R_1]_{n,n} \\ 0_{N-n,n} \end{bmatrix}}_{R}$$

with $Q_1^\top Q_1 = I$, R_1 upper triangular, $r_{ii} > 0$ if Φ is full-column rank.

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- Solve the following linear system through backward substitution to compute $\hat{\theta}_{\rm LS}$:

$$R_1 \hat{\theta}_{\rm LS} = Q_1^\top Y$$

Recursive linear least squares

- Estimate the parameters $\hat{\theta}_{\mathrm{LS}}$ recursively in time.
- If there is an estimate $\hat{\theta}_{\mathrm{LS}}(k-1)$ based on data up to time $k-1$, then $\hat{\theta}_{\mathrm{LS}}(k)$ is computed based on a "simple" update of $\hat{\theta}_{\mathrm{LS}}(k-1)$.
- No need to record all data up to time k (low memory requirement).
- Recursive LS can be easily modified to estimate time-varying parameters.

Recursive linear least squares

$$\hat{\theta}_{\mathrm{LS}}(k) = \left(\sum_{\ell=1}^{k} \varphi(\ell)\varphi^\top(\ell)\right)^{-1} \sum_{\ell=1}^{k} \varphi(\ell) y(\ell)$$

- $P(k) = \left(\sum_{\ell=1}^{k} \varphi(\ell)\varphi^\top(\ell)\right)^{-1}, \quad P^{-1}(k) = P^{-1}(k-1) + \varphi(k)\varphi^\top(k)$
- $\hat{\theta}_{\mathrm{LS}}(k) = P(k)\left(\sum_{\ell=1}^{k-1} \varphi(\ell)y(\ell) + \varphi(k)y(k)\right)$
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- $\hat{\theta}_{\mathrm{LS}}(k) = \hat{\theta}_{\mathrm{LS}}(k-1) + K(k)\varepsilon(k)$

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- $\hat{\theta}_{\mathrm{LS}}(k) = \hat{\theta}_{\mathrm{LS}}(k-1) + K(k)\varepsilon(k)$

$K(k)$: gain
$\varepsilon(k)$: error in the prediction of $y(k)$ based on $\hat{\theta}_{\mathrm{LS}}(k-1)$

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If the prediction error is "small", the estimate $\hat{\theta}_{\mathrm{LS}}(k-1)$ is "good" and should not be modified "very much"

- $\hat{\theta}_{\mathrm{LS}}(k) = \hat{\theta}_{\mathrm{LS}}(k-1) + \underbrace{P(k)\varphi(k)}_{K(k)} \Big(\underbrace{y(k) - \varphi^\top(k)\hat{\theta}_{\mathrm{LS}}(k-1)}_{\varepsilon(k)} \Big)$
- $\hat{\theta}_{\mathrm{LS}}(k) = \hat{\theta}_{\mathrm{LS}}(k-1) + K(k)\varepsilon(k)$

$K(k)$: gain
$\varepsilon(k)$: error in the prediction of $y(k)$ based on $\hat{\theta}_{\mathrm{LS}}(k-1)$

Recursive linear least squares

$$\hat{\theta}_{\mathrm{LS}}(k) = \hat{\theta}_{\mathrm{LS}}(k-1) + P(k)\varphi(k)\left(y(k) - \varphi^\top(k)\hat{\theta}_{\mathrm{LS}}(k-1)\right)$$
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Solution

$$P(k) = \left[P^{-1}(k)\right]^{-1} = \left[P^{-1}(k-1) + \varphi(k)\varphi^\top(k)\right]^{-1}$$

From Matrix Inversion Lemma:

$$P(k) = P(k-1) - \frac{P(k-1)\varphi(k)\varphi^\top(k)P(k-1)}{1 + \varphi^\top(k)P(k-1)\varphi(k)}$$

Recursive linear LS for real-time identification

- Identify (slowly) time-varying parameters (due to slow time-variation of the process)
- Useful for adaptive control
- Introduce forgetting factor $0 < \lambda \leq 1$ in the cost function:

$$\hat{\theta}(k) = \arg\min_{\theta} \sum_{\ell=1}^{k} \lambda^{k-\ell} \left(y(\ell) - \varphi^{\top}(\ell)\theta \right)^2$$

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Recursive LS with forgetting factor

$$\hat{\theta}(k) = \hat{\theta}(k-1) + P(k)\varphi(k)\left(y(k) - \varphi^\top(k)\hat{\theta}_{\rm LS}(k-1) \right)$$

$$P(k) = \frac{1}{\lambda}\left[P(k-1) - \frac{P(k-1)\varphi(k)\varphi^\top(k)P(k-1)}{\lambda + \varphi^\top(k)P(k-1)\varphi(k)} \right]$$

Estimate of FIR models through LS

- Use a Finite Impulse Response (FIR) model to describe the dynamical system $\mathcal{S}$ to be identified:

$$\mathcal{M} : \hat{y}(k, g) = \sum_{\ell=0}^{M} g(\ell) u(k - \ell)$$

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- $[R(N)]_{ij} = \left[\frac{1}{N} \sum_{k=1}^{N} \varphi(k) \varphi^{\top}(k) \right]_{ij} = \frac{1}{N} \sum_{k=1}^{N} u(k-i+1) u(k-j+1)$

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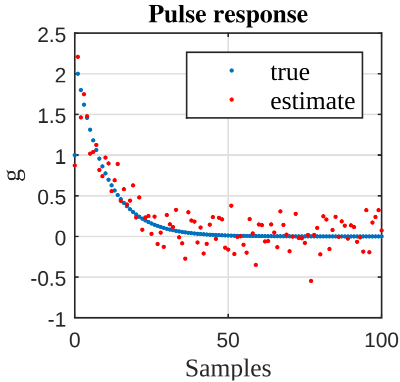
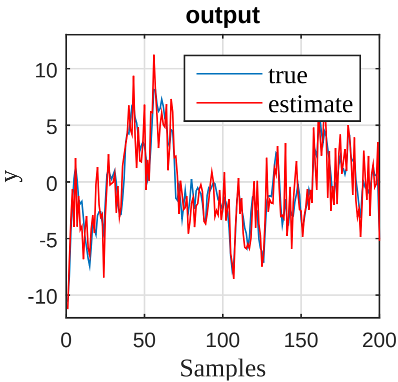
- $\lim_{N\to\infty} \hat{g} = g_o + R^* \overline{\mathbb{E}}\left[\varphi(k) v_o(k)\right] = g_o$
- Thus, $\hat{g}$ is a consistent estimate of g_o

Estimate of FIR models through LS: example

$N = 300$

$$SNR = 10 \log_{10} \left(\frac{\sum_{k=1}^{N} y^2(k)}{\sum_{k=1}^{N} v_o^2(k)} \right) = 6 \text{ db}$$



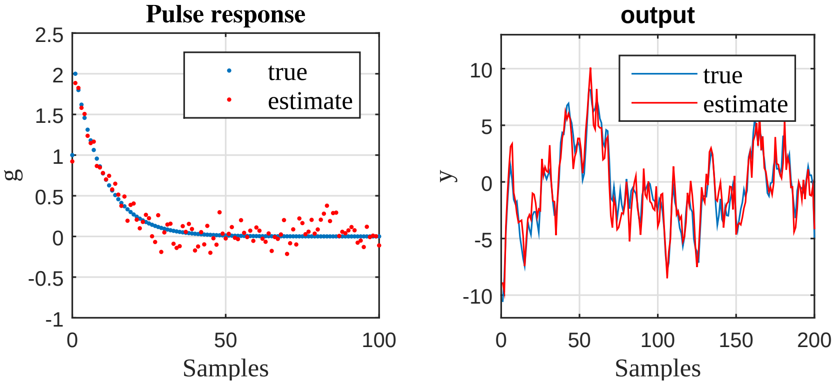


$$BFR = 1 - \frac{\sum_{k=1}^{N_v} (y(k) - \hat{y}(k))^2}{\sum_{k=1}^{N_v} (y(k) - \bar{y})^2} = 72\%$$

Estimate of FIR models through LS: example

$N = 500$

$$SNR = 10 \log_{10} \left(\frac{\sum_{k=1}^{N} y^2(k)}{\sum_{k=1}^{N} v_o^2(k)} \right) = 6 \text{ db}$$

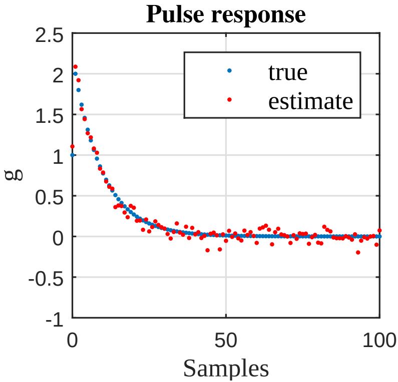


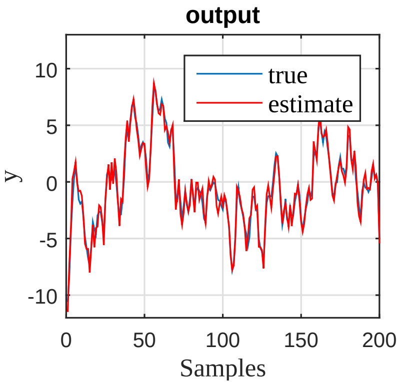
$$BFR = 1 - \frac{\sum_{k=1}^{N_v} (y(k) - \hat{y}(k))^2}{\sum_{k=1}^{N_v} (y(k) - \bar{y})^2} = 91\%$$

Estimate of FIR models through LS: example

$N = 1000$

$$SNR = 10 \log_{10} \left(\frac{\sum_{k=1}^{N} y^2(k)}{\sum_{k=1}^{N} v_o^2(k)} \right) = 6 \text{ db}$$



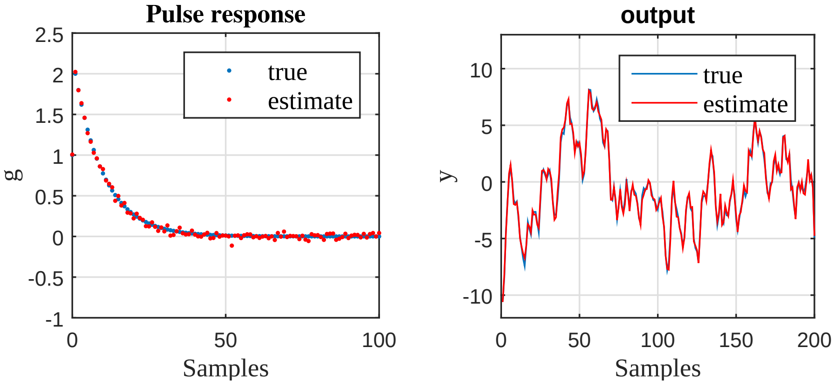


$$BFR = 1 - \frac{\sum_{k=1}^{N_v} (y(k) - \hat{y}(k))^2}{\sum_{k=1}^{N_v} (y(k) - \bar{y})^2} = 96\%$$

Estimate of FIR models through LS: example

$N = 5000$

$$SNR = 10\log_{10}\left(\frac{\sum_{k=1}^{N} y^2(k)}{\sum_{k=1}^{N} v_o^2(k)}\right) = 6 \text{ db}$$

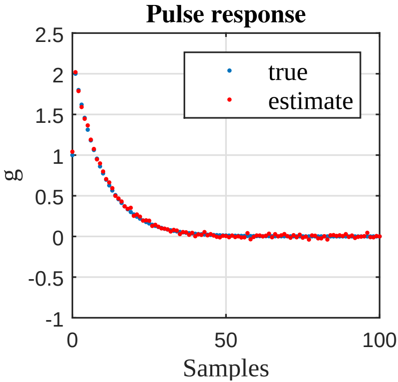


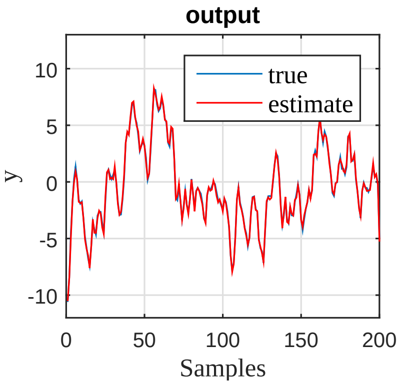
$$BFR = 1 - \frac{\sum_{k=1}^{N_v} (y(k) - \hat{y}(k))^2}{\sum_{k=1}^{N_v} (y(k) - \bar{y})^2} = 99\%$$

Estimate of FIR models through LS: example

$N = 10000$

$$SNR = 10\log_{10}\left(\frac{\sum_{k=1}^{N} y^2(k)}{\sum_{k=1}^{N} v_o^2(k)}\right) = 6 \text{ db}$$

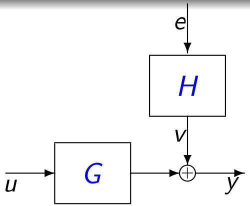




$$BFR = 1 - \frac{\sum_{k=1}^{N_v} (y(k) - \hat{y}(k))^2}{\sum_{k=1}^{N_v} (y(k) - \bar{y})^2} = 99.7\%$$

Estimate of ARX models through LS

$$y(k) = G(q)u(k) + H(q)e(k)$$



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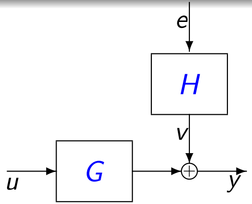

- AutoRegressive with eXogenous input (ARX) model structure:

$$G(q) = \frac{B(q)}{A(q)} = \frac{b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

$$H(q) = \frac{1}{A(q)} = \frac{1}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

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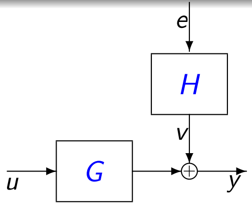

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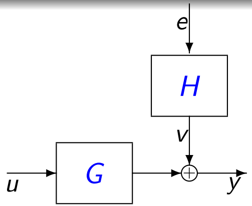
$$H(q) = \frac{1}{A(q)} = \frac{1}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

- Corresponding input/output relationship

$$y(k) = -a_1 y(k-1) - \cdots - a_{n_a} y(k-n_a) + b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) + e(k)$$

Estimate of ARX models through LS

$$y(k) = G(q)u(k) + H(q)e(k)$$



- AutoRegressive with eXogenous input (ARX) model structure:

$$G(q) = \frac{B(q)}{A(q)} = \frac{b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$
$$H(q) = \frac{1}{A(q)} = \frac{1}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

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- Output representation: $y(k) = \varphi^\top(k)\theta + e(k)$

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Does $\lim_{N\to\infty} \hat{\theta}_{\mathrm{LS}} = \theta_{\mathrm{o}}$?

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- $\lim_{N\to\infty} \hat{\theta}_{\mathrm{LS}} = \theta_{\mathrm{o}} + R^* \overline{\mathbb{E}}\left[\varphi(k)e(k)\right] = \theta_{\mathrm{o}}$
- Thus, $\hat{\theta}_{\mathrm{LS}}$ is a consistent estimate of θ_{o}

Estimate of LPV-ARX models through LS

Linear Parameter-Varying (LPV) systems

- Linear relationship between inputs and outputs:

$$y(k) = G(q^{-1}, p(k))u(k) + v(k)$$

- The input/output relationship changes over time according to a measurable signal p (called scheduling signal)

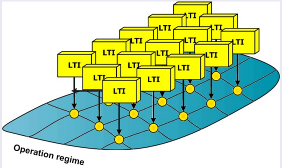

Figure provided by R. Tóth

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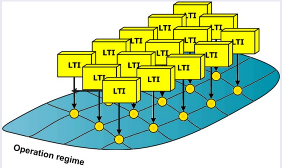

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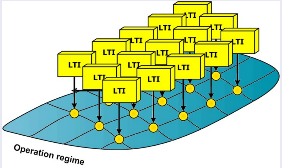

Figure provided by R. Tóth

LPV-ARX models

- $A(q^{-1}, p(k))y(k) = B(q^{-1}, p(k))u(k) + e(k), \quad e$ white

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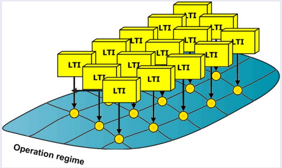

Figure provided by R. Tóth

LPV-ARX models

- $A(q^{-1}, p(k))y(k) = B(q^{-1}, p(k))u(k) + e(k), \quad e$ white
- $A(q^{-1}, p(k)) = 1 + \sum_{i=1}^{n_a} a_i(p(k))q^{-i}, \quad B(q^{-1}, p(k)) = \sum_{j=1}^{n_b} b_j(p(k))q^{-j}$

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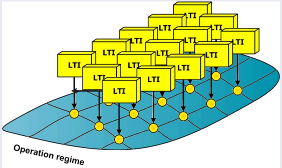

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- $a_i(p(k))$ and $b_j(p(k))$ are a-priori parametrized functions of $p(k)$ (e.g., polynomials):

$$a_i(p(k)) = a_{i,0} + \sum_{l=1}^{n_l} a_{i,l}p^l(k), \quad b_j(p(k)) = b_{j,0} + \sum_{l=1}^{n_l} b_{j,l}p^l(k)$$

Estimate of LPV-ARX models through LS

LPV-ARX models: example

- $A(q^{-1}, p(k))y(k) = B(q^{-1}, p(k))u(k) + e(k), \quad e \text{ white}$
- Example:

$$y(k) = -\left[a_{1,0} + a_{1,1}p(k)\right] y(k-1) + \left[b_{1,0} + b_{1,1}p(k)\right] u(k-1) + e(k)$$

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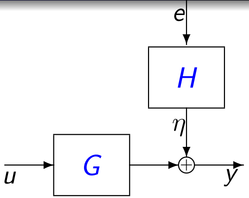
- Estimate θ through least-squares
- Consistency is guaranteed if e is white

Inconsistency of LS: OE case

$$y(k) = G(q)u(k) + H(q)e(k)$$



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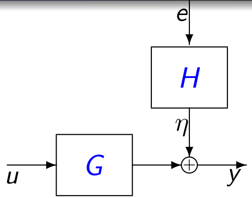

- Output Error (OE) model structure:

$$G(q) = \frac{B(q)}{A(q)} = \frac{b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

$$H(q) = 1$$

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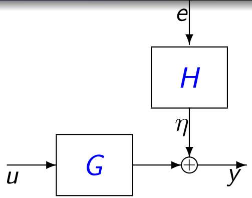

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Inconsistency of LS: OE case

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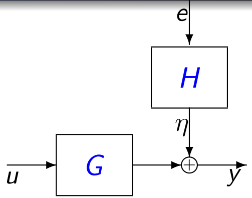

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- Output representation: $y(k) = \varphi^\top(k)\theta + \underbrace{e(k) + a_1 e(k-1) + \cdots + a_{n_a} e(k-n_a)}_{v(k)}$

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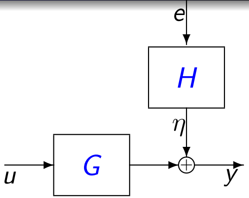

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$v(k)$ is not white!

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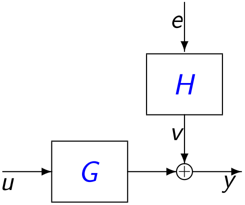
Instrumental Variable Methods

Instrumental Variables (IV)

$$y(k) = G(q)u(k) + H(q)e(k)$$



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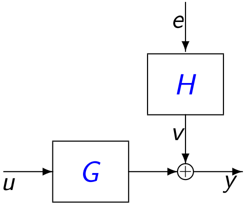

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- Instrumental Variables (IV) methods provide solutions to guarantee consistency also when $\{v(k)\}$ is correlated with the regressor

Instrumental Variables (IV): main idea

$$\hat{\theta}_{\mathrm{LS}} = \left(\Phi^\top \Phi\right)^{-1} \Phi^\top Y = \left(\sum_{k=1}^{N} \varphi(k)\varphi^\top(k)\right)^{-1} \sum_{k=1}^{N} \varphi(k) y(k)$$

Instrumental Variables Estimate

- Chose a vector $z(k)$, called instrument, with the same dimension of the regressor $\varphi(k)$ and such that

$$\mathbb{E}\left[z(k)v(k)\right] = 0 \quad \text{(i.e., } z(k) \text{ is not correlated with } v(k)\text{)}$$

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$$\left(Z^\top \Phi\right) \hat{\theta}_{\rm IV} = Z^\top Y \rightarrow R\hat{\theta}_{\rm IV} = Q^\top Z^\top Y, \quad [Q, R] = qr(Z^\top \Phi)$$

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3. Construct the instrument $z(k)$ as follows
$$z(k) = [-\hat{y}(k-1) \ \cdots \ -\hat{y}(k-n_{\rm a}) \ u(k-1) \ \cdots u(k-n_{\rm b})]^\top$$
4. Estimate the model parameters $\hat{\theta}$ through IV (consistent estimate)

Instrumental Variables: how to choose the instruments?

- The instrument $z(k)$ should be such that $\mathbb{E}[z(k)v(k)] = 0$
- The instrument $z(k)$ should be correlated with $\varphi(k)$
- In order to minimize the variance of the estimate, we would like to choose:
$$z(k) = [-y_{\rm o}(k-1) \ \cdots \ -y_{\rm o}(k-n_{\rm a}) \ u(k-1) \ \cdots u(k-n_{\rm b})]^\top$$
- Summarizing, $z(k)$ should be as much as possible correlated with the noise-free regressor, but not correlated with the residual $v(k)$.

Choice of the instrument: idea

1. Estimate the model parameters $\hat{\theta}$ through LS (biased estimate!)
2. Perform an open-loop simulation of the estimated model:
$$\hat{y}(k) = [-\hat{y}(k-1) \ \cdots \ -\hat{y}(k-n_{\rm a}) \ u(k-1) \ \cdots u(k-n_{\rm b})]^\top \hat{\theta}$$
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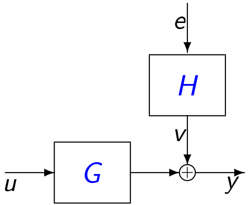
4. Estimate the model parameters $\hat{\theta}$ through IV (consistent estimate)
5. Repeat from step 2 until convergence

Prediction Error Methods

Description of Prediction Error Methods

$$y(k) = G(q,\theta)u(k) + H(q,\theta)e(k)$$
$$\mathbb{E}\left[e(t)e^{\top}(s)\right] = \Lambda_e \delta(s-t) \text{ (i.e., } e \text{ is white)}$$
$$G(0,\theta) = 0, \quad H(0,\theta) = I, \quad H^{-1}(q,\theta) \text{ is stable}$$



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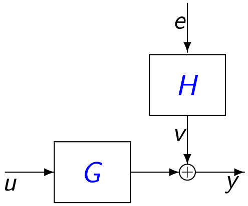

Linear predictor

$$\hat{y}(k|k-1;\theta) = L_y(q^{-1},\theta)y(k) + L_u(q^{-1},\theta)u(k)$$
$$L_y(0,\theta) = 0, \quad L_u(0,\theta) = 0$$

$\hat{y}(k|k-1;\theta)$ only depends on past input/output data

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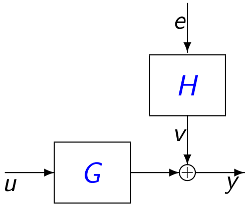

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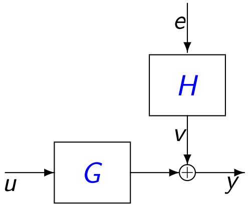

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$$\varepsilon(k,\theta) = y(k) - \hat{y}(k|k-1,\theta)$$

The estimated parameters $\hat{\theta}$ should make the prediction errors $\{\varepsilon(k,\theta)\}_{k=1}^{N}$ “small”

Description of Prediction Error Methods

Prediction Error Methods

- Choice of model structure (parametrization of $G(q, \theta)$ and $H(q, \theta)$ as a function of θ)
- Choice of predictor (define filters of $L_y(q^{-1}, \theta)$ and $L_u(q^{-1}, \theta)$)
- Choice of criterion $V_N(\theta)$ (scalar function of the prediction errors $\{\varepsilon(k, \theta)\}_{k=1}^{N}$ to assess the performance of the predictor)
- Estimate the parameters $\hat{\theta} = \arg\min_{\theta} V_N(\theta)$

SISO model structures

- AutoRegressive with Exogenous inputs (ARX) models

$$y(k) = \frac{B(q,\theta)}{A(q,\theta)}u(k) + \frac{1}{A(q,\theta)}e(k)$$

- AutoRegressive-Moving-Average with Exogenous inputs (ARMAX) models

$$y(k) = \frac{B(q,\theta)}{A(q,\theta)}u(k) + \frac{C(q,\theta)}{A(q,\theta)}e(k)$$

- Output Error (OE) models

$$y(k) = \frac{B(q,\theta)}{A(q,\theta)}u(k) + e(k)$$

- Box-Jenkins (BJ) models

$$y(k) = \frac{B(q,\theta)}{A(q,\theta)}u(k) + \frac{C(q,\theta)}{D(q,\theta)}e(k)$$

Choice of the predictor filters

Optimal predictor

Choose the prediction filters $L_y(q^{-1}, \theta)$ and $L_u(q^{-1}, \theta)$ providing the prediction error with smallest variance

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$$\begin{aligned}
y(k) =& G(q,\theta)u(k) + H(q,\theta)e(k) = G(q,\theta)u(k) + \left(H(q,\theta) - I\right) e(k) + e(k) = \\
=& G(q,\theta)u(k) + \left(H(q,\theta) - I\right) H^{-1}(q,\theta) \left(y(k) - G(q,\theta)u(k)\right) + e(k) = \\
=& \left[\left(I - H^{-1}(q,\theta)\right) y(k) + H^{-1}(q,\theta)G(q,\theta)u(k)\right] + e(k) = \\
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= \mathbb{E}\left[(z(k) - y^*(k))\,(z(k) - y^*(k))^\top \right] + \mathbb{E}\left[e(k)e^\top(k) \right] \succeq \Lambda_e
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$z(k)$ is the optimal predictor and $e(k)$ is the "optimal" prediction error

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Minimization criterion

Choice of the loss function $V_N(\theta)$

- Sample covariance matrix

$$R_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon(k, \theta)\varepsilon^{\top}(k, \theta)$$

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$$V_N(\theta) = h(R_N(\theta))$$

with h continuous monotonically increasing function defined on the set of positive semidefinite matrices:

$$h(Q + \Delta Q) \geq h(Q) \quad \forall Q, \Delta Q \succeq 0.$$

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Final estimate

$$\hat{\theta}_{\mathrm{PEM}} = \arg\min_{\theta} V_N(\theta)$$

PEM: Asymptotic analysis

What happens when $N \to \infty$?

- $\lim_{N\to\infty} R_N(\theta) = \bar{\mathbb{E}}\left[\varepsilon(k,\theta)\varepsilon^\top(k,\theta)\right] = R_\infty(\theta)$
- $\lim_{N\to\infty} h\left(R_N(\theta)\right) = h\left(R_\infty(\theta)\right) = V_\infty(\theta)$
- Convergence is uniform on a compact set Θ, i.e.,

$$\sup_{\theta\in\Theta} |V_N(\theta) - V_\infty(\theta)| \to 0$$

- $\lim_{N\to\infty} \hat{\theta}_{\mathrm{PEM}} = \theta^* = \arg\min_\theta V_\infty(\theta)$

PEM: Asymptotic analysis

Is $\hat{\theta}_{\mathrm{PEM}}$ a consistent estimate of θ_{o}?

- Let θ_{o} be the true system parameters:

$$y(k) = G(q,\theta_{\mathrm{o}})u(k) + H(q,\theta_{\mathrm{o}})e(k), \quad \mathbb{E}\left[e(t)e^{\top}(s)\right] = \Lambda_e\delta(s-t),$$
$$G(0,\theta_{\mathrm{o}}) = 0, \quad H(0,\theta_{\mathrm{o}}) = I, \; H^{-1}(q,\theta_{\mathrm{o}}) \;\; \text{stable}$$

- Thus:

$$\begin{aligned}\varepsilon(k,\theta) =& H^{-1}(q,\theta)\left(G(q,\theta_{\mathrm{o}})u(k) + H(q,\theta_{\mathrm{o}})e(k) - G(q,\theta)u(k)\right) = \\ =& H^{-1}(q,\theta)\left(G(q,\theta_{\mathrm{o}}) - G(q,\theta)\right)u(k) + H^{-1}(q,\theta)H(q,\theta_{\mathrm{o}})e(k) = \\ =& e(k) + \text{ terms independent of } e(k)\end{aligned}$$

- Thus: $R_\infty(\theta) = \bar{\mathbb{E}}\left[\varepsilon(k,\theta)\varepsilon^{\top}(k,\theta)\right] \geq \mathbb{E}\left[e(k)e^{\top}(k)\right] = \Lambda_e$
- θ_{o} is a minimizer of $h(R_\infty(\theta)) = V_\infty$.
- If $u(k)$ and $e(k)$ are not correlated, only the "true" parameters θ_{o} minimize $h(R_\infty(\theta)) = V_\infty$
- Thus $\lim\limits_{N\to\infty} \hat{\theta}_{\mathrm{PEM}} = \theta_{\mathrm{o}}$.

PEM: examples

$$\hat{y}(k|k-1,\theta) = \left(I - H^{-1}(q^{-1},\theta)\right) y(k) + H^{-1}(q^{-1},\theta) G(q^{-1},\theta) u(k)$$
$$\varepsilon(k,\theta) = e(k) = H^{-1}(q^{-1},\theta) \left(y(k) - G(q^{-1},\theta) u(k)\right)$$

Predictor for ARX models

$$y(k) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} u(k) + \frac{1}{A(q^{-1},\theta)} e(k), \quad e \text{ white}$$

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- Optimal predictor:

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Linear regression representation

- PEM estimate:

$$\hat{\theta}_{\text{PEM}} = \min_{\theta} \sum_{k=1}^{N} \left(y(k) - \hat{y}(k|k-1,\theta)\right)^2$$

PEM: examples

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Linear regression representation

- PEM estimate:

$$\hat{\theta}_{\text{PEM}} = \min_{\theta} \sum_{k=1}^{N} (y(k) - \hat{y}(k|k-1,\theta))^2 = \min_{\theta} \sum_{k=1}^{N} \left(y(k) - \varphi^\top(k)\theta\right)^2 = \left(\Phi^\top\Phi\right)^{-1}\Phi^\top Y$$

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- PEM estimate:

$$\hat{\theta}_{\mathrm{PEM}} = \min_{\theta} \sum_{k=1}^{N} (y(k) - \hat{y}(k|k-1,\theta))^2 = \min_{\theta} \sum_{k=1}^{N} \left(y(k) - \varphi^\top(k)\theta\right)^2 = \left(\Phi^\top \Phi\right)^{-1} \Phi^\top Y$$

LS method and PEM coincide!

PEM: examples

$$\hat{y}(k|k-1,\theta) = \left(I - H^{-1}(q^{-1},\theta)\right) y(k) + H^{-1}(q^{-1},\theta) G(q^{-1},\theta) u(k)$$
$$\varepsilon(k,\theta) = e(k) = H^{-1}(q^{-1},\theta) \left(y(k) - G(q^{-1},\theta)u(k)\right)$$

Predictor for OE models

$$y(k) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} u(k) + e(k), \quad e \text{ white}$$

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$$y(k) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} u(k) + e(k), \quad e \text{ white}$$

- Optimal predictor:

$$\begin{aligned}\hat{y}(k|k-1,\theta) =& \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} u(k) = \\ =& - a_1 \hat{y}(k-1|k-2) - \cdots - a_{n_a} \hat{y}(k-n_a|k-n_a-1) + \\ &+ b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) = \\ =& \hat{\varphi}^\top(k)\theta\end{aligned}$$

PEM: examples

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$\hat{\varphi}^\top(k)$ depends on the noise-free past outputs $\hat{y}(k-i|k-i-1,\theta) = \dfrac{B(q^{-1},\theta)}{A(q^{-1},\theta)} u(k-i)$

PEM: numerical optimization

- PEM estimate:

$$\hat{\theta}_{\mathrm{PEM}} = \arg\min_{\theta} V_N(\theta) = \arg\min_{\theta} h\left(\frac{1}{N}\sum_{k=1}^{N}\varepsilon(k,\theta)\varepsilon^{\top}(k,\theta)\right)$$

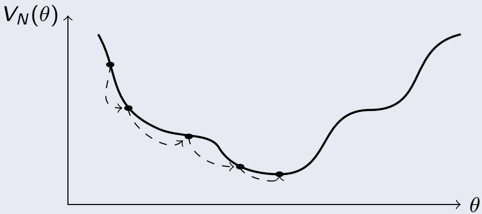
- The solution cannot be always computed analytically
- Numerical iterative algorithms for non-convex optimization should be used:

  i initialize with an initial estimate $\hat{\theta}^{(0)}$

  ii update: $\hat{\theta}^{(i+1)} = f(\hat{\theta}^{(i)})$ (the estimate is iteratively refined)

  iii we would like that the estimate converges to the optimum $\hat{\theta}_{\mathrm{PEM}}$:

$$\hat{\theta}^{(0)} \to \hat{\theta}^{(1)} \to \hat{\theta}^{(2)} \to \cdots \to \hat{\theta}_{\mathrm{PEM}}$$



PEM: numerical optimization

Gradient method

- choose an **initial condition** $\hat{\theta}^{(0)}$;
- **iterate**
  - (i) *line search*: choose a positive step size $t > 0$
  - (ii) *update*: $\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - t \left. \frac{\partial V_N(\theta)}{\partial \theta} \right|_{\theta=\hat{\theta}^{(i)}}$
- **until** stopping criterion is satisfied (typically: $\left\| \frac{\partial V_N(\theta)}{\partial \theta} \right\| \leq \epsilon$)
- in case of scalar output, $V_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon^2(k,\theta)$ and $\frac{\partial V_N(\theta)}{\partial \theta} = \frac{2}{N} \sum_{k=1}^{N} \varepsilon(k,\theta) \frac{\partial \varepsilon(k,\theta)}{\partial \theta}$
- it converges (slowly) to the global optimum if $V_N(\theta)$ is convex
- in case of non-convex $V_N(\theta)$, convergence to the global minimum is not guaranteed

Exact line search

$$t = \arg\min_{t>0} V_N\left(\hat{\theta}^{(i)} + t\Delta\theta\right), \text{ with } \Delta\theta = - \left. \frac{\partial V_N(\theta)}{\partial \theta} \right|_{\theta=\hat{\theta}^{(i)}}$$

PEM: numerical optimization

Gauss-Newton method

- choose an **initial condition** $\hat{\theta}^{(0)}$;
- **iterate**
  (i) *line search*: choose a positive step size $t > 0$
  (ii) *update*: $\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - t\left(\nabla^2 V_N(\hat{\theta}^{(i)})\right)^{-1} \nabla V_N\left(\hat{\theta}^{(i)}\right)$
- **until** stopping criterion is satisfied. Typically: $\left|\nabla V_N\left(\hat{\theta}^{(i)}\right)^\top \left(\nabla^2 V_N(\hat{\theta}^{(i)})\right)^{-1} \nabla V_N\left(\hat{\theta}^{(i)}\right)\right| \leq \epsilon$
- in case of scalar output, $V_N(\theta) = \dfrac{1}{N}\sum_{k=1}^{N} \varepsilon^2(k,\theta)$ and

$$\nabla V_N(\theta) = \frac{2}{N}\sum_{k=1}^{N} \varepsilon(k,\theta)\frac{\partial \varepsilon(k,\theta)}{\partial \theta}, \quad \nabla^2 V_N(\theta) = \frac{2}{N}\sum_{k=1}^{N} \frac{\partial \varepsilon(k,\theta)}{\partial \theta}\frac{\partial \varepsilon^\top(k,\theta)}{\partial \theta} + \frac{2}{N}\sum_{k=1}^{N} \frac{\partial^2 \varepsilon(k,\theta)}{\partial \theta^2}\varepsilon(k,\theta)$$

PEM: numerical optimization

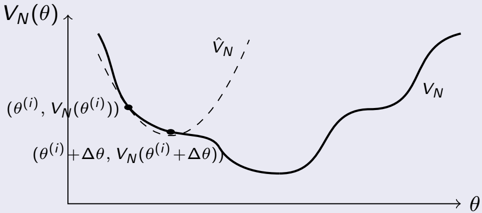
$$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} + t\Delta\theta, \text{ with } \Delta\theta = -\left(\nabla^2 V_N(\hat{\theta}^{(i)})\right)^{-1} \nabla V_N\left(\hat{\theta}^{(i)}\right)$$

Gauss-Newton method: interpretation

$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} + \Delta\theta$ minimizes the second order approximation:

$$\hat{V}_N(\theta^{(i)} + \Delta\theta) = V_N(\theta^{(i)}) + \nabla V_N(\theta^{(i)})^\top \Delta\theta + \frac{1}{2}\Delta\theta^\top \nabla^2 V_N(\theta^{(i)})\Delta\theta$$

The minimum of the quadratic function above is achieved at $\Delta\theta = -\left(\nabla^2 V_N(\hat{\theta}^{(i)})\right)^{-1} \nabla V_N\left(\hat{\theta}^{(i)}\right)$



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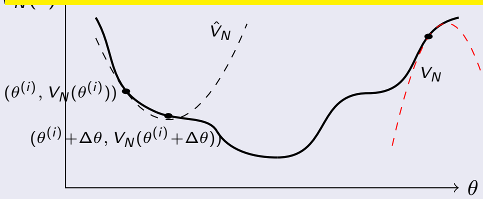
Careful: if the Hessian is not positive definite, we move to the "wrong" direction



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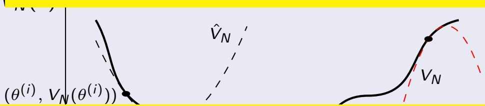

Hessian approximation:

$$\nabla^2 V_N(\theta) = \frac{2}{N}\sum_{k=1}^{N} \frac{\partial\varepsilon(k,\theta)}{\partial\theta}\frac{\partial\varepsilon^\top(k,\theta)}{\partial\theta} + \frac{2}{N}\sum_{k=1}^{N} \frac{\partial^2\varepsilon(k,\theta)}{\partial\theta^2}\varepsilon(k,\theta) \approx \frac{2}{N}\sum_{k=1}^{N} \frac{\partial\varepsilon(k,\theta)}{\partial\theta}\frac{\partial\varepsilon^\top(k,\theta)}{\partial\theta} + \underbrace{\delta I}_{\text{regularization}} \succ 0$$

PEM: numerical optimization

Example: evaluation of the gradient for ARMAX models

- ARMAX model: $y(k) = \dfrac{B(q^{-1},\theta)}{A(q^{-1},\theta)}u(k) + \dfrac{C(q^{-1},\theta)}{A(q^{-1},\theta)}e(k)$
- prediction error $\varepsilon(k,\theta)$: $C(q^{-1},\theta)\varepsilon(k,\theta) = A(q^{-1},\theta)y(k) - B(q^{-1},\theta)u(k)$

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- prediction error $\varepsilon(k,\theta)$: $C(q^{-1},\theta)\varepsilon(k,\theta) = A(q^{-1},\theta)y(k) - B(q^{-1},\theta)u(k)$
- compute derivatives of both left and right hand of the above equation:

$$C(q^{-1},\theta)\frac{\partial\varepsilon(k,\theta)}{\partial a_i} = y(k-i) \Rightarrow \frac{\partial\varepsilon(k,\theta)}{\partial a_i} = \frac{1}{C(q^{-1},\theta)}y(k-i)$$

$$C(q^{-1},\theta)\frac{\partial\varepsilon(k,\theta)}{\partial b_i} = -u(k-i) \Rightarrow \frac{\partial\varepsilon(k,\theta)}{\partial b_i} = -\frac{1}{C(q^{-1},\theta)}u(k-i)$$

$$\varepsilon(k-i,\theta) + C(q^{-1},\theta)\frac{\partial\varepsilon(k,\theta)}{\partial c_i} = 0 \Rightarrow \frac{\partial\varepsilon(k,\theta)}{\partial c_i} = -\frac{1}{C(q^{-1},\theta)}\varepsilon(k-i,\theta)$$

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- Thus:

$$\frac{\partial\varepsilon(k,\theta)}{\partial\theta} = [y_F(k-1)\cdots y_F(k-n_a)\ \ -u_F(k-1)\cdots -u_F(k-n_b)\ \ -\varepsilon_F(k-1)\cdots -\varepsilon_F(k-n_c)]^\top$$

with:

$$y_F(k) = \frac{1}{C(q^{-1},\theta)}y(k),\quad u_F(k) = \frac{1}{C(q^{-1},\theta)}u(k),\quad \varepsilon_F(k) = \frac{1}{C(q^{-1},\theta)}\varepsilon(k)$$