University of Trento

Prof. Alberto Bemporad June 27, 2011

Automatic Control 2: Solutions

Exercise 1 (10 points)

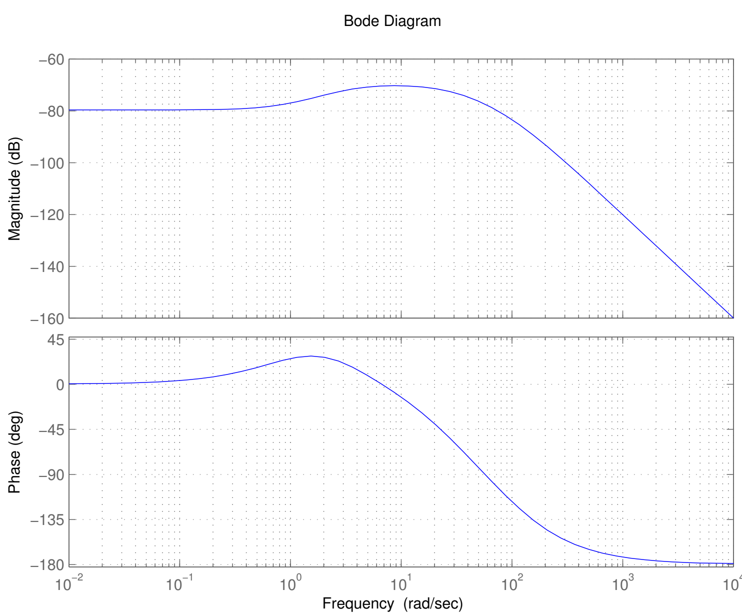
The Bode form of $P(s)$ is the following

$$P(s) = \frac{5}{48000} \frac{(1+\frac{1}{5}s)(1+s)}{(1+\frac{1}{4}s)^2(1+\frac{1}{30}s)(1+\frac{1}{100}s)}$$

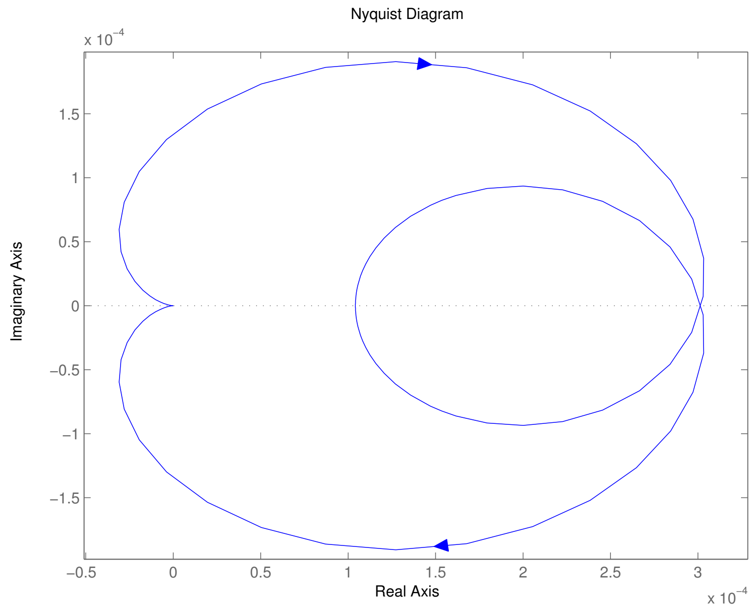
A table of the magnitude and phase contribution of the basic components of $P(s)$ is the following

	Magnitude dB	Phase deg
K_B	$20\log_{10}(5/48000) = -79.64$	0
$(s+5)$	$+20$ dB/dec, $\omega_{z_1} = 5$	$+\frac{\pi}{2}$
$(s+1)$	$+20$ dB/dec, $\omega_{z_2} = 1$	$+\frac{\pi}{2}$
$(s+4)^2$	-40 dB/dec, $\omega_{p_1} = 4$	$-\pi$
$(s+30)$	-20 dB/dec, $\omega_{p_2} = 30$	$-\frac{\pi}{2}$
$(s+100)$	-20 dB/dec, $\omega_{p_3} = 100$	$-\frac{\pi}{2}$

The real Bode diagram is depicted in the following Figure



The Nyquist diagram is shown in the following figure



As the nominal plant is asymptotically stable, exploiting the Nyquist criterion, the stability of the closed loop is ensured iff there are no (clockwise) encirclements of the point -1, while moving the point $-1/K$. The closed loop is stable iff $K > 0$.

<u>Exercise</u> 2 (10 points)

The design specifications can be translated as follows:

- $M_r \simeq 1.5$ dB
  By recalling the approximate formulae $M_p = \frac{2.3 - M_r}{1.25}$, where M_r is not in dB, the desired phase margin is $M_p \simeq 51$ deg
- $t_r \leq 0.01$ s
  Since $t_r B_3 \simeq 3$ and $\omega_c = [0.5 \ldots 0.8]\, B_3$, assuming $t_r = 0.01$ s, the desired bandwidth is $B_3 = 300$ dB and $\omega_c = [150 \ldots 240]$ rad/s. A good choice is $\omega_c = 180$ rad/s

By looking at the phase and the magnitude of $P(j\omega_c)$, the gap to compensate can be calculated as follows

$$\angle(P(j180)) = -176.81 \text{ deg}, \ |P(j180)|_{\mathrm{dB}} = -88.63 \text{ dB}$$

$$\Delta M = 0 \text{ dB} - (-88.63) \text{ dB}, \ \Delta\phi = M_p - (180 - 176.81) \simeq 48 \text{ deg}$$

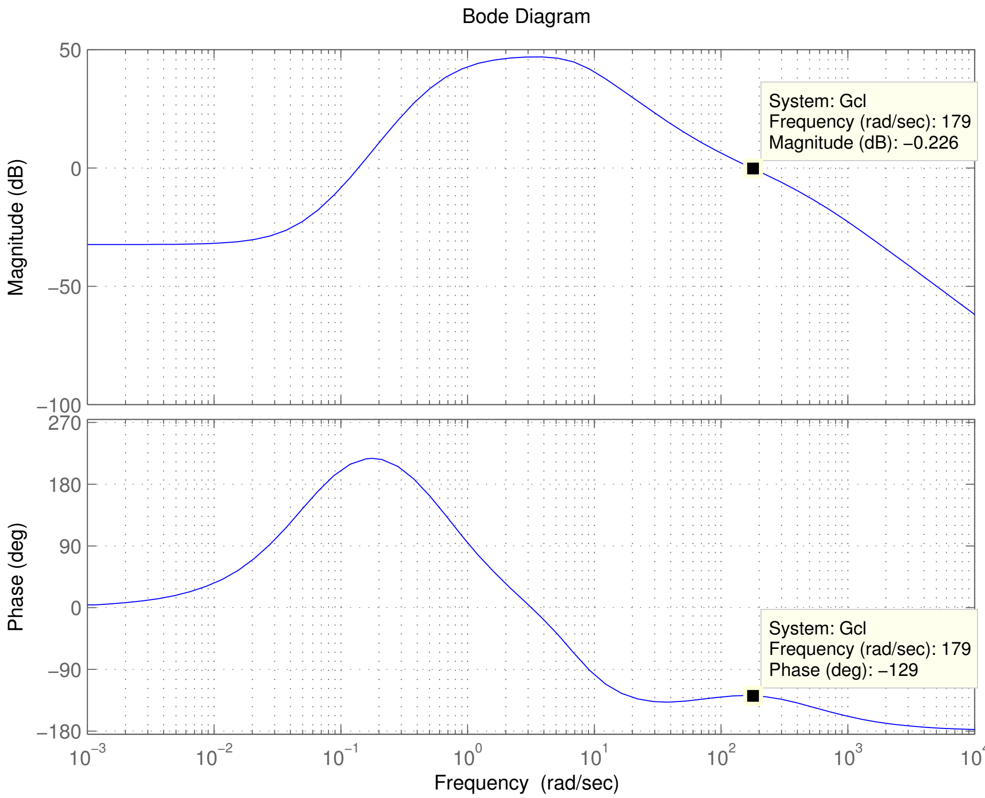
In order to satisfy the phase specification, we choose a *lead* network with $\alpha_1 = 0.15$, centered in w_c for which $\tau_1 = \frac{1}{w_c\sqrt{\alpha_1}} = 0.0143$. This network increases the phase for about 48 deg and the magnitude for about 1.35 dB.
The phase specification is satisfied, while the new magnitude specification is $\Delta M' = \Delta M - 1.35 \simeq 80$ dB. In order to shift the magnitude diagram while preserving the phase diagram at $\omega_c = 180$ rad/s, four *lead* network are chosen at $\omega_{low} = \omega_c/1000$ rad/s, with $\alpha_2 = 0.1$ and $\tau_2 = \frac{1}{w_{low}\sqrt{\alpha_2}} = 17.5682$.
The corresponding network functions have the form

$$C_{lead}(s) = \frac{1 + \tau_i s}{1 + \alpha_i \tau_i s}, \quad i = 1, 2$$

At this point, the design specifications are verified for the closed loop transfer function

$$F(s) = \frac{L(s)}{1 + L(s)}, \text{ where } L(s) = C_{lead_1}(s) C_{lead_2}(s)^4 G(s)$$

In the following Figure is shown the Bode diagram of $L(s)$



Exercise 3 (8 points)

The resulting autonomous system for $u(t) = 0$ is

$$\dot{x}(t) = -x(t) - x^3(t)$$

for which the origin is an equilibrium point. Taking for instance $V(x) = x^2/2$, which is positive definite for all $x \in \mathbb{R}$ (i.e. $V(x) > 0\ \forall x \neq 0,\ V(0) = 0$), one obtains $\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = x(-x - x^3) = -x^2 - x^4$, which is negative definite for all $x \in \mathbb{R}$ (i.e. $\dot{V}(x) < 0\ \forall x \neq 0,\ \dot{V}(0) = 0$). Therefore, $V(x)$ is a Lyapunov function for the given system, which proves that the origin is an asymptotically stable equilibrium point with domain of attraction equal to $\mathbb{R}$.

As for the feedback linearization, the system can be represented as $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, with $f(x(t)) = -x(t) - x^3(t)$, and $g(x(t)) = sin(x(t)) + 2$. Defining $v(t) = f(x(t)) + g(x(t))u(t)$ one obtains

$$\dot{x}(t) = v(t)$$

which is a linear system (more precisely, an integrator). A scalar system with the only pole in -1 is $\dot{x}(t) = -x(t)$, and then it is sufficient to define $v(t) = -x(t)$ as the state-feedback control law (i.e. $v(t) = Kx(t)$, with $K = -1$). The resulting control law $u(t) = \frac{v(t) - f(x(t))}{g(x(t))}$ is

$$u(t) = \frac{x^3(t)}{2 + \sin(x(t))}$$

Another choice leading to an easy solution is to take $f(x(t)) = -x^3(t)$, and $g(x(t)) = sin(x(t)) + 2$. Defining $v(t) = f(x(t)) + g(x(t))u(t)$ one obtains

$$\dot{x}(t) = -x(t) + v(t)$$

which is again a linear system. To obtain $\dot{x}(t) = -x(t)$ one can define $v(t) = 0$ as the state-feedback control law (i.e. $K = 0$). The resulting control law $u(t) = \frac{v(t) - f(x(t))}{g(x(t))}$ is clearly the same as with the previous choice:

$$u(t) = \frac{x^3(t)}{2 + \sin(x(t))}$$

Exercise 4 (3 points)

For the given nonlinear system, let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be positive definite in a ball B_ϵ around the origin, $\epsilon > 0$, $V \in C^1(\mathbb{R})$. If the function

$$\dot{V}(x) = \nabla V(x)'\dot{x} = \nabla V(x)' f(x)$$

is negative definite on B_ϵ, then the origin is an asymptotically stable equilibrium point with *domain of attraction* B_ϵ ($\lim_{t \to +\infty} x(t) = 0$ for all $x(0) \in B_\epsilon$). Such a function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is called a *Lyapunov function* for the system $\dot{x} = f(x)$.

Therefore, the existence of a Lyapunov function is a sufficient condition for the asymptotic stability of an equilibrium point.