UNIVERSITY OF TRENTO

Prof. Alberto Bemporad January 27, 2011

AUTOMATIC CONTROL 2: SOLUTIONS

Exercise 1 (10 points)

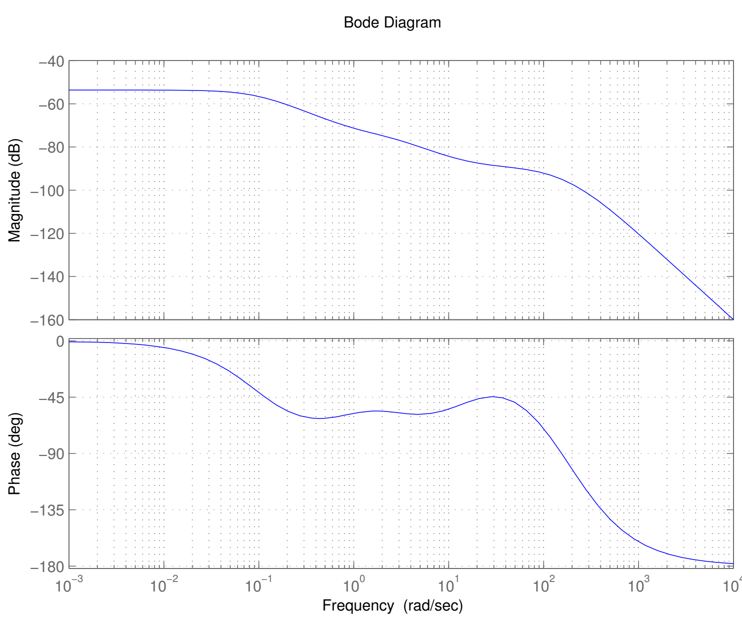
The transfer function $P(s)$ can be rewritten in the following Bode form

$$P(s) = \frac{15}{7200}\frac{(1+s)(1+\frac{1}{15}s)}{(1+\frac{1}{0.1}s)(1+\frac{1}{2.5}s)(1+\frac{1}{120}s)(1+\frac{1}{240}s)}$$

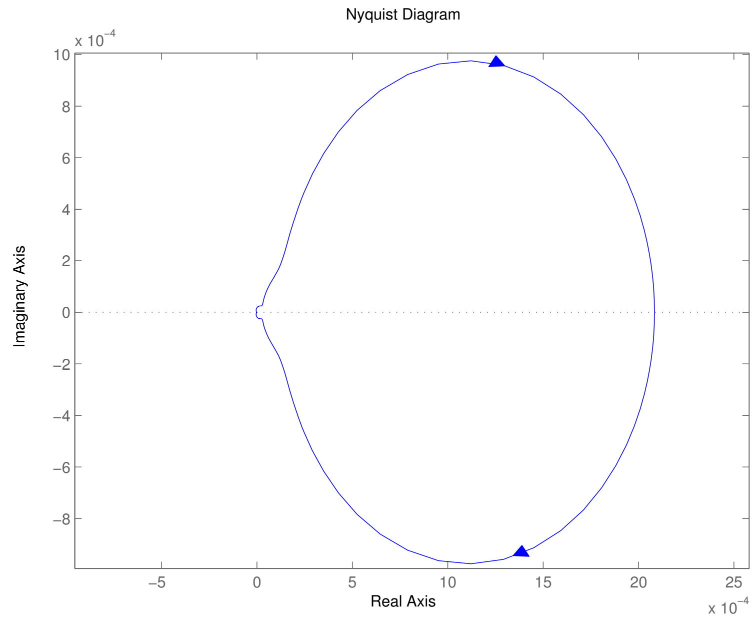
The following table summarizes the contributions of magnitude and phase of each basic component of $P(s)$:

	Magnitude [dB]	Phase [rad]
K_B	$20\ \log_{10}(\frac{15}{7200}) \simeq -53.6248$	0
$(s+1)$	$+20$ dB/dec, $\omega_{z_1} = 1$	$+\frac{\pi}{2}$
$(s+15)$	-20 dB/dec, $\omega_{z_2} = 15$	$+\frac{\pi}{2}$
$(s+0.1)$	-20 dB/dec, $\omega_{p_1} = 0.1$	$-\frac{\pi}{2}$
$(s+2.5)$	-20 dB/dec, $\omega_{p_2} = 2.5$	$-\frac{\pi}{2}$
$(s+120)$	-20 dB/dec, $\omega_{p_3} = 120$	$-\frac{\pi}{2}$
$(s+240)$	-20 dB/dec, $\omega_{p_4} = 240$	$-\frac{\pi}{2}$

The real Bode diagram is depicted in the following figure



The Nyquist diagram is shown in the following figure



As the open-loop transfer function $KP(s)$ is asymptotically stable, by Nyquist's criterion closed-loop stability is ensured if and only if the Nyquist diagram of $KP(s)$ does not have clockwise encirclements of the point -1, or in other words the Nyquist diagram of $P(s)$ does not have clockwise encirclements of the point $-1/K + j0$. Therefore, by restricting K to only assume positive values, closed-loop stability is always ensured.

Exercise 2 (10 points)

The design specifications are translated as follows:

1. Steady-state specifications.
   Since $P(s)$ is of *type* 0, we must add one integrator in the loop function to satisfy the requested steady-state specification

2. $\hat{s} \leq 0.1$.
   By recovering the approximate formulae $M_r = \frac{\hat{s}+1}{0.85}$ and $M_p = \frac{2.3 - M_r}{1.25}$ the desired phase margin is $M_p \simeq 46$ deg

3. $t_r \leq 0.018$ s.
   Since $t_r B_3 \simeq 3$ and $\omega_c = [0.5 \ldots 0.8]\, B_3$, the desired bandwidth is $B_3 \simeq 167$ dB and $\omega_c = [83.3 \ldots 133.3]$. A good choice is $\omega_c = 100$ rad/s

The transfer function to be shaped is $G(s) = \frac{P(s)}{s}$. By looking at the phase and the magnitude of $G(j\omega_c)$, the gap to compensate is

$$\angle(G(j100)) = -263.71 \text{ deg}, \;\; |G(j100)|_{\text{dB}} = -126.05 \text{ dB}$$

$$\begin{aligned} \Delta M &= 0 - (-126.05) = 126.05 \text{ dB} \\ \Delta \phi &= M_p - (180 - 263.71) \simeq 130 \text{ deg} \end{aligned}$$

In order to satisfy the phase specification, we first insert two lead networks with $\alpha_1 = 0.05$, centered in w_c for which $\tau_1 = \frac{1}{w_c \sqrt{\alpha_1}} = 0.0447$ s. Each network increases the phase by about 64 deg and the magnitude by about 13 dB.

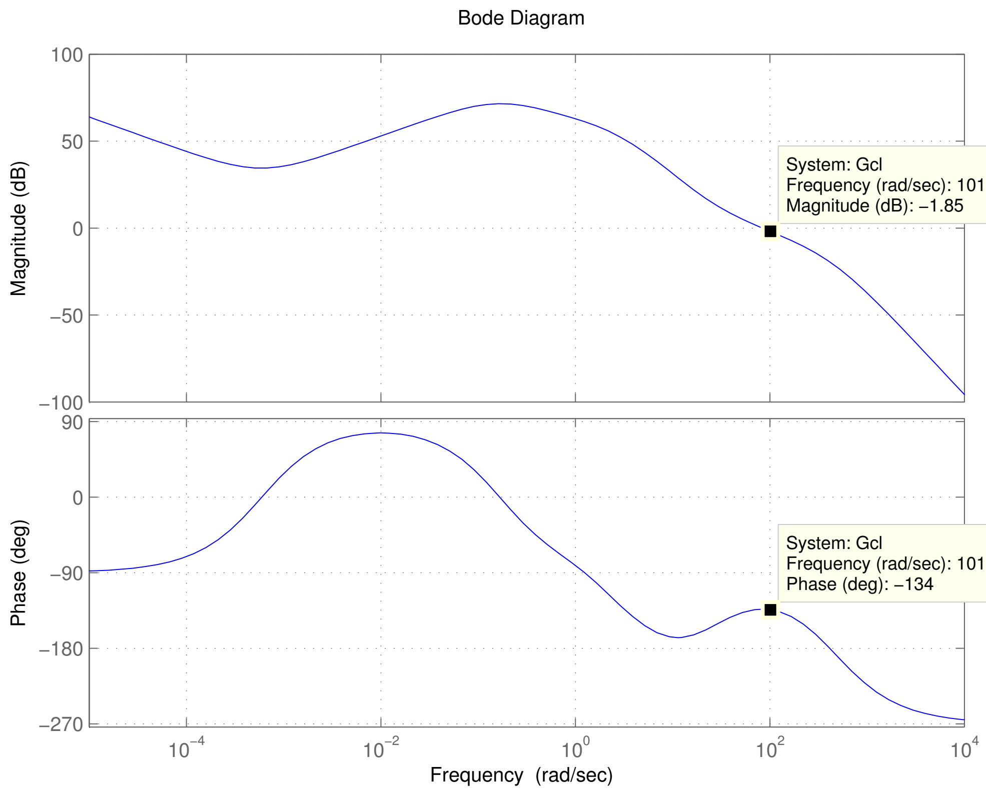
After satisfying the phase specification, the new specification on magnitude becomes $\Delta M' = \Delta M - 26 = 100$ dB. In order to shift the magnitude diagram while preserving the phase diagram at $\omega_c = 100$ rad/s, a second couple of lead networks is placed with $\omega_{low} = 1/100$ rad/s, $\alpha_2 = 0.0035$, and $\tau_2 = \frac{1}{w_{low}\sqrt{\alpha_2}} \simeq 1690.3$ s. The lead networks have transfer functions

$$C_i(s) = \frac{1+\tau_i s}{1+\alpha_i \tau_i s}, \ i = 1, 2$$

The design specifications are verified for the closed-loop transfer function

$$W(s) = \frac{L(s)}{1+L(s)}, \ \text{where } L(s) = C_1^2(s) C_2^2(s) G(s)$$

The following figure shows the Bode diagram of $L(s)$, showing that the crossover frequency is roughly 100 rad/s, and correspondingly the phase has increased from the original -263.71 deg to -134 deg, i.e., an increase of 129.71 deg.



<u>Exercise</u> 3 (8 points)

As for system (a) it is immediate to see that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

has two eigenvalues in 1. As a consequence, the system is unstable and there exist no Lyapunov functions. Analyzing system (b), setting $V(x) = x'Px$, with $P = P' \succ 0$ and $x = [x_1 \ \ x_2]'$, and imposing the negative definiteness of $\dot{V}(x) = \dot{x}'Px + x'P\dot{x}$, one obtains that $\dot{V}(x)$ is negative definite if and only if, for some $Q = Q' \succ 0$, $A'P + PA = -Q$. Now, we take $Q = I$, and defining $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ we obtain

$$\begin{bmatrix} -2p_{11} & p_{11} - 2p_{12} \\ p_{11} - 2p_{12} & 2p_{12} - 2p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

leading to

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

which is positive definite. Therefore, the origin is a globally asymptotically stable equilibrium point for the system.

<u>**Exercise** 4</u> (3 points)

The four steps of the general black-box system identification procedure are:

- Experiment design: decide what kind of input excitation $u(k)$ to apply
- Model structure: choose a class of models (for instance, BJ, OE, ARX, ARMAX) in order to fit the data
- Fit criterion between data and model: choose the model within that class (i.e. choose the parameter vector using least squares methods)
- Validation criterion: decide if the identified model is good enough to reproduce the dynamics of the process.