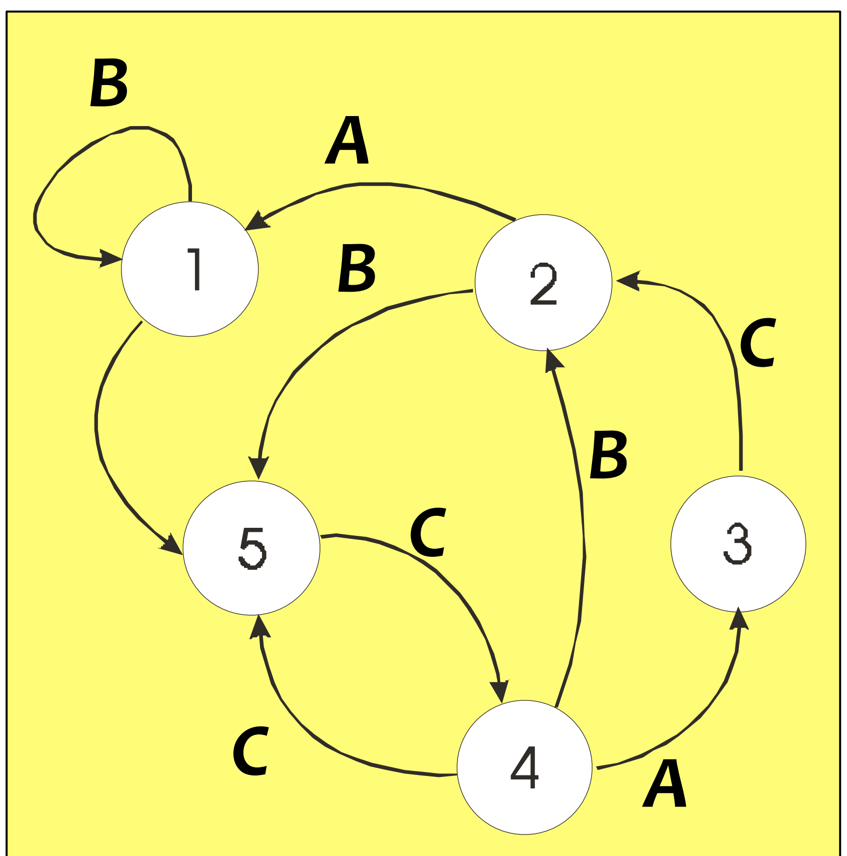
Hybrid Systems

Hybrid Systems

$$x \in \{1,2,3,4,5\}$$
$$u \in \{A,B,C\}$$



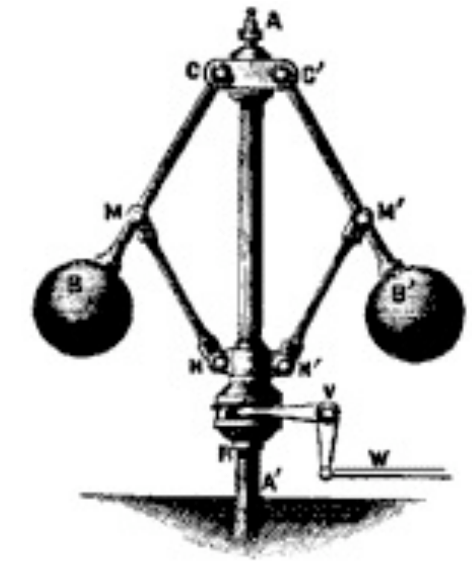
Computer Science

Finite state machines

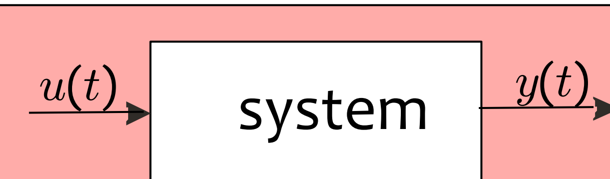
Control Theory

Continuous dynamical systems

Hybrid systems



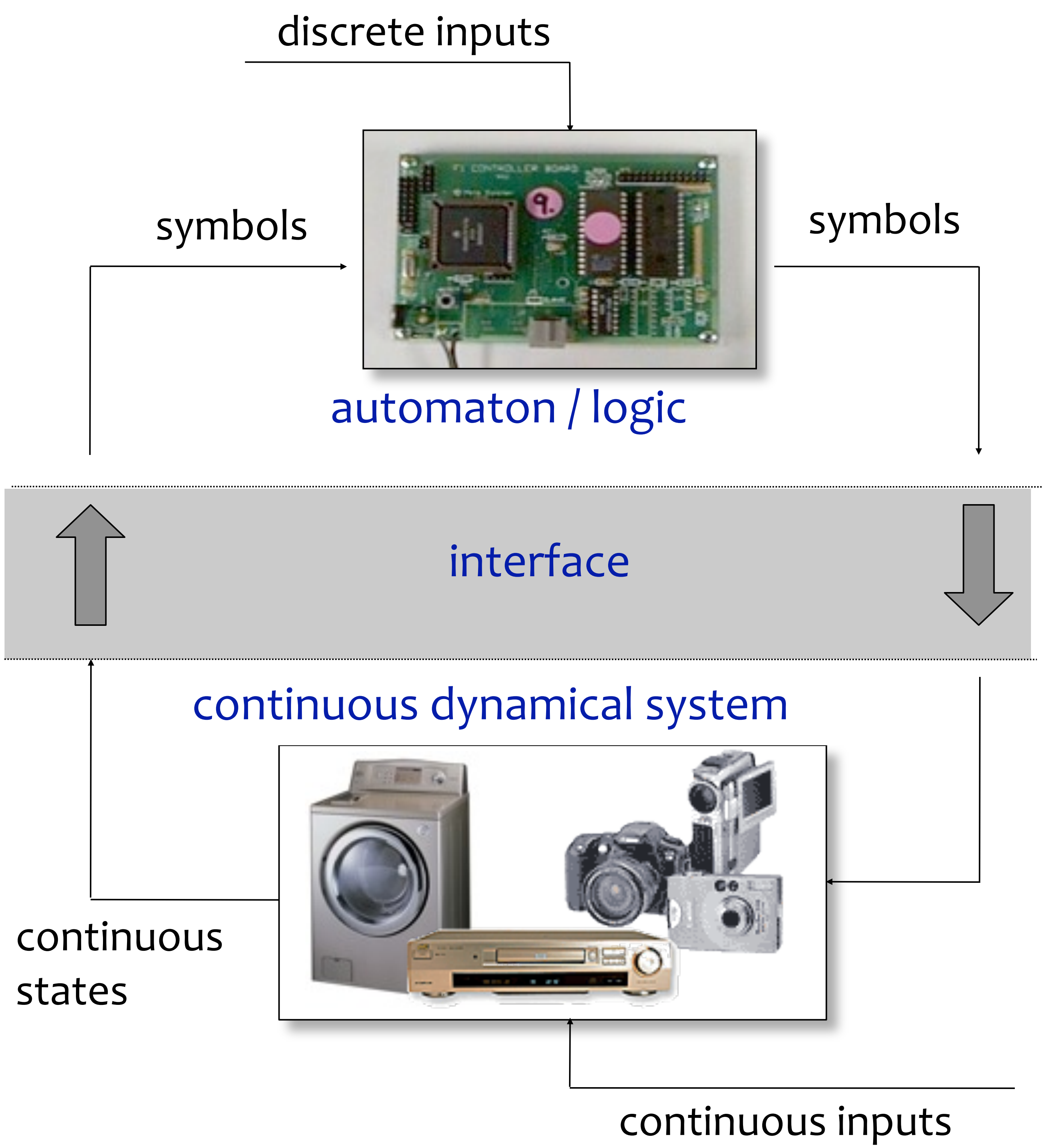
$$x \in \mathbb{R}^n$$
$$u \in \mathbb{R}^m$$
$$y \in \mathbb{R}^p$$



$$\begin{cases} \frac{dx(t)}{dt} &= f(x(t),u(t)) \\ y(t) &= g(x(t),u(t)) \end{cases}$$

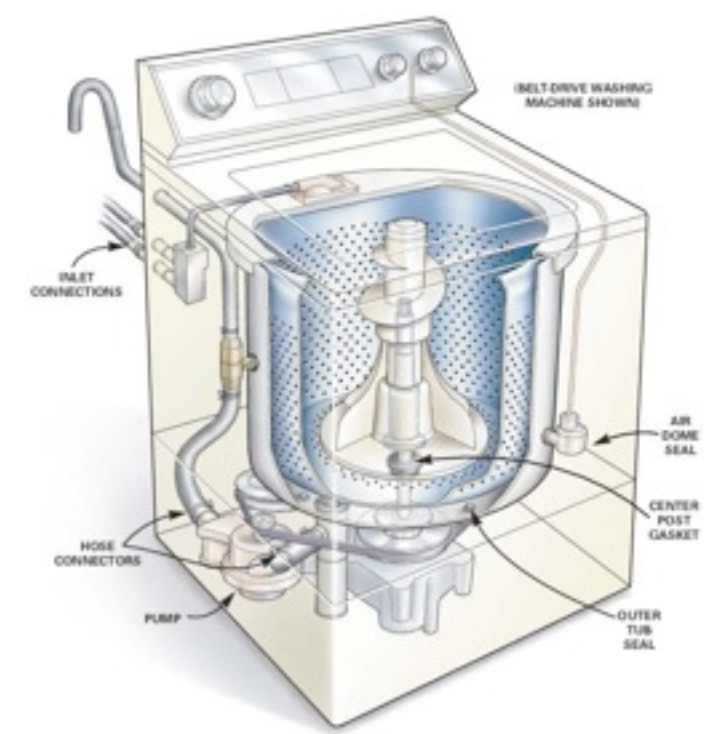
$$\begin{cases} x(k+1) &= f(x(k),u(k)) \\ y(k) &= g(x(k),u(k)) \end{cases}$$

Embedded Systems

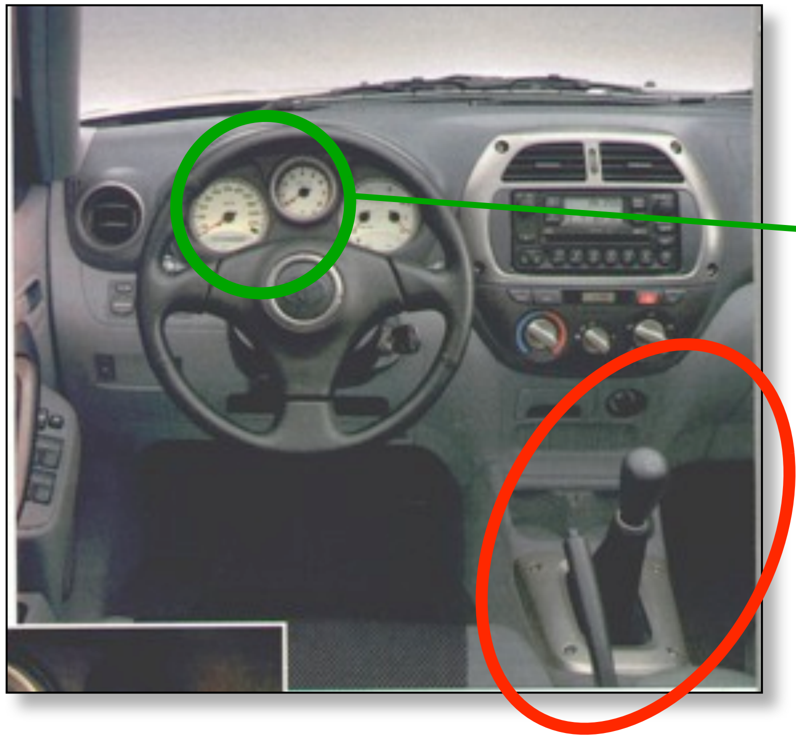


- Automobiles
- Industrial processes
- Consumer electronics
- Home appliances
- …

Example:

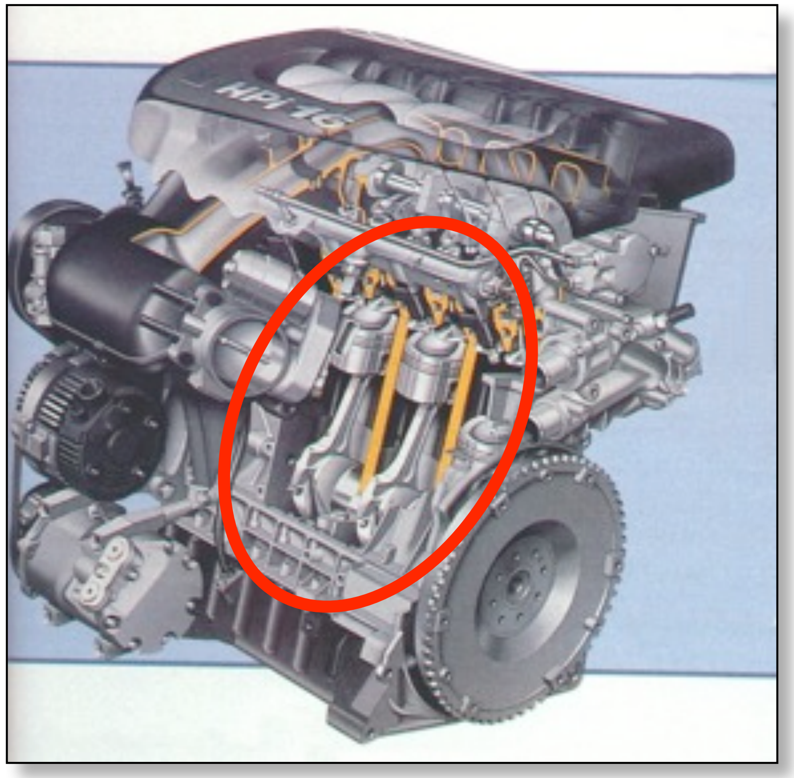


Motivation: “Intrinsically Hybrid”



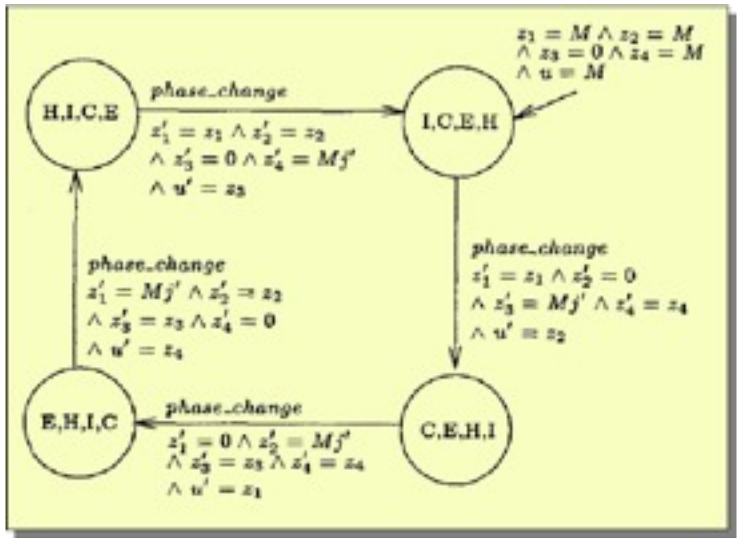
- Transmission

discrete command (R,N,1,2,3,4,5) + continuous dynamical variables (velocities, torques)



- Four-stroke engines

Automaton, dependent on power train motion



"Intrinsically Hybrid" Systems

Discrete input (gear 1,2,3,4,N) + Continuous inputs (brakes, gas, clutch) + Continuous dynamical states (velocities, torques, air-flows, fuel level)

Example of Hybrid Control Problem

Cruise control problem

GOAL:

command gear ratio, gas pedal, and brakes to **track** a desired speed and minimize consumptions

CHALLENGES:

- continuous **and** discrete inputs
- dynamics depends on gear
- nonlinear torque/speed maps

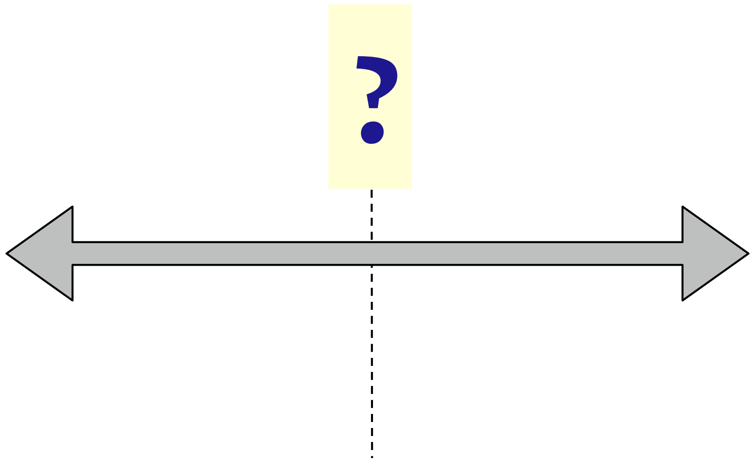
Key Requirements for Hybrid Models

- **Descriptive** enough to capture the behavior of the system
  - continuous dynamics (physical laws)
  - logic components (switches, automata, software code)
  - interconnection between logic and dynamics

- **Simple** enough for solving *analysis* and *synthesis* problems

$$\begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$$

linear systems

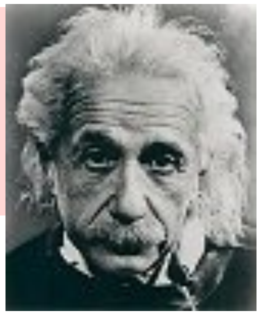


$$\begin{cases} x' = f(x,u,t) \\ y = g(x,u,t) \end{cases}$$

nonlinear systems

linear hybrid systems

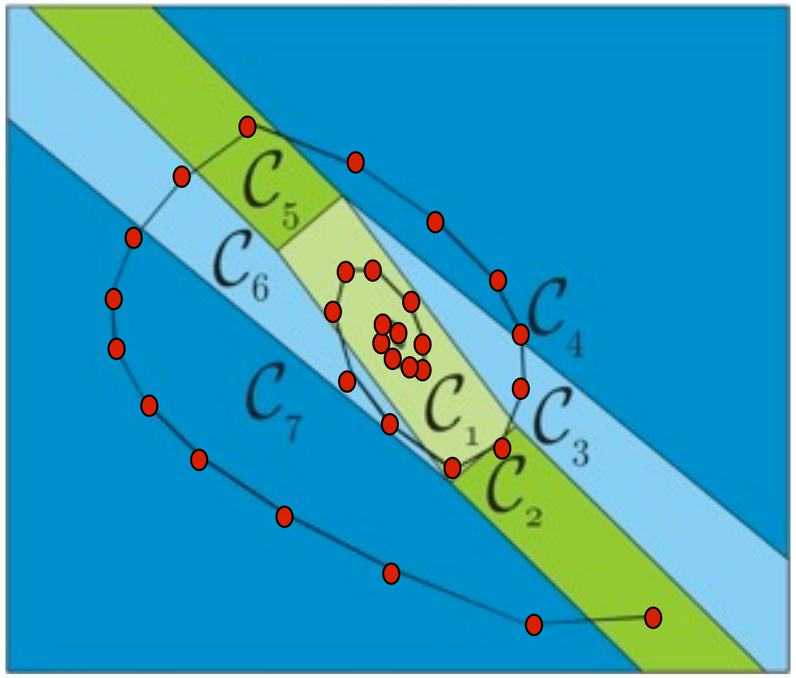
"Make everything as simple as possible, but not simpler."
— Albert Einstein



Piecewise Affine Systems

$$\begin{aligned} x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ i(k) \ \text{s.t.} &\ \ H_{i(k)}x(k) + J_{i(k)}u(k) \le K_{i(k)} \end{aligned}$$

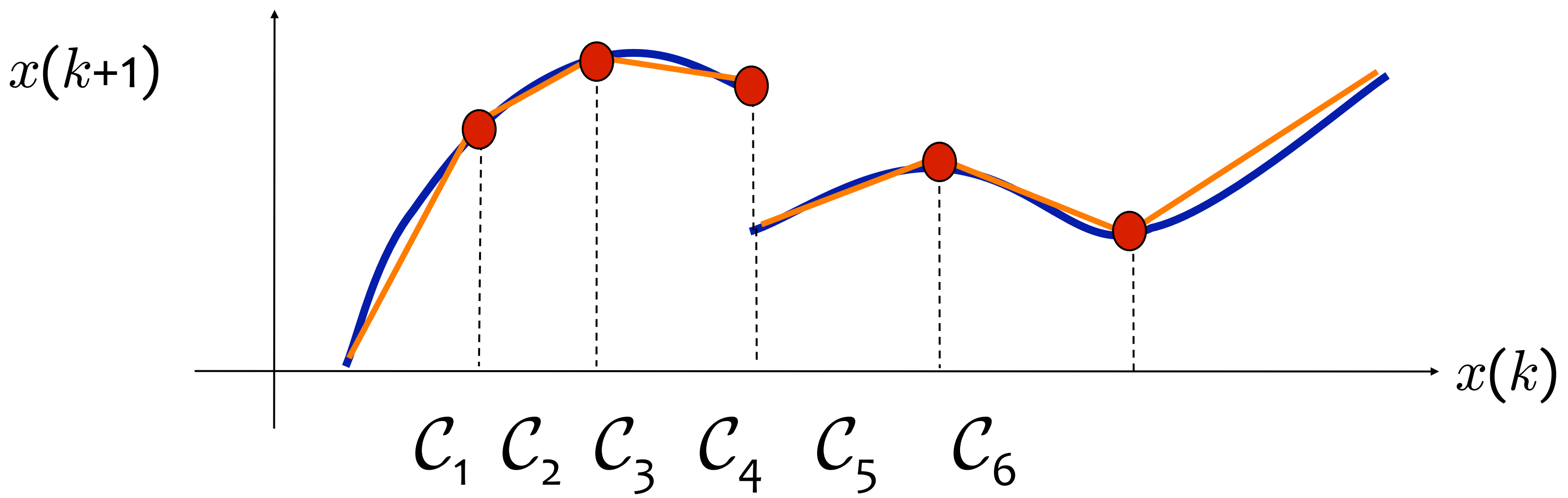
state+input space



(Sontag 1981)

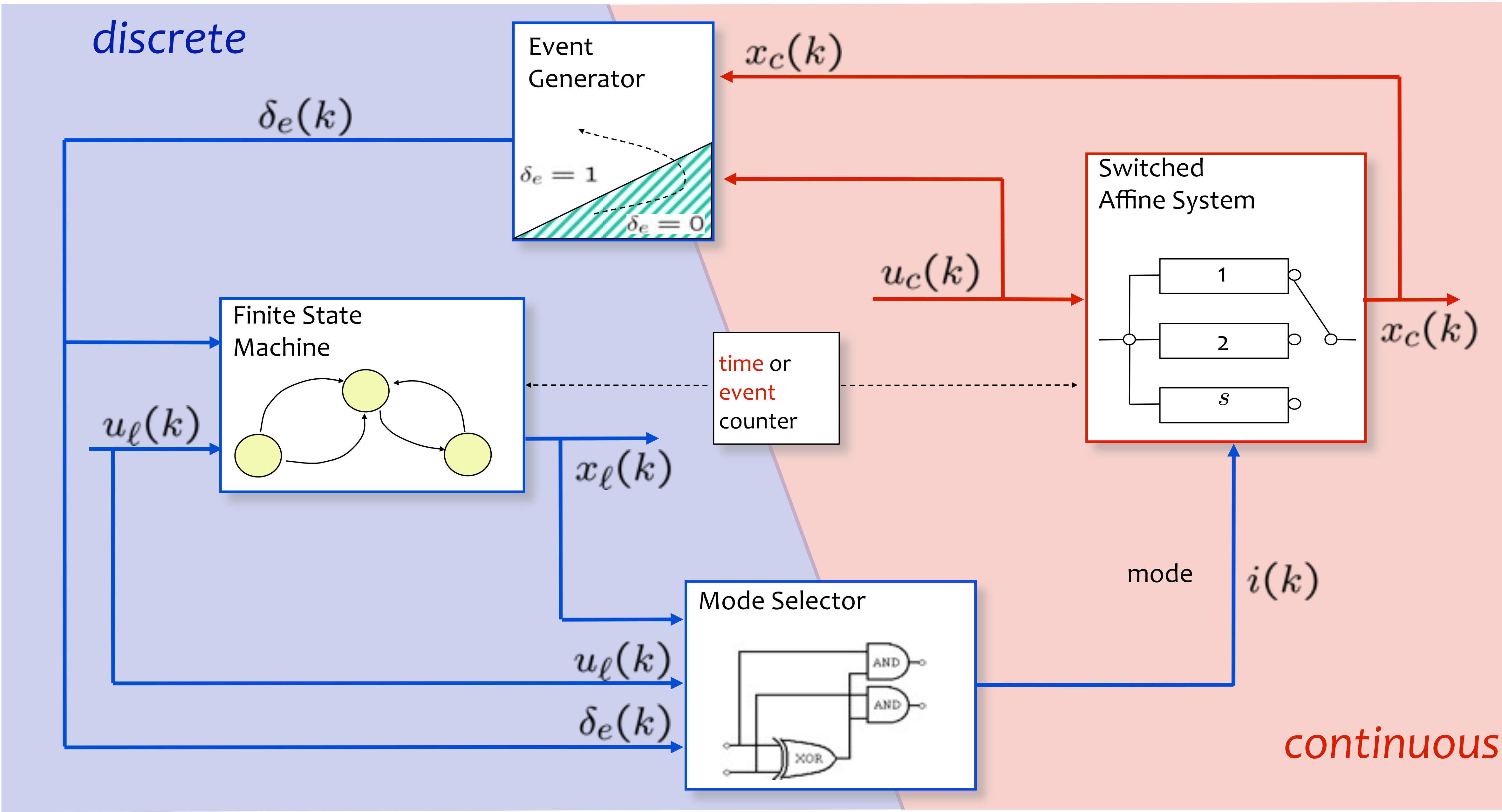
$$x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^p$$
$$i(k) \in \{1, \ldots, s\}$$

Can approximate nonlinear and/or discontinuous dynamics arbitrarily well



Discrete Hybrid Automaton

(Torrisi, Bemporad, 2004)



$x_\ell \in \{0,1\}^{n_b}$ = binary states

$u_\ell \in \{0,1\}^{m_b}$ = binary inputs

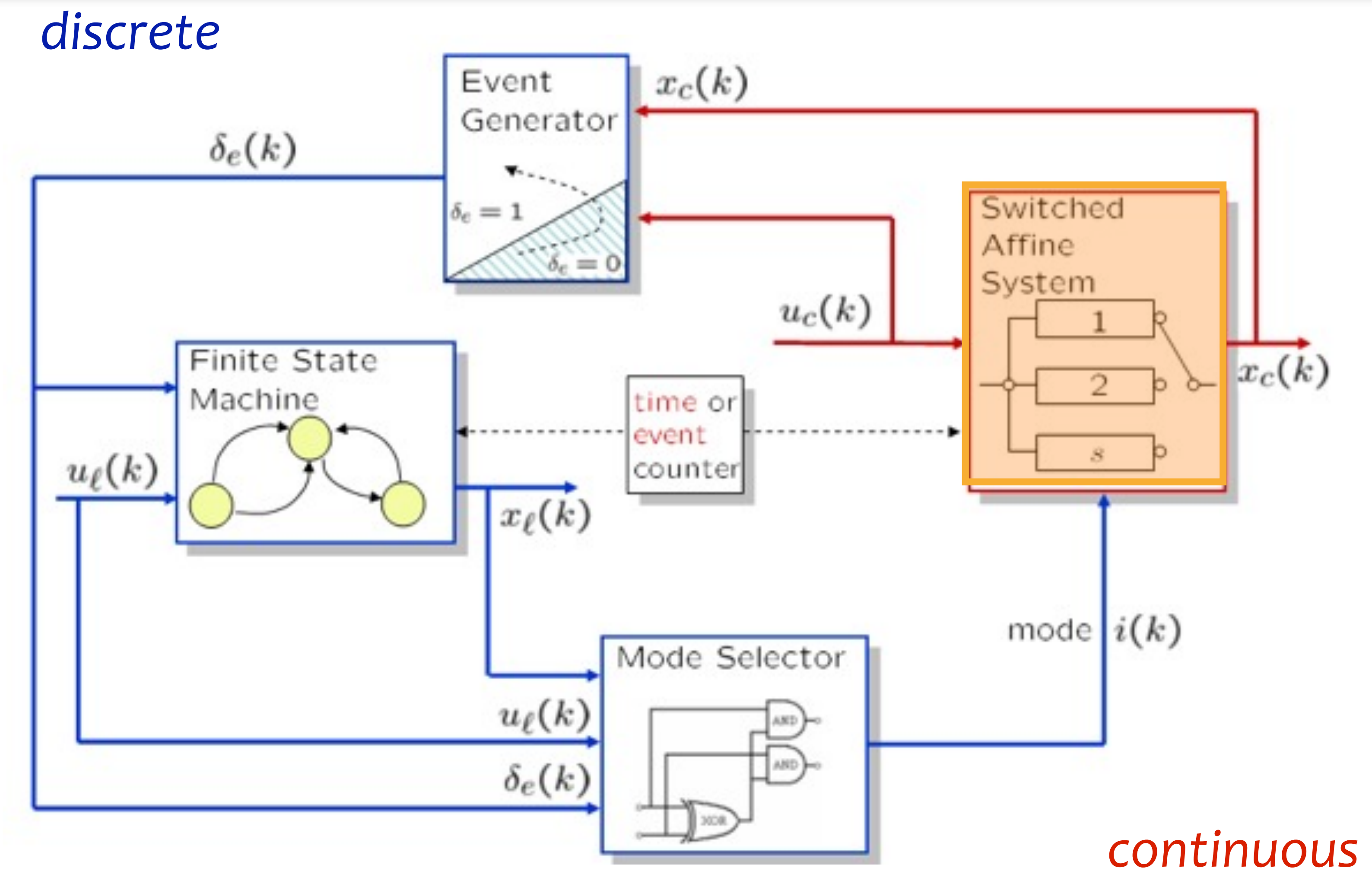
$\delta_e \in \{0,1\}^{n_e}$ = event variables

$x_c \in \mathbb{R}^{n_c}$ = continuous states

$u_c \in \mathbb{R}^{m_c}$ = continuous inputs

$i \in \{1, 2, \ldots, s\}$ = current mode

Switched Affine System



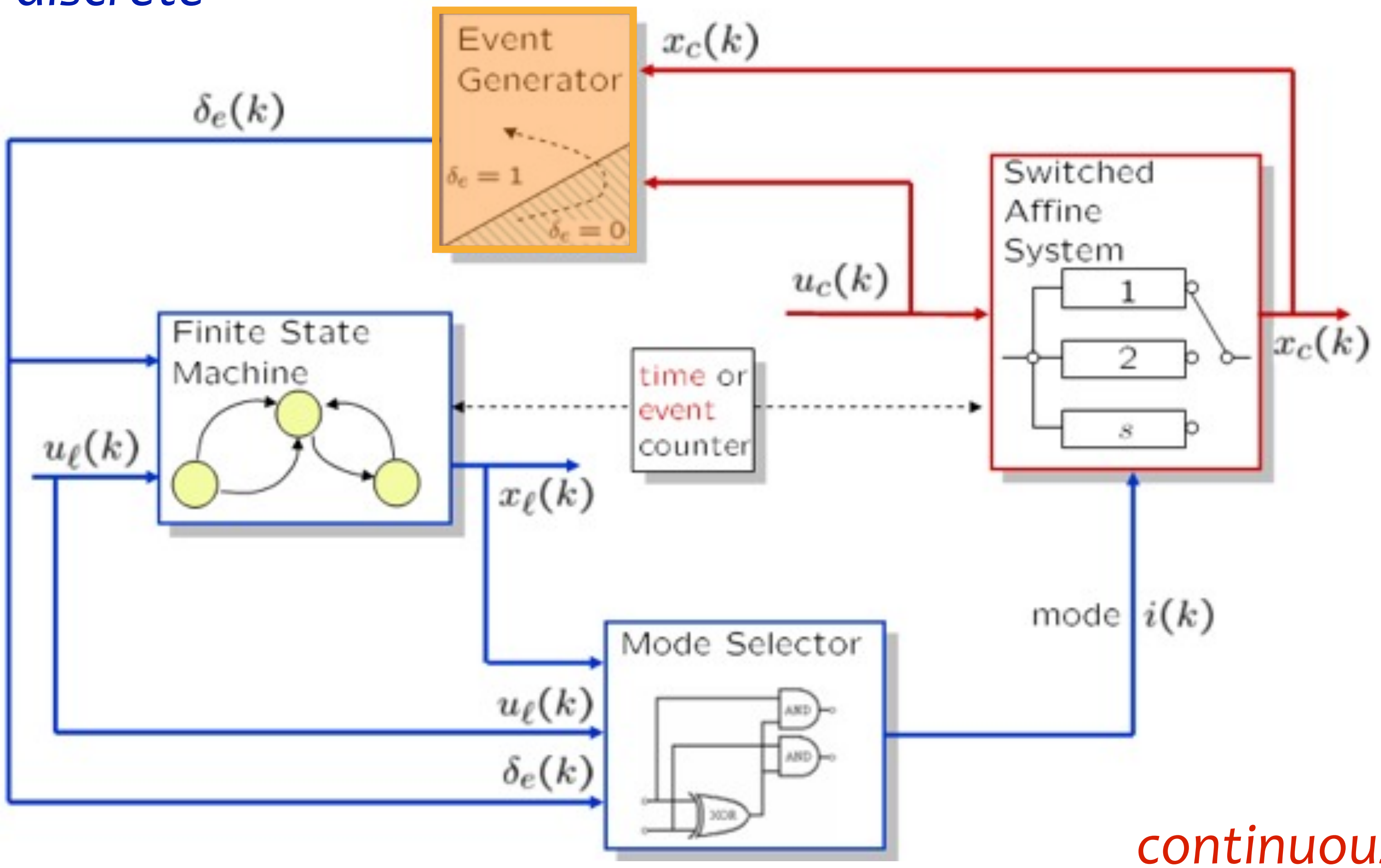
The affine dynamics depend on the current mode $i(k)$:

$$x_c(k+1) = A_{i(k)} x_c(k) + B_{i(k)} u_c(k) + f_{i(k)}$$

$$x_c \in \mathbb{R}^{n_c}, \;\; u_c \in \mathbb{R}^{m_c}$$

Event Generator

discrete



continuous

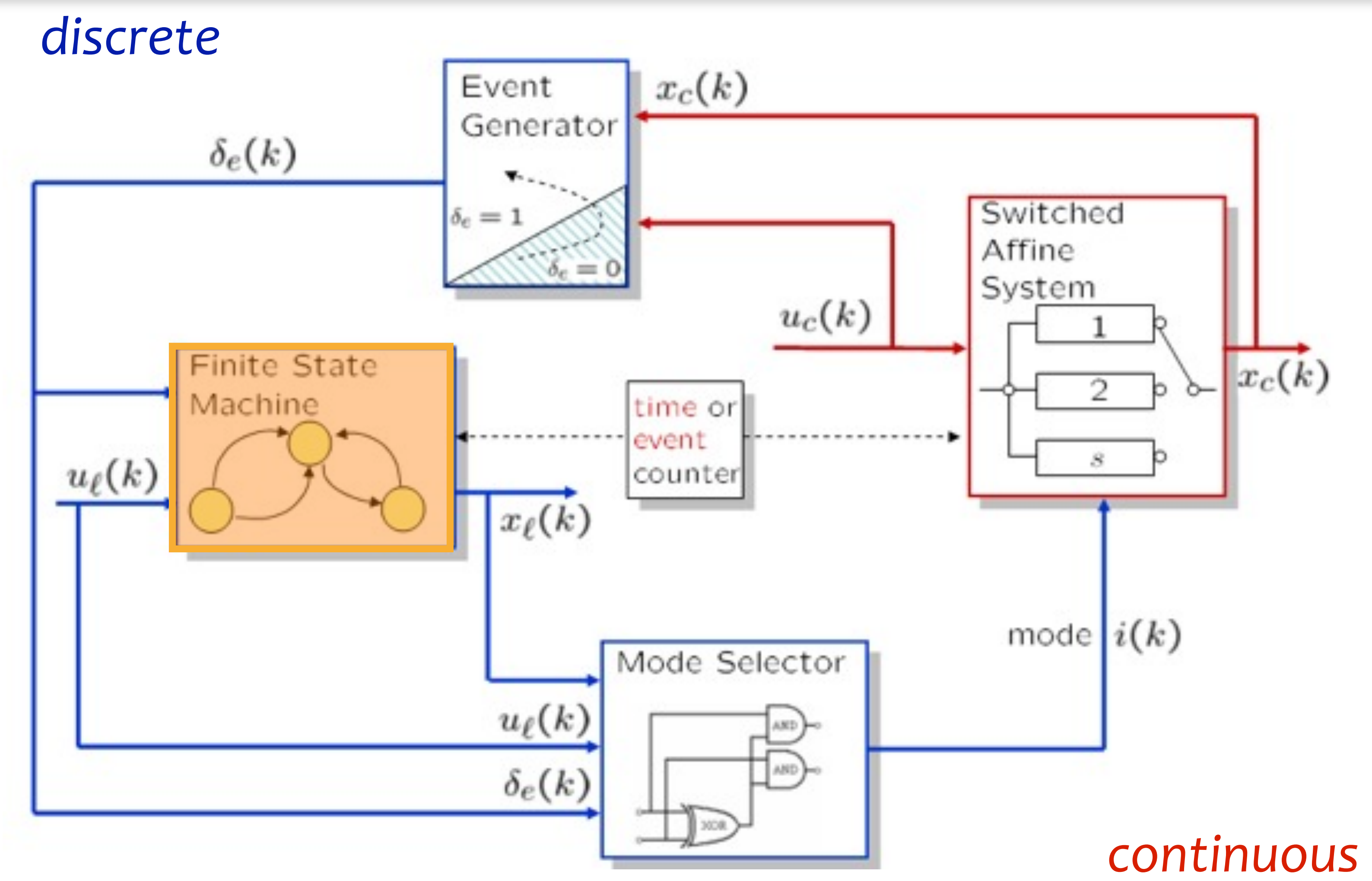
Event variables are generated by linear threshold conditions over continuous states, continuous inputs, and time:

$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) + K^i u_c(k) \leq W^i]$$

$$x_c \in \mathbb{R}^{n_c}, \;\; u_c \in \mathbb{R}^{m_c}, \;\; \delta_e \in \{0,1\}^{n_e}$$

Example: $[\delta(k)=1] \leftrightarrow [x_c(k) \geq 0]$

Finite State Machine



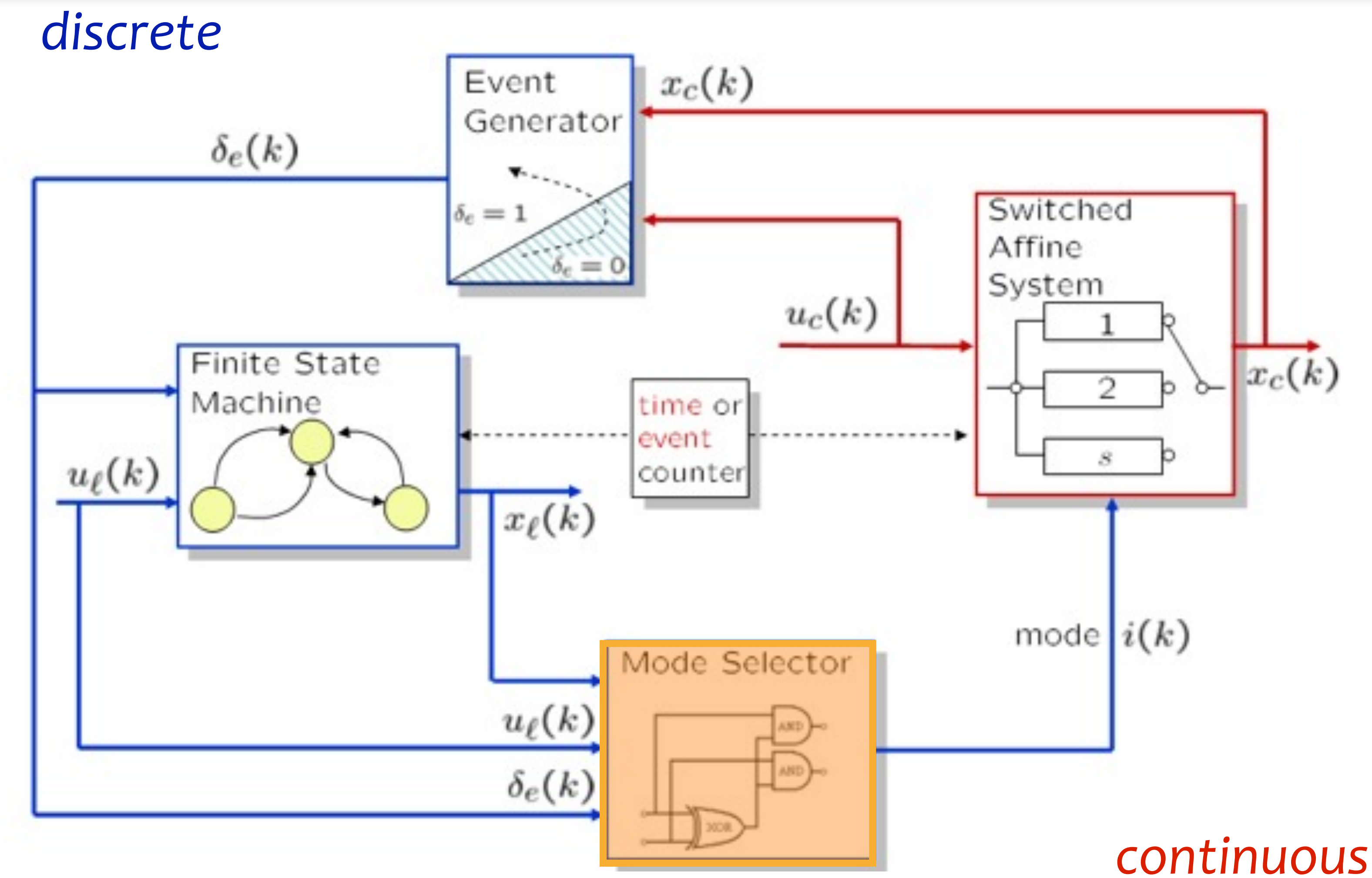
The binary state of the finite state machine evolves according to a Boolean state update function:

$$x_\ell(k+1) = f_B(x_\ell(k), u_\ell(k), \delta_e(k))$$

$x_\ell \in \{0,1\}^{n_\ell},\ u_\ell \in \{0,1\}^{m_\ell},\ \delta_e \in \{0,1\}^{n_e}$

Example: $x_\ell(k+1) = \neg\delta_e(k) \vee (x_\ell(k) \wedge u_\ell(k))$

Mode Selector



The mode selector can be seen as the output function of the discrete dynamics

The active mode $i(k)$ is selected by a Boolean function of the current binary states, binary inputs, and event variables:

$$i(k) = f_{\mathsf{M}}(x_\ell(k), u_\ell(k), \delta_e(k))$$

$x_\ell \in \{0,1\}^{n_\ell}, \ u_\ell \in \{0,1\}^{m_\ell}, \ \delta_e\{0,1\}^{n_e}$

Example:

$$i(k) = \begin{bmatrix} \neg u_\ell(k) \vee x_\ell(k) \\ u_\ell(k) \wedge x_\ell(k) \end{bmatrix}$$

u_ℓ / x_ℓ	0	1
0	$i = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$	$i = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$
1	$i = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$	$i = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$

the system has 3 modes

$$X_1 \vee X_2 = \text{TRUE} \longrightarrow \delta_1 + \delta_2 \geq 1, \quad \delta_1, \delta_2 \in \{0,1\}$$

(Glover 1975, Williams 1977, Hooker 2000)

0. Given a Boolean statement

$$F(X_1, X_2, \ldots, X_n) = \text{TRUE}$$

1. Convert to Conjunctive Normal Form (CNF):

$$\bigwedge_{j=1}^{m} \left(\bigvee_{i \in P_j} X_i \bigvee_{i \in N_j} \bar{X}_i \right) = \text{TRUE}$$

2. Transform into inequalities:

$$\begin{array}{rcl} \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) & \geq & 1 \\ \vdots & \vdots & \vdots \\ \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) & \geq & 1 \end{array}$$

⇒ polyhedron

$$A\delta \leq b, \quad \delta \in \{0,1\}^n$$

Any logic proposition can be translated into linear integer inequalities

Logic $\rightarrow$ Inequalities: Symbolic Approach

Example: $$F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$$

1. Convert to Conjunctive Normal Form (CNF):
   (see e.g.: `http://www.oursland.net/aima/propositionApplet.html`
   or just search [CNF + applet] on Google ...)

$$(X_3 \vee \neg X_1 \vee \neg X_2) \wedge (X_1 \vee \neg X_3) \wedge (X_2 \vee \neg X_3)$$

2. Transform into inequalities:

$$\begin{cases} \delta_3 + (1 - \delta_1) + (1 - \delta_2) \geq 1 \\ \delta_1 + (1 - \delta_3) \geq 1 \\ \delta_2 + (1 - \delta_3) \geq 1 \end{cases}$$

Boolean statement

$F(X_1, X_2, \ldots, X_n) = \text{TRUE}$ $\Longrightarrow$

polyhedron

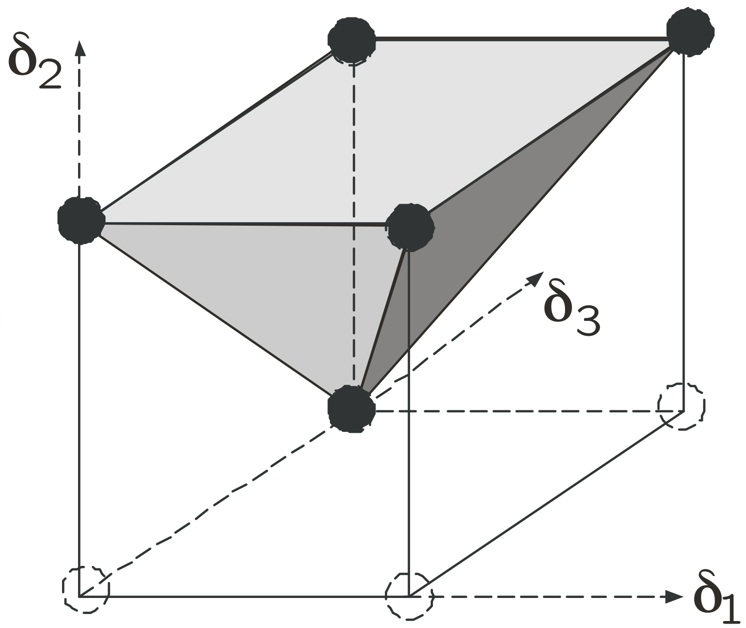
$A\delta \leq b, \quad \delta \in \{0,1\}^n$

The polytope $P = \{\delta : \; A\delta \leq b\}$ is the convex hull of the rows of the truth table T associated with formula $F(X_1, \ldots, X_N)$

T:

X_1	X_2	$\ldots$	X_N
0	0	$\ldots$	1
0	1	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	$\ldots$	0

$\Longrightarrow$



$$\Longrightarrow \quad P : \; \{\delta : \; A\delta \leq b\}, \; \delta \in \{0,1\}^n$$

Convex hull algorithms: `cdd`, `lrs`, `qhull`, `chD`, `Hull`, `Porto`

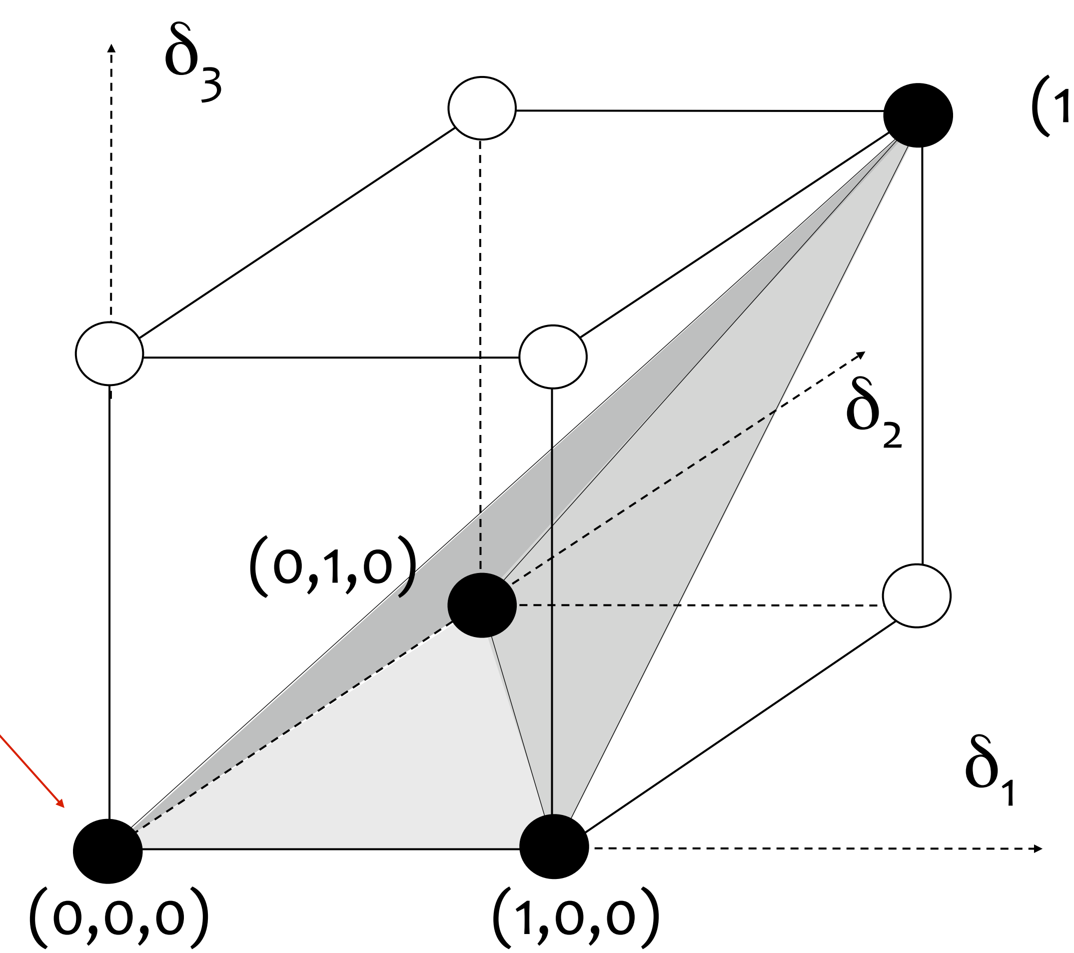
Logic → Inequalities: Geometric Approach

Example: logic "AND"

$F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$

T:

X_1	X_2	X_3
0	0	0
0	1	0
1	0	0
1	1	1



Key idea:
White points cannot be in the hull of black points

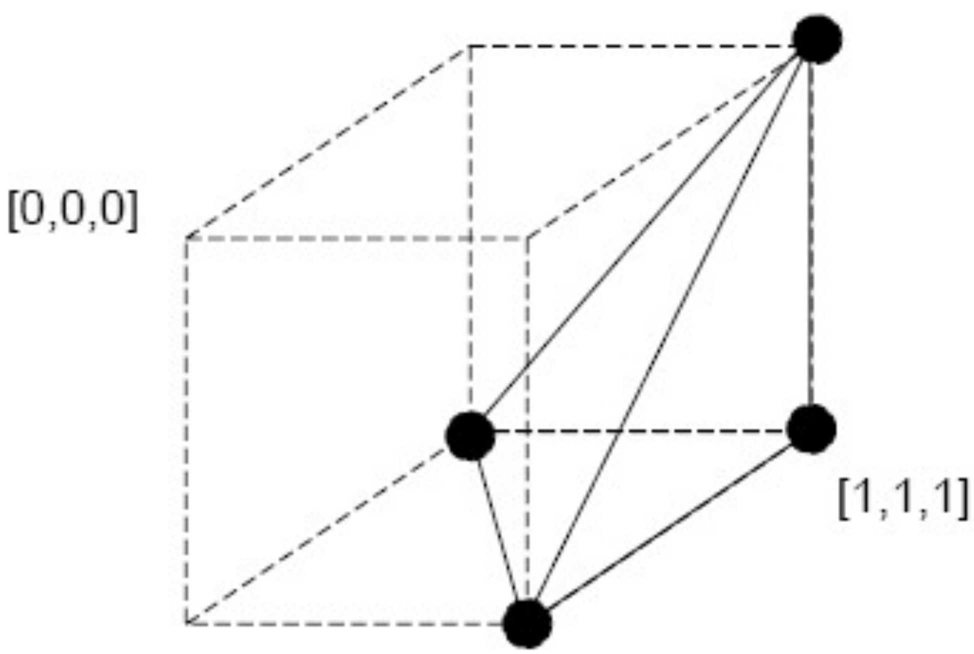
$$\text{conv}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \left\{\delta : \begin{array}{rl} -\delta_1 + \delta_3 & \leq 0 \\ -\delta_2 + \delta_3 & \leq 0 \\ \delta_1 + \delta_2 - \delta_3 & \leq 1 \end{array}\right\}$$

Convex hull algorithms: **cdd**, **lrs**, **qhull**, **chD**, **Hull**, **Porto**

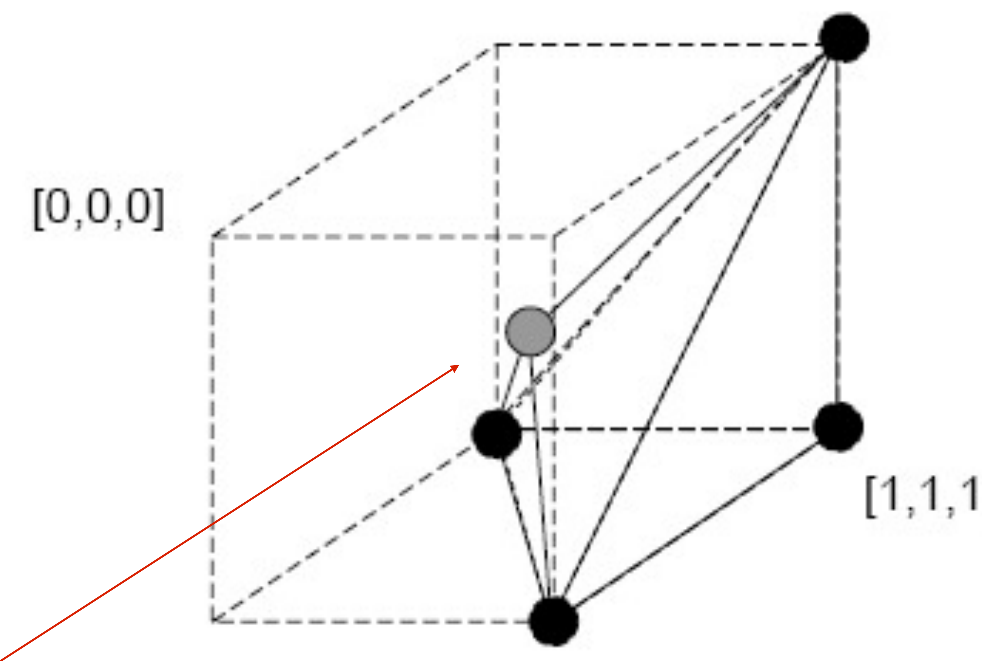
Geometric vs. Symbolic Approach

- The polyhedron obtained via convex hull is the smallest one
- The one obtained via CNF may be larger. Example:

$$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) = \text{TRUE}$$



convex hull



CNF

spurious vertex
in (0.5,0.5,0.5)

Note: no other example with 3 vars but $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge (\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3)$

Big-M Technique (iff)

- Consider the *if-and-only-if* condition

$$[\delta = 1] \leftrightarrow [a'x_c - b \leq 0]$$

$$x_c \in \mathcal{X} \subset \mathbb{R}^{n_c}$$
$$\delta \in \{0, 1\}$$

- Assume $\mathcal{X}$ bounded and let M and m such that

$$M > a'x_c - b, \ \forall x_c \in \mathcal{X}$$
$$m < a'x_c - b, \ \forall x_c \in \mathcal{X}$$

- The *if-and-only-if* condition is equivalent to

$$\begin{cases} a'x_c - b & \leq & M(1 - \delta) \\ a'x_c - b & > & m\delta \end{cases}$$

Big-M Technique (if-then-else)

- Consider the *if-then-else* condition

$$z = \begin{cases} a_1' x_c + f_1 & \text{if } \delta = 1 \\ a_2' x_c + f_2 & \text{otherwise} \end{cases}$$

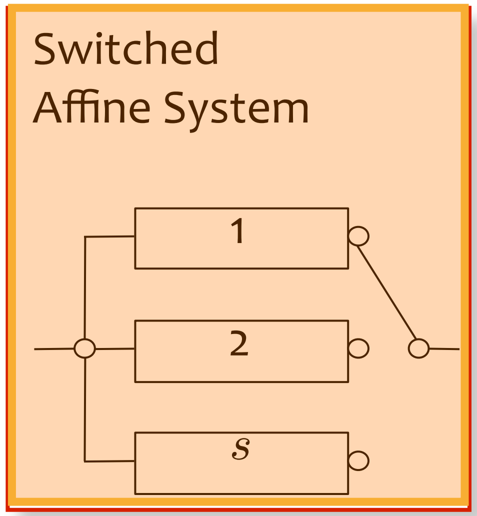
$$x_c \in \mathcal{X} \subset \mathbb{R}^{n_c}$$
$$\delta \in \{0, 1\}$$
$$z \in \mathbb{R}$$

- Assume $\mathcal{X}$ bounded and let M_1, M_2 and m_1, m_2 such that

$$M_1 > a_1' x_c + f_1 > m_1, \ \forall x_c \in \mathcal{X}$$
$$M_2 > a_2' x_c + f_2 > m_2, \ \forall x_c \in \mathcal{X}$$

- The *if-then-else* condition is equivalent to

$$\begin{cases} (m_1 - M_2)(1-\delta) + z & \leq & a_1 x_c + f_1 \\ (m_2 - M_1)(1-\delta) - z & \leq & -a_1 x_c - f_1 \\ (m_2 - M_1)\delta + z & \leq & a_2 x_c + f_2 \\ (m_1 - M_2)\delta - z & \leq & -a_2 x_c - f_2 \end{cases}$$



The state-update equation can be rewritten as a difference equation + *if-then-else* conditions:

$$z_1(k) = \begin{cases} A_1 x_c(k) + B_1 u_c(k) + f_1, & \text{if } i(k) = 1 \\ 0, & \text{otherwise}, \end{cases}$$

$$\vdots$$

$$z_s(k) = \begin{cases} A_s x_c(k) + B_s u_c(k) + f_s, & \text{if } i(k) = s, \\ 0, & \text{otherwise}, \end{cases}$$

$$x_c(k+1) = \sum_{i=1}^{s} z_i(k)$$

where $z_i(k) \in \mathbb{R}^{n_c}, i = 1, \ldots, s$

Output equations $y_c(k)=C_i x_c(k)+D_i u_c(k)+g_i$ admit a similar transformation.

Logic and Inequalities

$X_1 \vee X_2 = \text{TRUE} \longrightarrow \delta_1 + \delta_2 \geq 1, \quad \delta_1, \delta_2 \in \{0,1\}$

(Glover 1975, Williams 1977, Hooker 2000)

Any logic statement
$f(X) = \text{TRUE}$

$$\bigwedge_{j=1}^{m} \left(\vee_{i \in P_j} X_i \vee_{i \in N_j} \neg X_i \right) \quad \text{(CNF)}$$

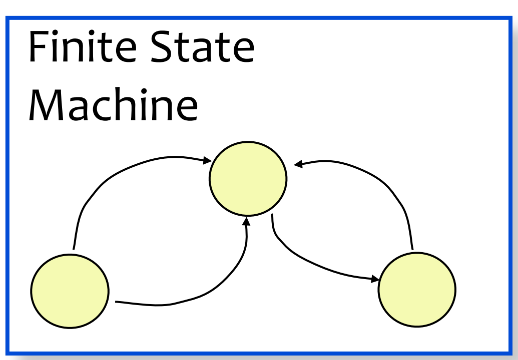
$N_j, P_j \subseteq \{1, \ldots, n\}$

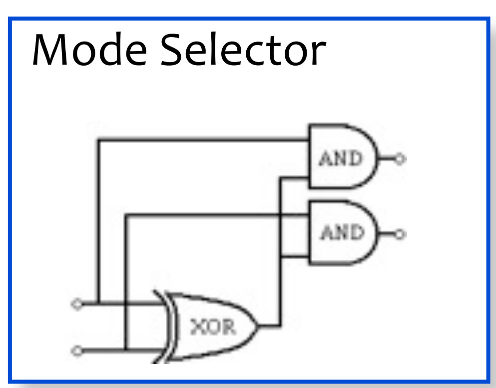
$$\begin{cases} 1 \leq \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) \\ \vdots \\ 1 \leq \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) \end{cases}$$

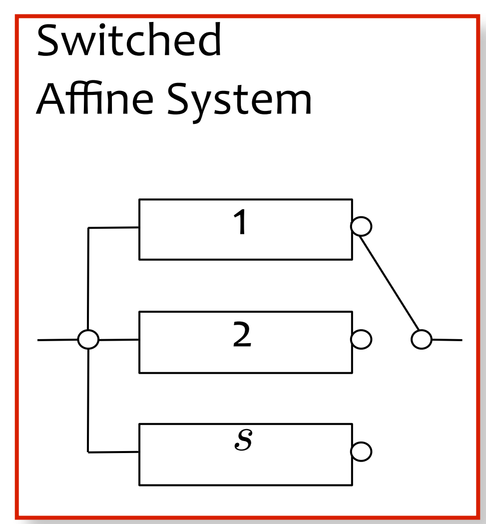
$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) \leq W^i] \qquad \begin{cases} H^i x_c(k) - W^i \leq M^i(1 - \delta_e^i) \\ H^i x_c(k) - W^i > m^i \delta_e^i \end{cases}$$

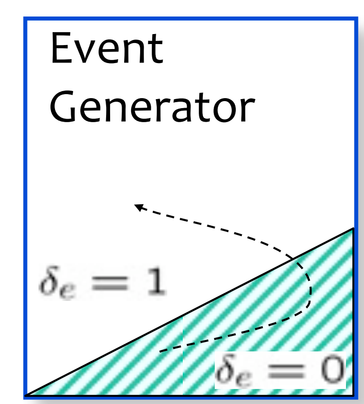
IF $[\delta = 1]$ THEN $z = a_1^T x + b_1^T u + f_1$
ELSE $z = a_2^T x + b_2^T u + f_2$

$$\begin{cases} (m_1 - M_2)(1 - \delta) + z \leq a_1 x + b_1 u + f_1 \\ (m_2 - M_1)(1 - \delta) - z \leq -a_1 x - b_1 u - f_1 \\ (m_2 - M_1)\delta + z \leq a_2 x + b_2 u + f_2 \\ (m_1 - M_2)\delta - z \leq -a_2 x - b_2 u - f_2 \end{cases}$$



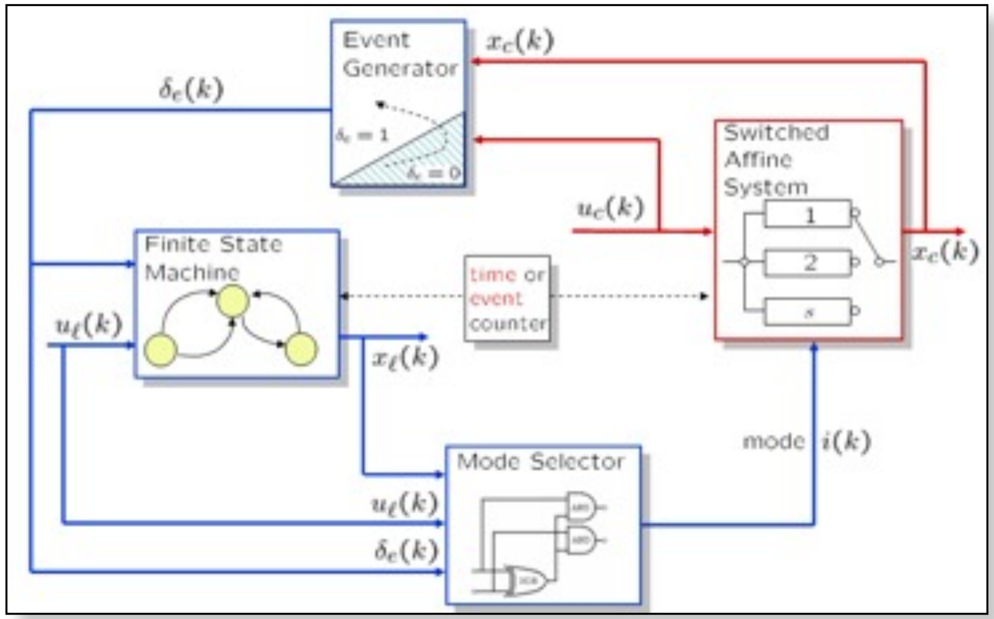






Mixed Logical Dynamical Systems

Discrete Hybrid Automaton



HYSDEL

(Torrisi, Bemporad, 2004)

Mixed Logical Dynamical (MLD) Systems

(Bemporad, Morari 1999)

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) &= Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\ E_2\delta(k) + E_3z(k) &\leq E_4x(k) + E_1u(k) + E_5 \end{aligned}$$

Continuous and binary variables

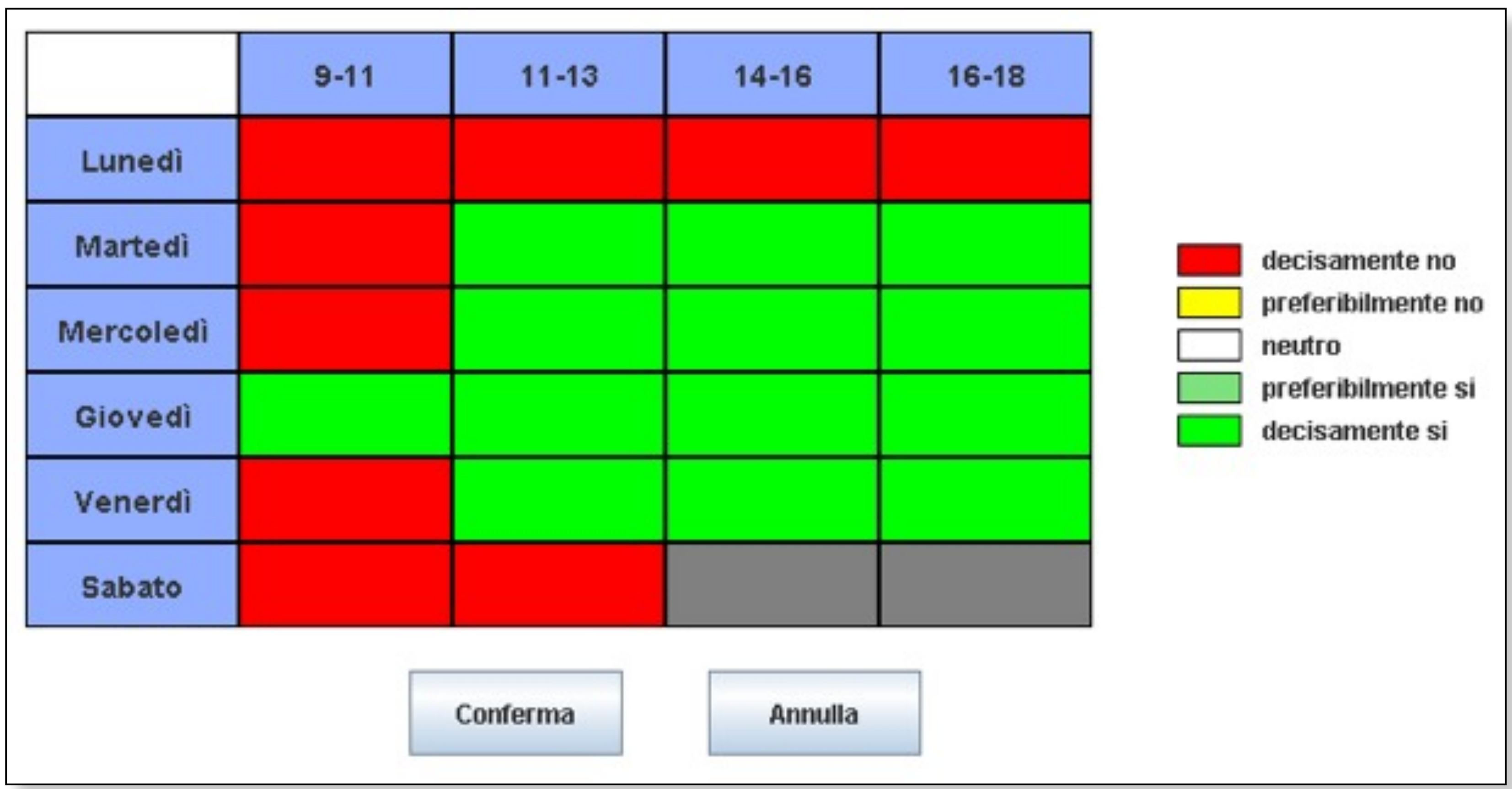
$$x \in \mathbb{R}^{n_r} \times \{0,1\}^{n_b},\ u \in \mathbb{R}^{m_r} \times \{0,1\}^{m_b}$$
$$y \in \mathbb{R}^{p_r} \times \{0,1\}^{p_b},\ \delta \in \{0,1\}^{r_b},\ z \in \mathbb{R}^{r_r}$$

- Computationally oriented (mixed-integer programming)
- Suitable for controller synthesis, verification, ...

Mixed-integer models in Operations Research

Translation of logical relations into linear inequalities is heavily used in operations research (OR) for solving complex decision problems by using mixed-integer (linear) programming (MIP)

Example: Timetable generation (for demanding professors …)



Cost function:
sum of professors' preferences

Constraints:
professors [students] cannot teach [take] two courses at the same time, etc.

Mixed-integer models in Operations Research

Translation of logical relations into linear inequalities is heavily used in operations research (OR) for solving complex decision problems by using mixed-integer (linear) programming (MIP)

Example: Timetable generation (for demanding professors …)

CPU time: 0.2 s

	8	9	10	11	12	13	14	15	16	17	18	19
lun		Sistemi Operativi (*18)					Misure per la Automazione (*7)		Ingegneria del Software (*18)			
							Basi di Dati (*18)					
mar		Basi di Dati (*3)		Sistemi Operativi (*3)			Robotica ed Automazione di Processo (*18)					
mer		Robotica ed Automazione di Processo (*8)		Misure per la Automazione (*7)				Laboratorio di Robotica e Realtà Virtuale (*15)				
		Ingegneria del Software (*18)										
gio		Basi di Dati (*3)					Sistemi Operativi (*5)					
		Laboratorio di Robotica e Realtà Virtuale (*15)										
ven		Robotica ed Automazione di Processo (*8)					Misure per la Automazione (*7)					
		Ingegneria del Software (*18)										
sab												

Effort: 5% mathematical problem setup (MILP model)
35% database & web interfaces
60% deal with professors' complaints, complaints, complaints …

Mixed-integer models in Operations Research

Translation of logical relations into linear inequalities is heavily used in operations research (OR) for solving complex decision problems by using mixed-integer (linear) programming (MIP)

Example: Optimal multi-period investments for maintenance and upgrade of electrical energy distribution networks

(Bemporad, Muñoz, Piazzesi, 2006)

Enel

A Simple Example

- System:

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if} \quad x(k) \geq 0 \\ -0.8x(k) + u(k) & \text{if} \quad x(k) < 0 \end{cases}$$

$$-10 \leq x(k) \leq 10$$

- Associate $[\delta(k) = 1] \leftrightarrow [x(k) \geq 0]$ and transform

→
$$\begin{array}{rcl} x(k) & \geq & m(1-\delta(k)) \\ x(k) & \leq & -\epsilon + (M+\epsilon)\delta(k) \end{array}$$

$M = -m = 10$

$\epsilon > 0$ "small"

- Then $x(k+1) = 1.6\delta(k)x(k) - 0.8x(k) + u(k)$

$z(k) = \delta(k)x(k)$ →
$$\begin{array}{rcl} z(k) & \leq & M\delta(k) \\ z(k) & \geq & m\delta(k) \\ z(k) & \leq & x(k) - m(1-\delta(k)) \\ z(k) & \geq & x(k) - M(1-\delta(k)) \end{array}$$

$\delta(k) \in \{0, 1\}$

- Rewrite as a linear equation

→
$$x(k+1) = 1.6z(k) - 0.8x(k) + u(k)$$

(Note: nonlinear system $x(k+1) = 0.8|x(k)| + u(k)$)

HYSDEL

(HYbrid Systems DEscription Language)

- Describe *hybrid systems:*

– Automata
– Logic
– Lin. Dynamics
– Interfaces
– Constraints

(Torrisi, Bemporad, 2004)

- Automatically generate MLD models in Matlab

Download: http://www.dii.unisi.it/hybrid/toolbox

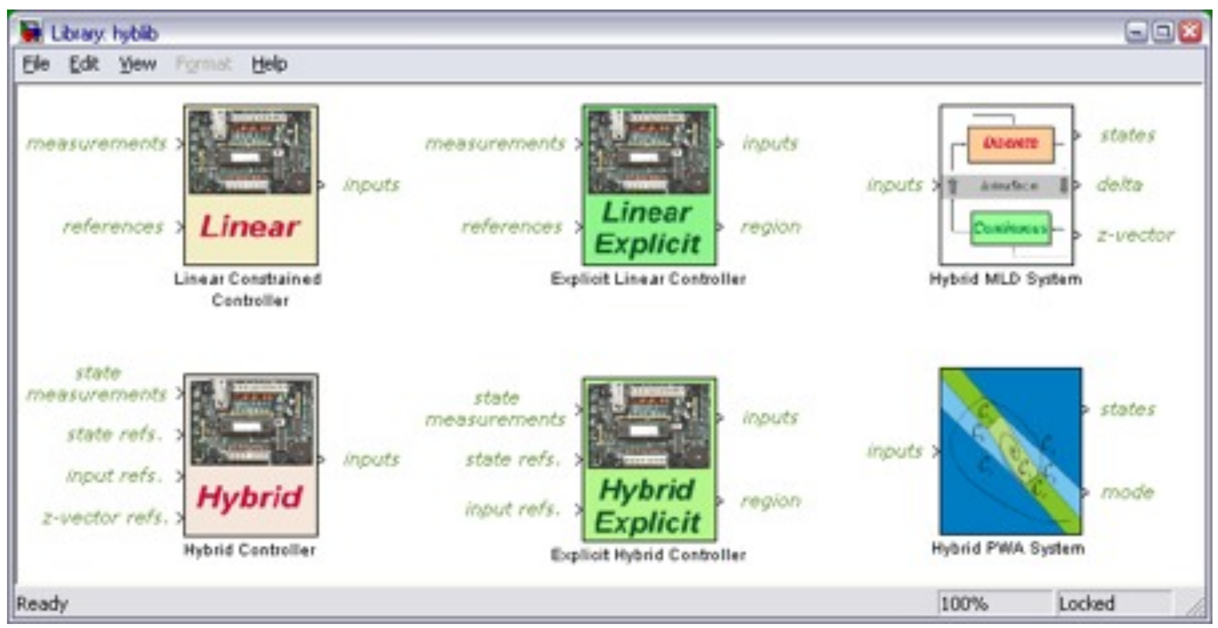
Reference: http://control.ethz.ch/~hybrid/hysdel

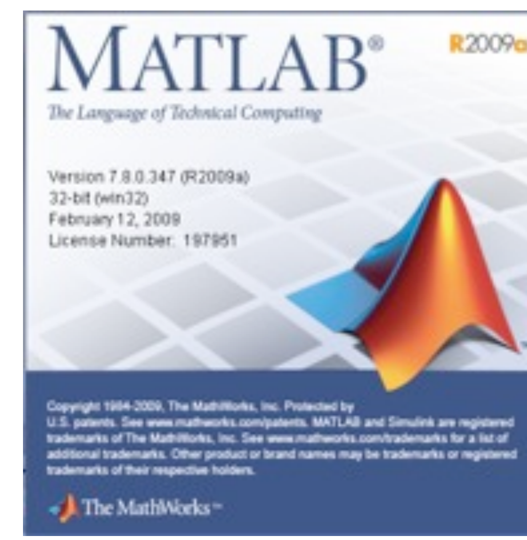
Hybrid Toolbox for Matlab

Features:

(Bemporad, 2003-2009)

- Hybrid models (MLD and PWA): design, simulation, verification
- Control design for linear systems w/ constraints and hybrid systems (on-line optimization via QP/MILP/MIQP)
- Explicit control (via multi-parametric programming)
- C-code generation
- Simulink library





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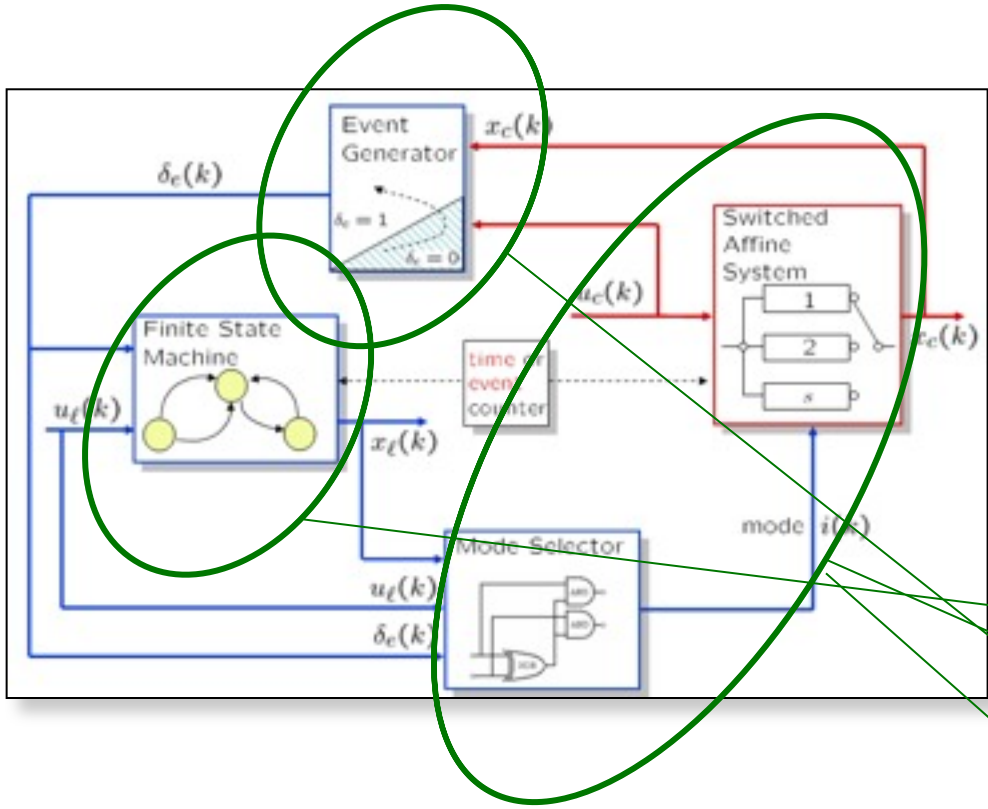
DHA and HYSDEL Models



Additional relations constraining system's variables

```
SYSTEM name {
   INTERFACE {
      STATE {
         REAL xc [xmin,xmax];
         BOOL xl;  }
      INPUT {
         REAL uc [umin,umax];
         BOOL ul;  }
      PARAMETER {
         REAL param1 = 1;}
   } /* end of interface */

   IMPLEMENTATION {
      AUX { BOOL d;
            REAL z;  }

      AUTOMATA { xl = xl & ~ul; }

      DA { z = { IF d THEN 2*xc ELSE -xc };   }

      AD { d = xc - 1 <= 0; }

      CONTINUOUS {
         xc = z; }

      MUST {
         xc + uc <= 2;
         ~(xl & ul);  }
   } /* end implementation */
} /* end system */
```

Example 1: Definition of Event Vars.

$$[\delta = 1] \leftrightarrow [h \geq h_{\text{max}}]$$

$\delta \in \{0, 1\}$

$h \in \mathbb{R}$

```
AD { delta = hmax - h <= 0; }
```

Example 2: Nonlinear (PWA) Functions

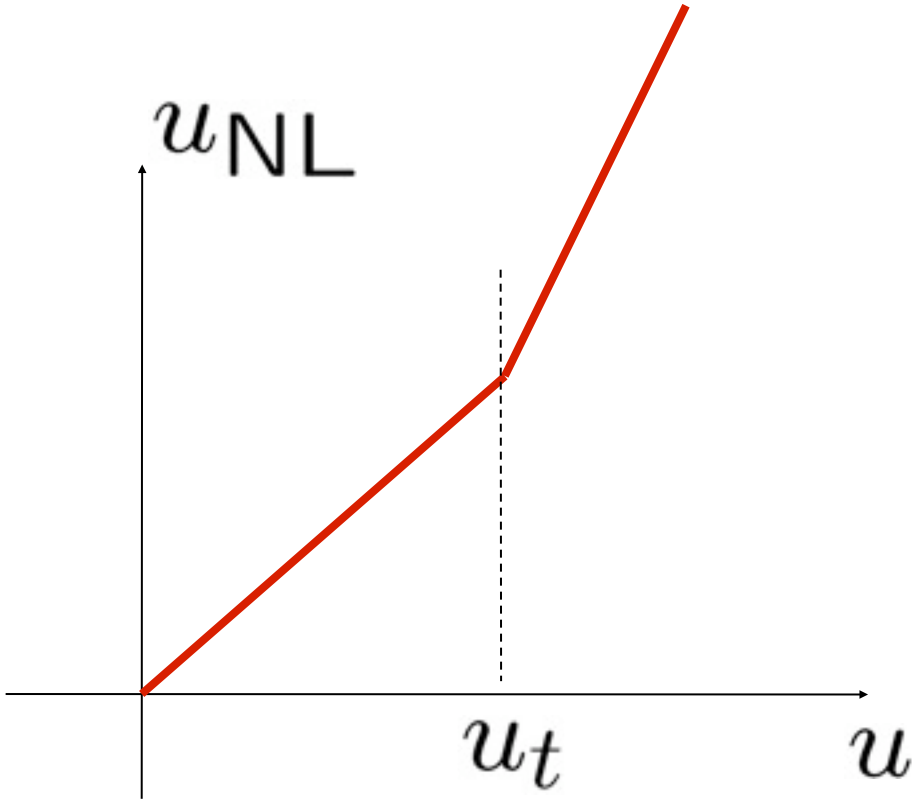
Nonlinear amplification unit

$$u_{\rm NL}(k) = \begin{cases} u(k) & \text{if } u(k) < u_t \\ 2.3u(k) - 1.3u_t & \text{if } u(k) \geq u_t \end{cases}$$

```
DA { unl = { IF th THEN 2.3*u - 1.3*ut
                   ELSE u }; }


AD { th = ut - u <= 0; }
```

$$[t_h = 1] \leftrightarrow [u \geq u_t]$$



Example 3: Logical relations

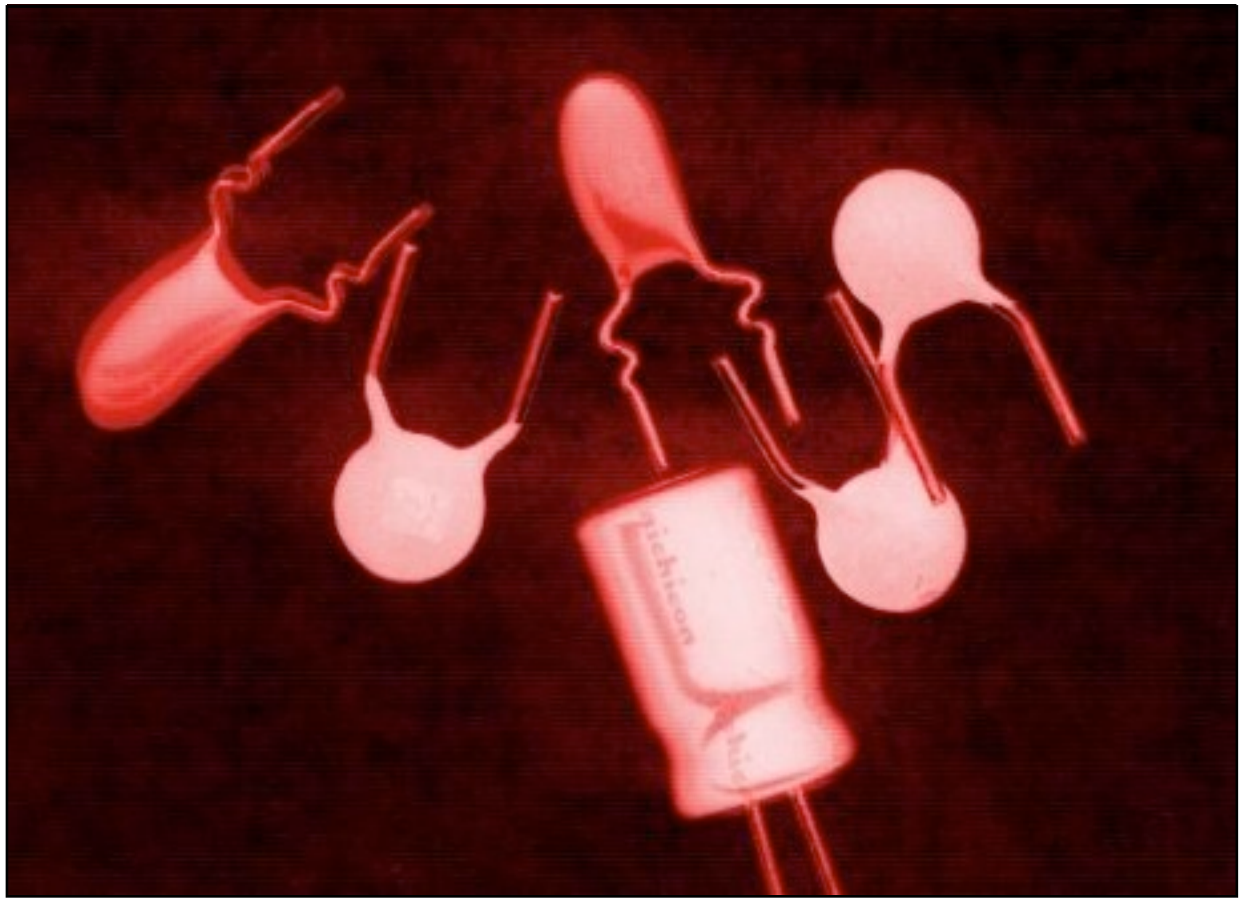
Rule: brake if there is an alarm signal, but only if the train is not on fire in a tunnel

$$\delta_{\text{brake}}, \delta_{\text{alarm}}, \delta_{\text{tunnel}}, \delta_{\text{fire}} \in \{0, 1\}$$

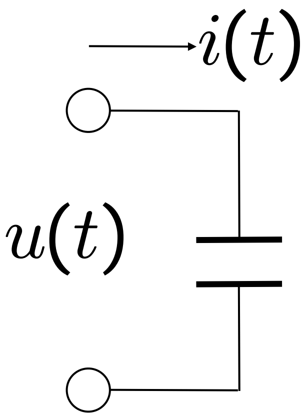
$$\delta_{\text{brake}} = \delta_{\text{alarm}} \wedge \neg(\delta_{\text{tunnel}} \wedge \delta_{\text{fire}})$$

```
LOGIC {
   brake =  alarm & ~(tunnel & fire);
}
```

Example 4: Continuous dynamics



$$i(t) = C\frac{du(t)}{dt}$$



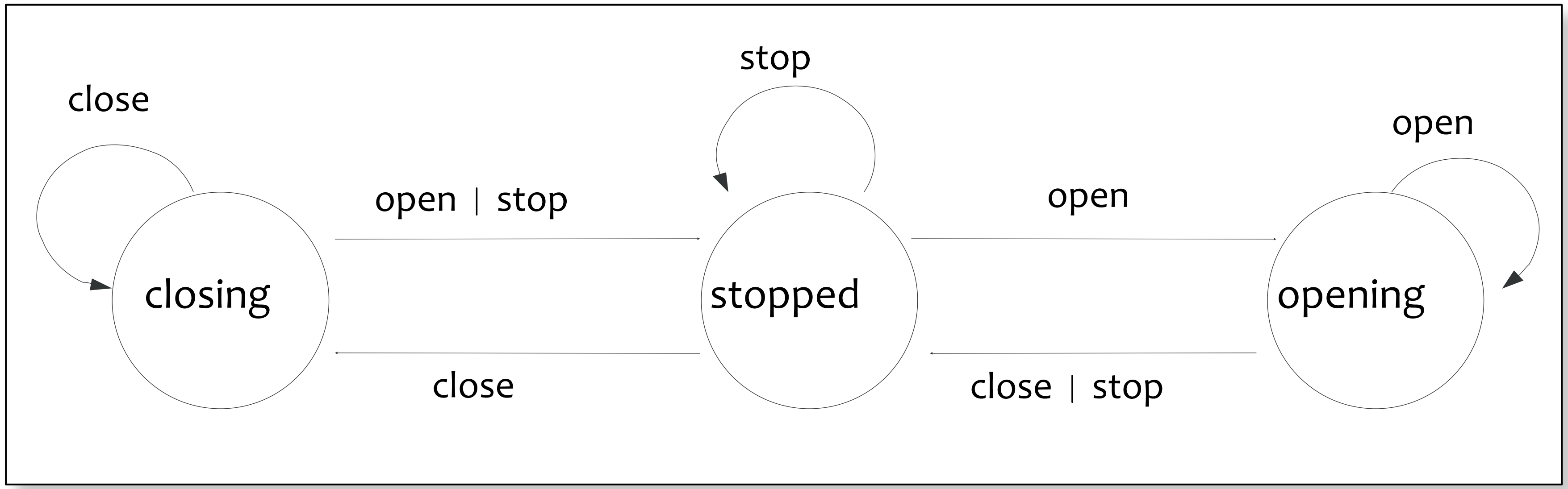
Apply forward difference rule:

$$u((k+1)T_s) = u(kT_s) + \frac{T}{C} i(kT_s)$$

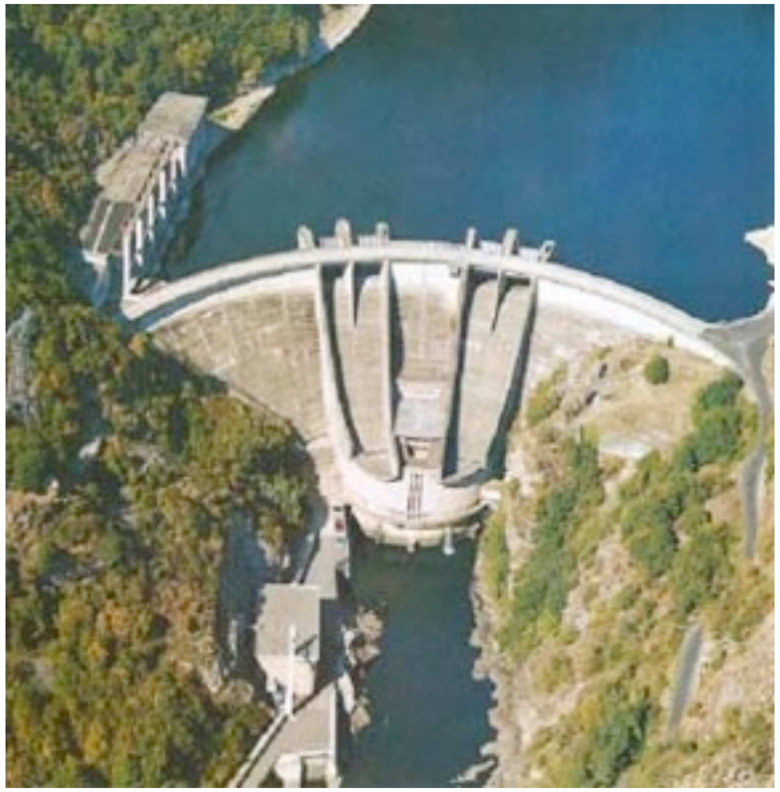
```
CONTINUOUS {
        u = u + Ts*iC*i;
}
```

Note: **iC** = 1/C is used due to a bug in HYSDEL, that cannot handle division by a scalar parameter.

Example 5: Automaton



Flow control through a dam



binary inputs: $u_{\text{open}}, u_{\text{close}}, u_{\text{stop}} \in \{0, 1\}$

binary states: $x_{\text{opening}}, x_{\text{closing}}, x_{\text{stopped}} \in \{0, 1\}$

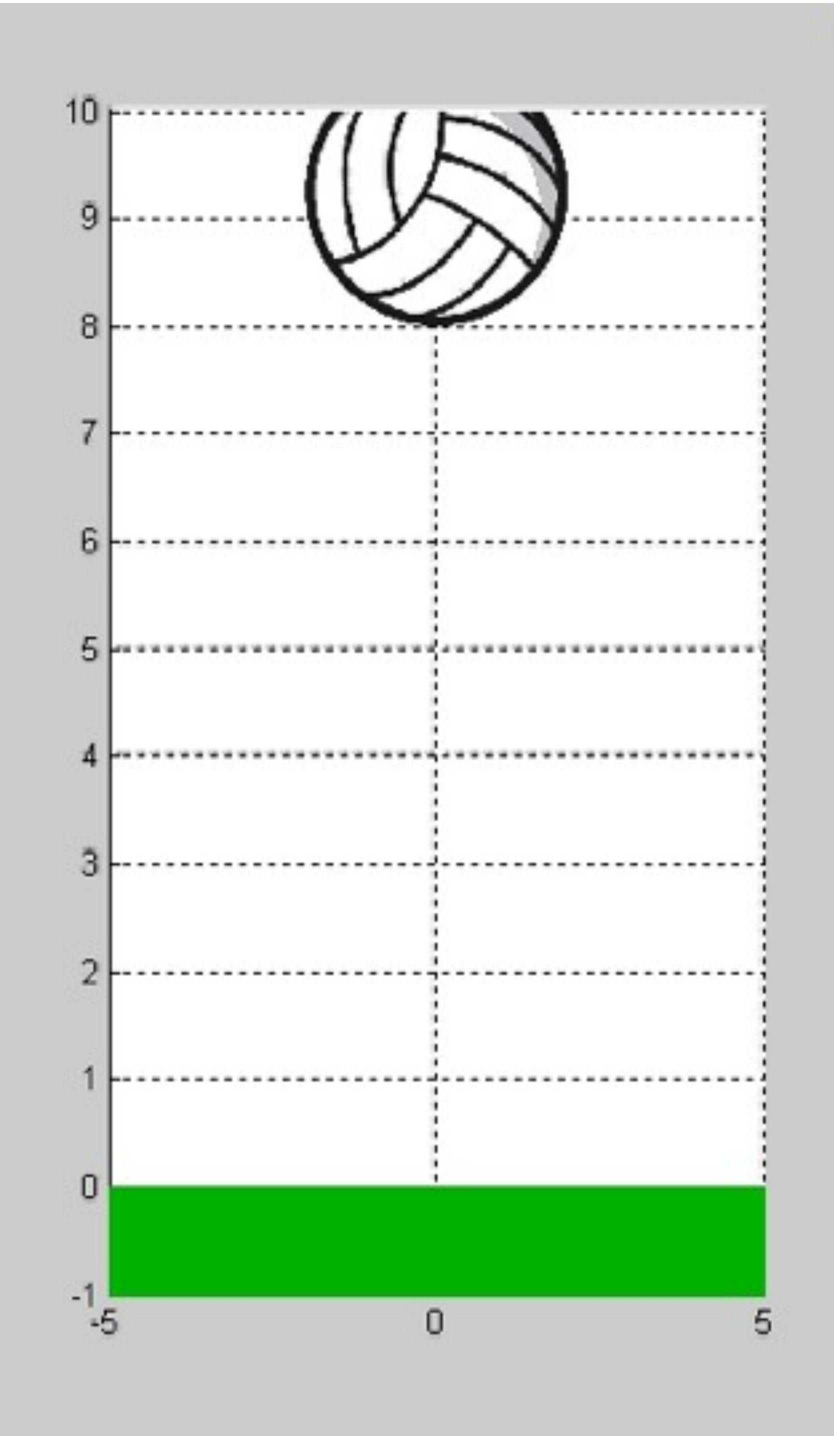
```
AUTOMATA {
   xclosing = (uclose & xclosing) | (uclose & xstopped);
   xstopped = ustop | (uopen & xclosing) | (uclose & xopening);
   xopening = (uopen & xstopped) | (uopen & xopening);
}
```

Example 6: Impose a constraint

$$0 \le h(k) \le h_{\max}$$

```
MUST {
     h - hmax <= 0;
     -h        <= 0;
}
```

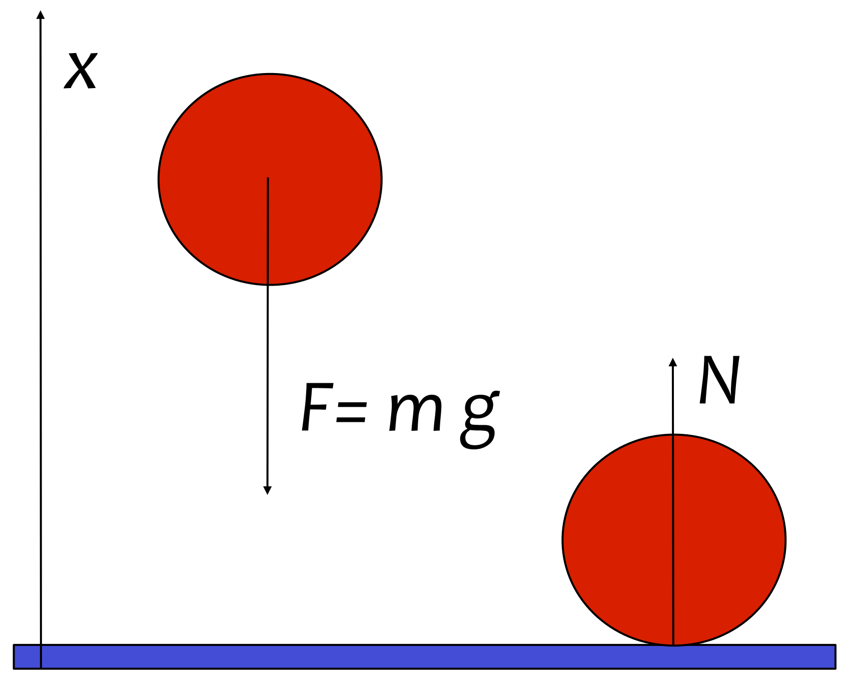
Example: Bouncing Ball



$$\ddot{x} = -g$$

$$x \leq 0 \Rightarrow \dot{x}(t^+) = -(1-\alpha)\dot{x}(t^-)$$

$$\alpha \in [0, 1]$$

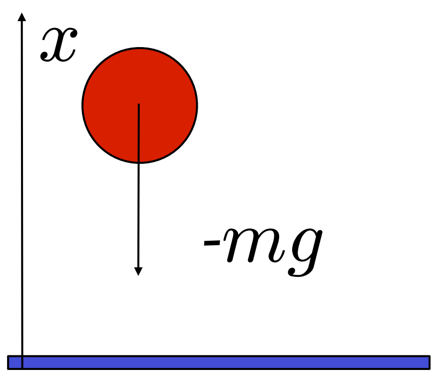


How to model the bouncing ball as a discrete-time hybrid system ?

Bouncing Ball – Time Discretization

$x(k) > 0$:

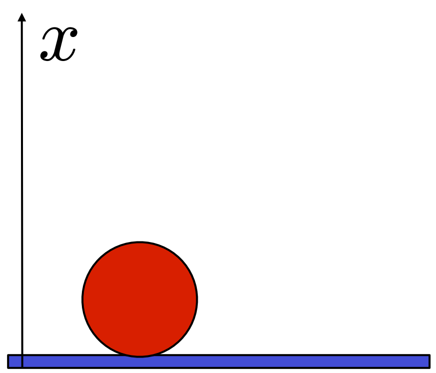
$$v(k) \approx \frac{x(k) - x(k-1)}{T_s} \qquad -g = \ddot{x}(k) \approx \frac{v(k) - v(k-1)}{T_s}$$



$$\begin{cases} v(k+1) & = v(k) - T_s g \\ x(k+1) & = x(k) + T_s v(k+1) \\ & = x(k) + T_s v(k) - T_s^2 g \end{cases}$$

$x(k) \leq 0$:

$$v(k) = -(1-\alpha)v(k-1) \qquad \begin{aligned} x(k+1) & = x(k-1) \\ & = x(k) - T_s v(k) \end{aligned}$$



$$\begin{cases} v(k+1) & = -(1-\alpha)v(k) \\ x(k+1) & = x(k) - T_s v(k) \end{cases}$$

HYSDEL - Bouncing Ball

```
SYSTEM bouncing_ball {
INTERFACE {
/* Description of variables and constants */
        STATE { REAL height [-10,10];
                REAL velocity [-100,100];    }


        PARAMETER {
                REAL g;
                REAL alpha;  /* 0=elastic, 1=completely anelastic */
                REAL Ts; }
}
IMPLEMENTATION {
        AUX {   REAL hnext;
                REAL vnext;
                BOOL negative;    }


        AD {    negative = height <= 0; }


        DA {    hnext = {  IF negative THEN height-Ts*velocity
                        ELSE height+Ts*velocity-Ts*Ts*g};
                vnext = {  IF negative THEN -(1-alpha)*velocity
                        ELSE velocity-Ts*g};   }
        CONTINUOUS {
                height   = hnext;
                velocity = vnext;}
}}
```

go to demo /**demos/hybrid/bball.m**

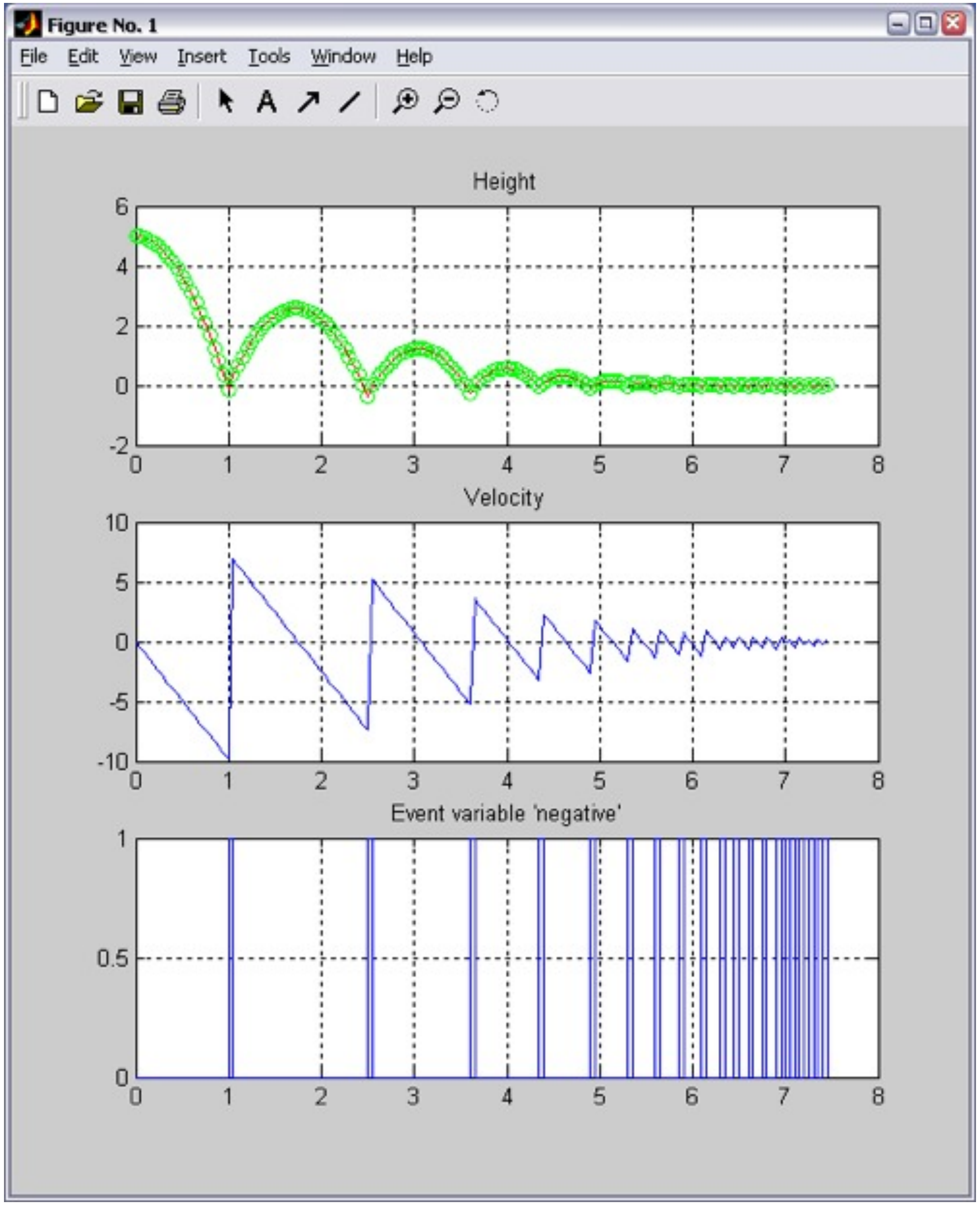
Bouncing Ball

```
>>Ts=0.05;
>>g=9.8;
>>alpha=0.3;

>>S=mld('bouncing_ball',Ts);

>>N=150;
>>U=zeros(N,0);
>>x0=[5 0]';

>>[X,T,D]=sim(S,x0,U);
```

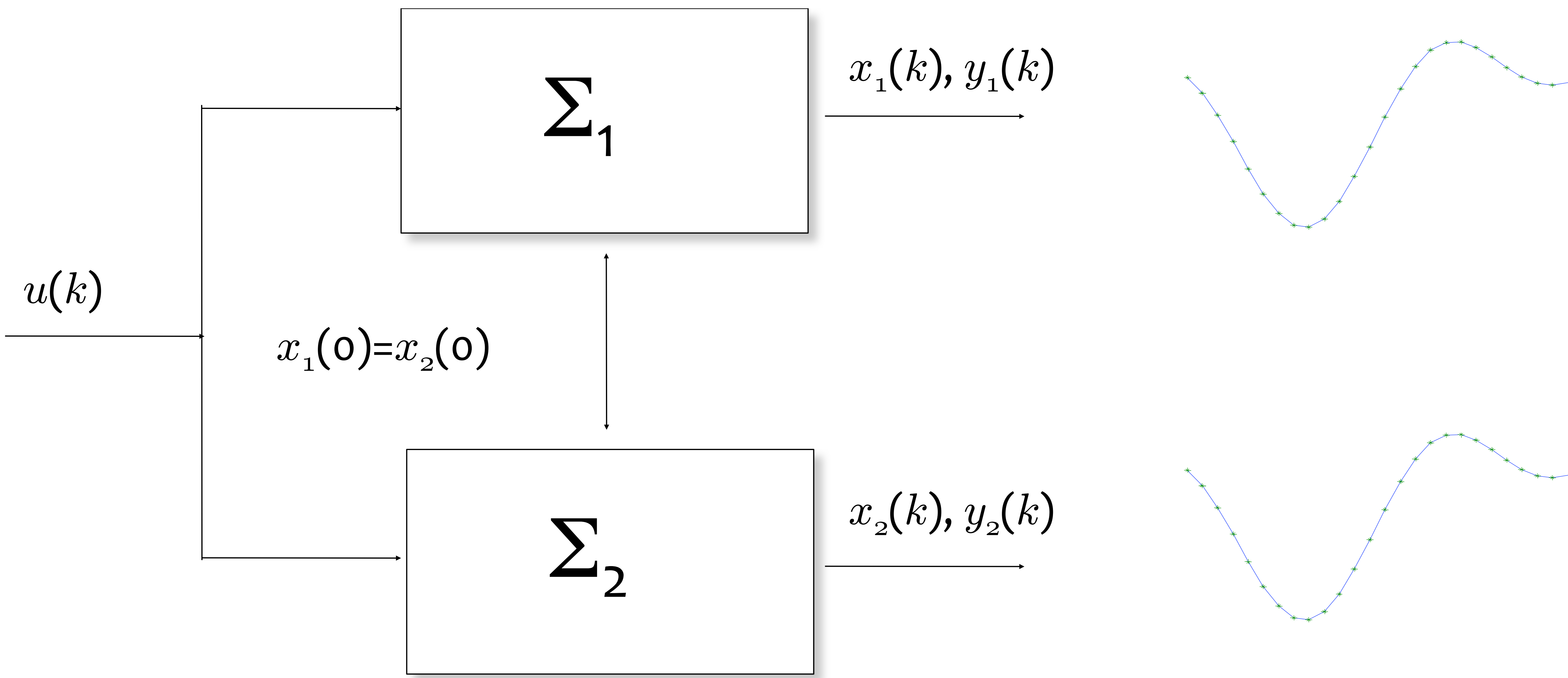


Note: no Zeno effects in discrete time !

Realization and Transformations (State-Space Hybrid Models)

Equivalences of Hybrid Models

Definition 1 *Two hybrid systems Σ_1, Σ_2 are* equivalent *if for all initial conditions $x_1(0) = x_2(0)$ and input $\{u_1(k)\}_{k\in\mathbb{Z}_+} = \{u_2(k)\}_{k\in\mathbb{Z}_+}$ then $x_1(k) = x_2(k)$ and $y_1(k) = y_2(k)$, for all $k \in \mathbb{Z}_+$.*



MLD and PWA Systems

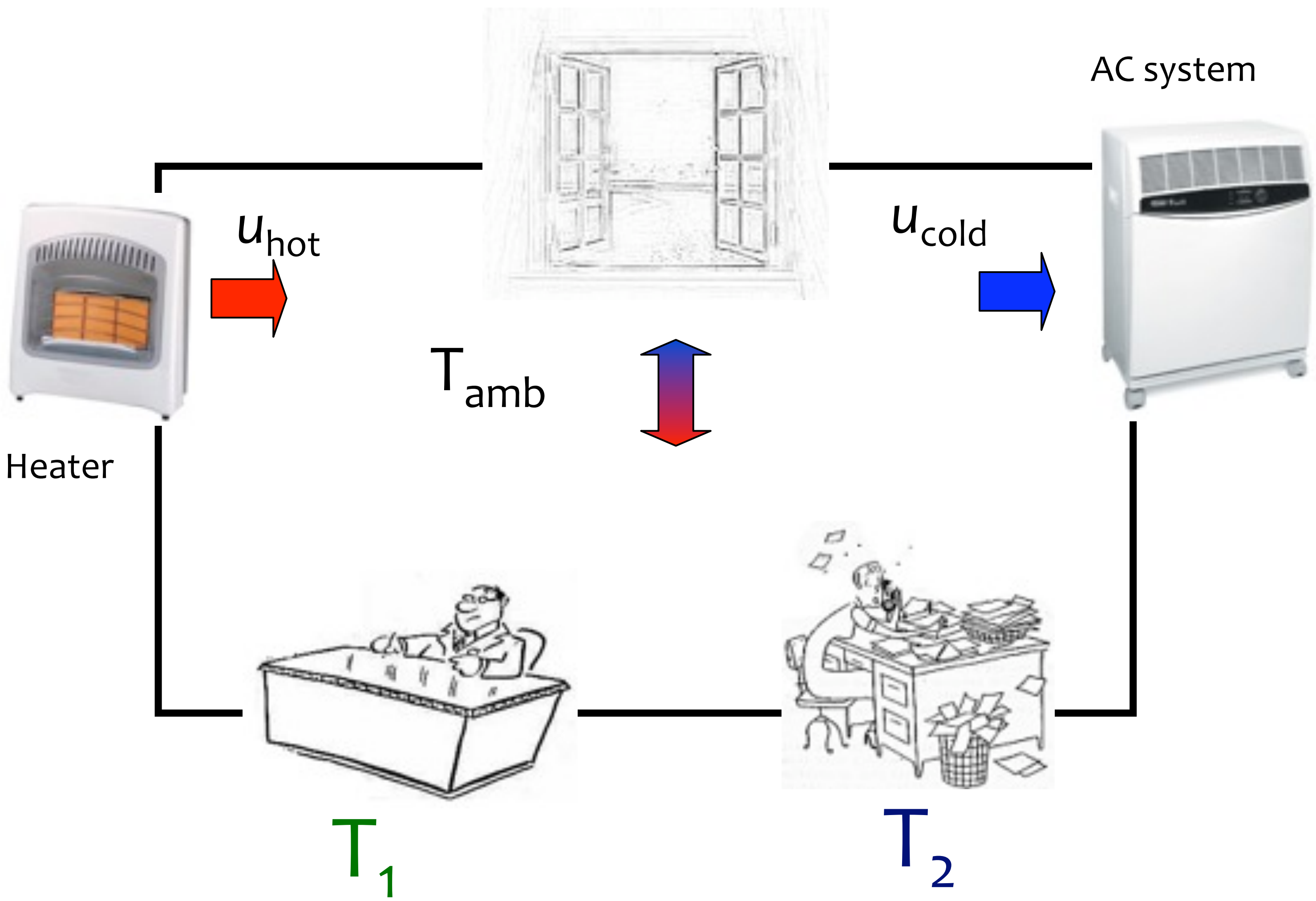
Theorem MLD systems and PWA systems are equivalent

(Bemporad, Ferrari-Trecate, Morari, *IEEE TAC,2000*)

- Proof is constructive: given an MLD system it returns its equivalent PWA form
- Drawback: it needs the enumeration of all possible combinations of binary states, binary inputs, and δ variables
- Most of such combinations lead to empty regions
- Efficient algorithms are available for converting MLD models into PWA models avoiding such an enumeration:

- A. Bemporad, "*Efficient Algorithms for Converting Mixed Logical Dynamical Systems into an Equivalent Piecewise Affine Form*", IEEE Trans. Autom. Contr., 2004.
- T. Geyer, F.D. Torrisi and M. Morari, "*Efficient Mode Enumeration of Compositional Hybrid Models*", HSCC'03

Example: Room Temperature



Hybrid dynamics

- #1 turns the heater (A/C) on whenever he is cold (hot)
- If #2 is cold he turns the heater on, unless #1 is hot
- If #2 is hot he turns A/C on, unless #2 is cold
- Otherwise, heater and A/C are off

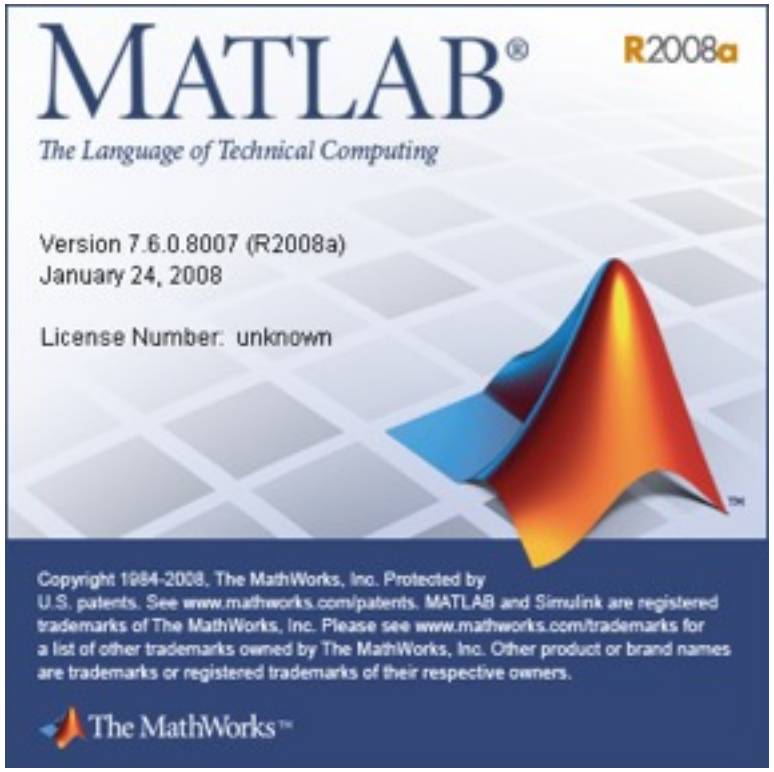
- $\dot{T}_1 = -\alpha_1(T_1 - T_{\text{amb}}) + k_1(u_{\text{hot}} - u_{\text{cold}})$ (body temperature dynamics of #1)
- $\dot{T}_2 = -\alpha_2(T_2 - T_{\text{amb}}) + k_2(u_{\text{hot}} - u_{\text{cold}})$ (body temperature dynamics of #2)

go to demo **`/demos/hybrid/heatcool.m`**

HYSDEL Model

```
SYSTEM heatcool {

INTERFACE {
    STATE { REAL T1 [-10,50];
            REAL T2 [-10,50];
          }
    INPUT { REAL Tamb [-10,50];
          }
    PARAMETER {
        REAL Ts, alpha1, alpha2, k1, k2;
        REAL Thot1, Tcold1, Thot2, Tcold2, Uc, Uh;
          }
}
IMPLEMENTATION {
        AUX { REAL uhot, ucold;
              BOOL hot1, hot2, cold1, cold2;
        }
        AD  { hot1 = T1>=Thot1;
              hot2 = T2>=Thot2;
              cold1 = T1<=Tcold1;
              cold2 = T2<=Tcold2;
        }
        DA  { uhot = {IF cold1 | (cold2 & ~hot1) THEN Uh ELSE 0};
              ucold = {IF hot1 | (hot2 & ~cold1) THEN Uc ELSE 0};
        }
        CONTINUOUS { T1 = T1+Ts*(-alpha1*(T1-Tamb)+k1*(uhot-ucold));
                     T2 = T2+Ts*(-alpha2*(T2-Tamb)+k2*(uhot-ucold));
        }
    }
}
```



Hybrid Toolbox for Matlab

http://www.dii.unisi.it/hybrid/toolbox

`>>S=mld('heatcoolmodel',Ts)` get the MLD model in Matlab

`>>[XX,TT]=sim(S,x0,U);` simulate the MLD model

Hybrid MLD Model

- MLD model

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \\
y(k) &= Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) \\
E_2\delta(k) + E_3z(k) &\leq E_1u(k) + E_4x(k) + E_5
\end{aligned}
$$

- 2 continuous states: (temperatures T_1,T_2)
- 1 continuous input: (room temperature T_{amb})
- 2 auxiliary continuous vars: (power flows u_{hot}, u_{cold})
- 6 auxiliary binary vars: (4 thresholds + 2 for OR condition)
- 20 mixed-integer inequalities

Possible combination of integer variables: $2^6 = 64$

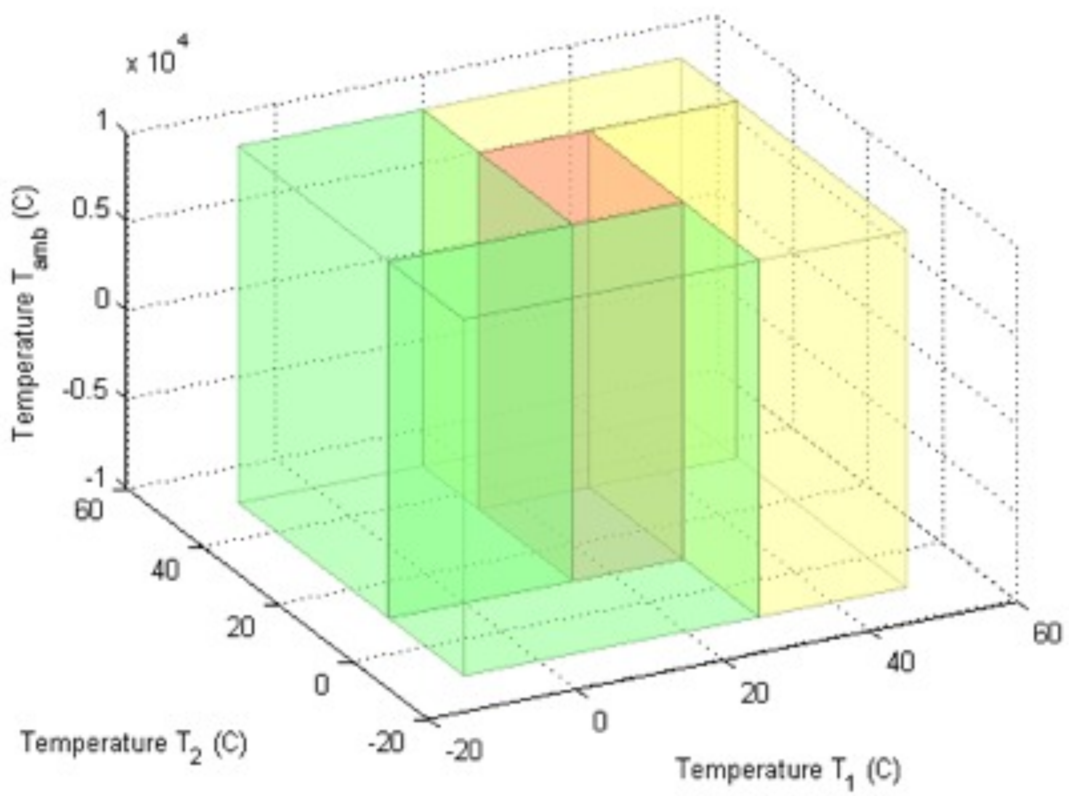
Hybrid PWA Model

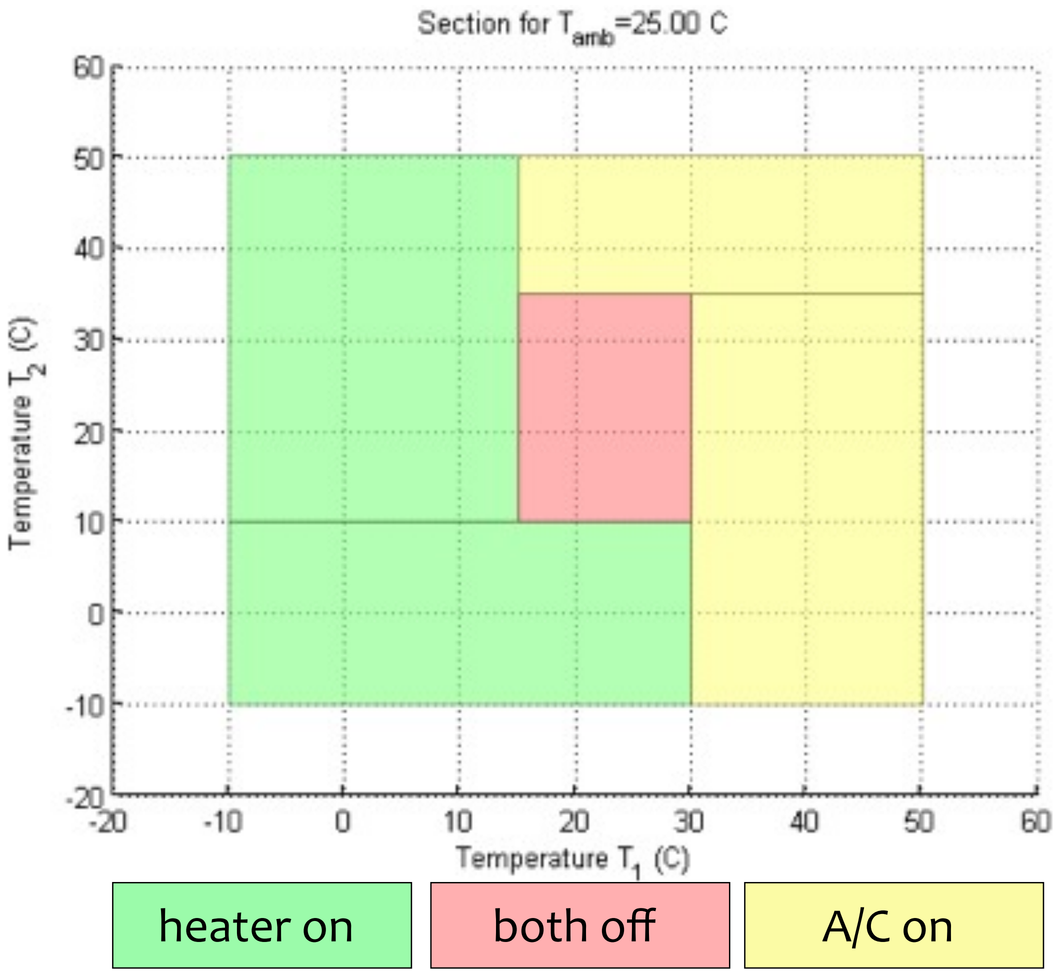
- PWA model

```
>>P=pwa(S);
```

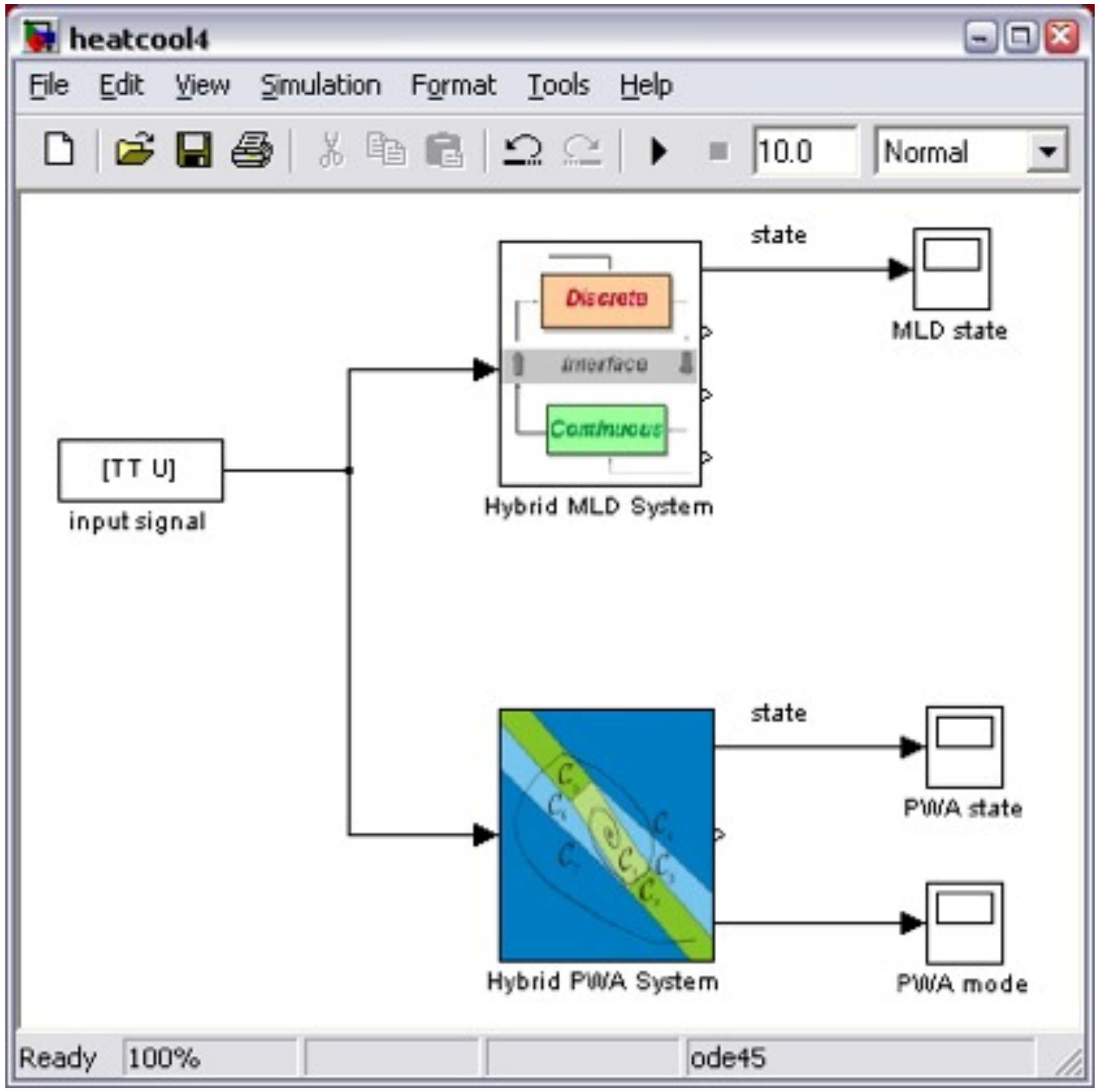
$$\begin{aligned} x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ i(k) \ \text{s.t.} & \ H_{i(k)}x(k) + J_{i(k)}u(k) \le K_{i(k)} \end{aligned}$$

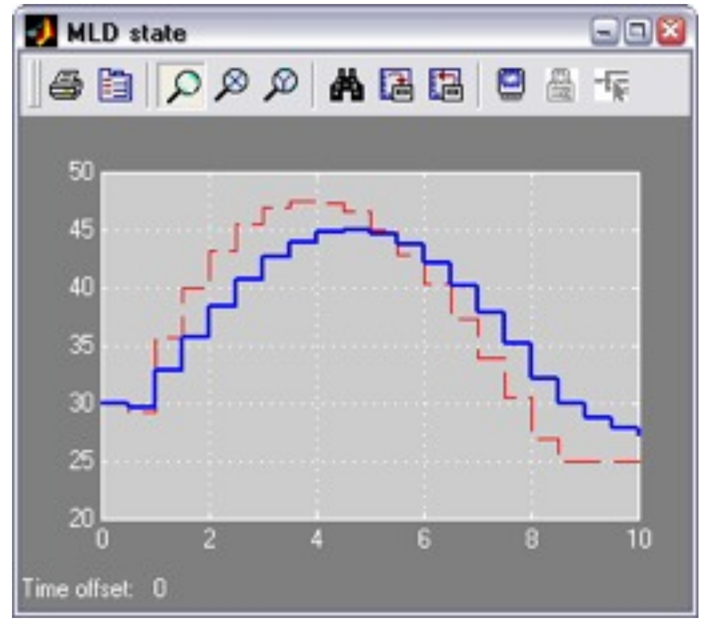
- 2 continuous states:
  (temperatures T_1,T_2)
- 1 continuous input:
  (room temperature T_{amb})

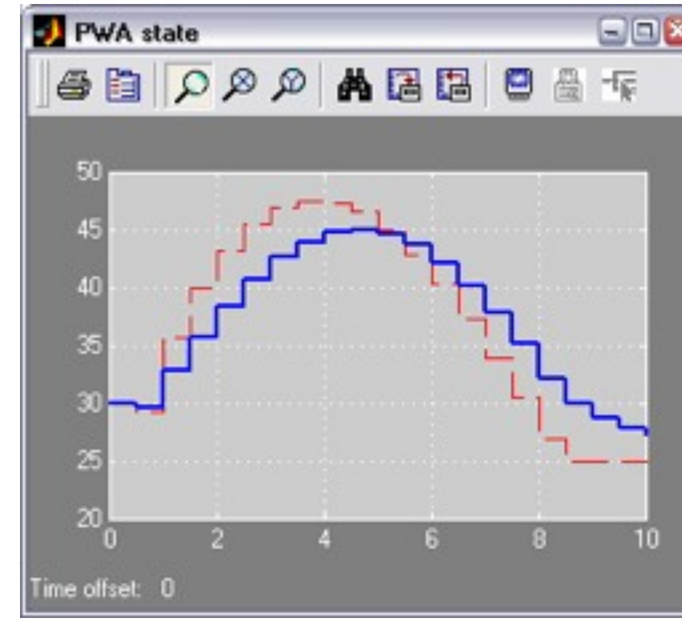


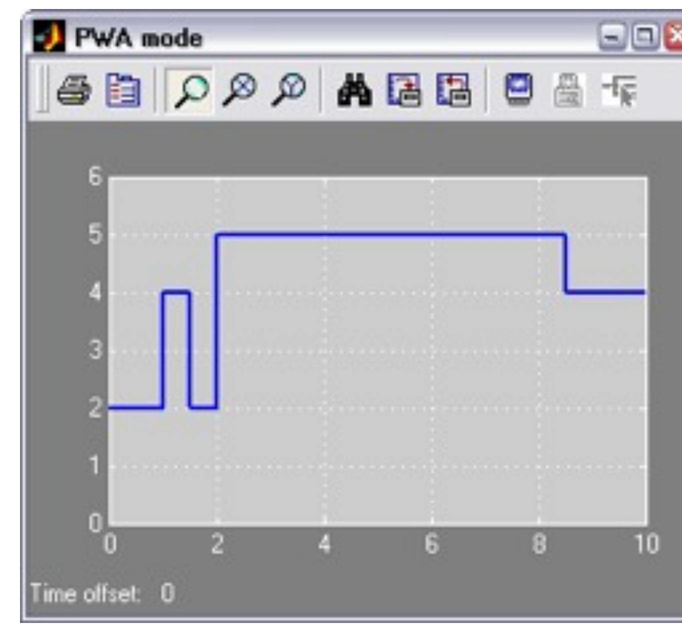


- 5 polyhedral regions
  (partition does not depend on input)









MLD and PWA models are equivalent

Using MLD→PWA for Model Checking

- Assume plant and controller can be modeled as DHA:

  - Plant = PWA approximation (e.g.: NL switched model)

  - Controller = switched linear controller (e.g: a combinations of threshold conditions, logics, linear feedback laws, ...)

- Write HYSDEL model, convert to MLD, then to PWA

- The resulting PWA map tells you how the closed-loop behaves in different regions of the state-space

Why are we interested in MLD/PWA models ?

Many problems of analysis:

- Stability (Johansson, Rantzer, 1998)
- Safety / Reachability (Torrisi, Bemporad, 2001) (Bemporad, Ferrari, Morari, 2000)
- Observability (Bemporad, Ferrari, Morari, 2000)
- Passivity (Bemporad, Bianchini, Brogi, 2006)
- Well-posedness (Heemels, 1999)

Many problems of synthesis:

- Controller design (Bemporad, Morari, 1999)
- Filter design (state estimation/fault detection) (Bemporad, Mignone, Morari, 1999) (Ferrari, Mignone, Morari, 2002)

can be solved through mathematical programming

(However, all these problems are NP-hard !)