University of Trento

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Automatic Control 2: Solutions

Exercise 1 (10 points)

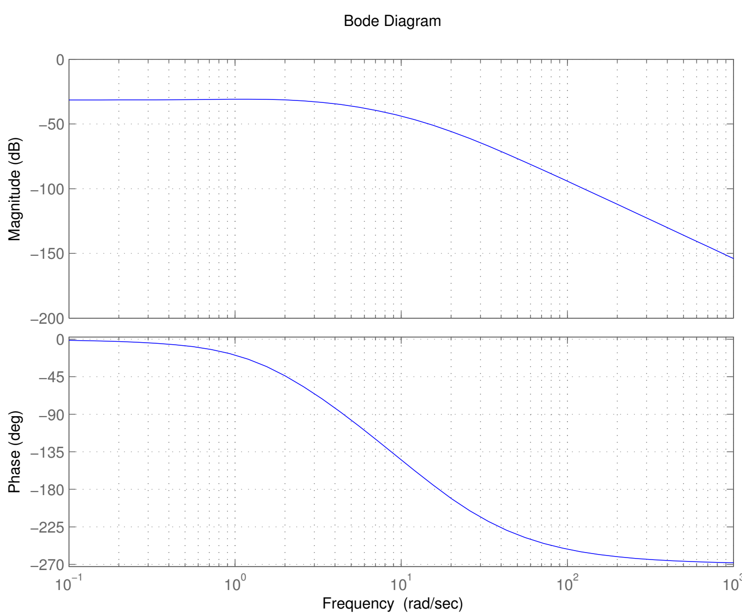
The transfer function $P(s)$ can be rewritten approximately in the following Bode form

$$P(s) = \frac{2}{75}\frac{1+s}{(1+\frac{1}{3}s)(1+0.75s)(1+0.1s)(1+0.054s)}$$

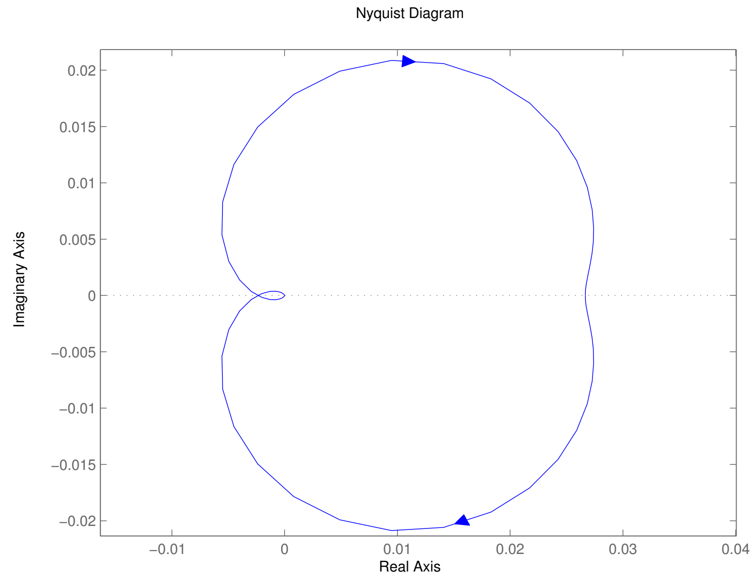
The following table summarizes the contributions of magnitude and phase of each basic component of $P(s)$:

	Magnitude [dB]	Phase [rad]
K_B	$20\log_{10}(\frac{20}{75}) \simeq -31.4806$	0
$(s+1)$	$+20$ dB/dec, $\omega_z = 1$	$+\frac{\pi}{2}$
$(s+3)$	-20 dB/dec, $\omega_{p_1} = 3$	$-\frac{\pi}{2}$
$(s+1.34)$	-20 dB/dec, $\omega_{p_2} \simeq 1.34$	$-\frac{\pi}{2}$
$(s+10)$	-20 dB/dec, $\omega_{p_3} = 10$	$-\frac{\pi}{2}$
$(s+18.66)$	-20 dB/dec, $\omega_{p_4} \simeq 18.5$	$-\frac{\pi}{2}$

The real Bode diagram is depicted in the following figure



The Nyquist diagram is shown in the following figure



As the open-loop transfer function $KP(s)$ is asymptotically stable, by Nyquist's criterion closed-loop stability is ensured if and only if the Nyquist diagram of $KP(s)$ does not have clockwise encirclements of the point -1, or in other words the Nyquist diagram of $P(s)$ does not have clockwise encirclements of the point $-1/K + j0$. Therefore, by restricting K to only assume positive values, closed-loop stability is ensured by $-\frac{1}{K} < \mathrm{Re}[P(j\omega^*)] = -0.0023$, or equivalently $K < 435$.

<u>Exercise</u> 2 (10 points)

The design specifications must be translated as follows.

1. Steady-state specifications.
   Since $P(s)$ is of *type* 0, we must add one integrator in the loop function to satisfy the requested steady-state specification

2. $\hat{s} \leq 0.1$.
   By recovering the approximate formulas $M_r = \frac{\hat{s}+1}{0.85}$ and $M_p = \frac{2.3-M_r}{1.25}$ the desired phase margin is $M_p \simeq 46$ deg

3. $t_r \leq 0.2$ s.
   Since $t_r B_3 \simeq 3$ and $\omega_c = [0.5 \dots 0.8]\, B_3$, assuming $t_r = 0.2$ s, the desired bandwidth is $B_3 = 1.2$ dB and $\omega_c = [7.5 \dots 12]$. A good choice is $\omega_c = 9$ rad/s

The transfer function to be shaped is $G(s) = \frac{P(s)}{s}$. By looking at the phase and the magnitude of $G(j\omega_c)$, the gap to compensate is

$$\angle(G(j9)) = -317 \text{ deg}, \;\; |G(j9)|_{\text{dB}} = -55 \text{ dB}$$

$$\begin{aligned} \Delta M &= 0 - (-55) = 55 \text{ dB} \\ \Delta\phi &= M_p - (180 - 317) \simeq 183 \text{ deg} \end{aligned}$$

In order to satisfy the phase specification, we first insert two lead networks with $\alpha_1 = 0.05$, centered in w_c for which $\tau_1 = \frac{1}{w_c\sqrt{\alpha_1}} \simeq 0.5$ s. Each network increases the phase by about 64 deg and the magnitude by about 13 dB. We need a third lead network with $\alpha_2 = 0.1$, centered in w_c for which $\tau_2 = \frac{1}{w_c\sqrt{\alpha_1}} \simeq 0.35$ s. This network increases the phase by about 55 deg and the magnitude by about 10 dB.

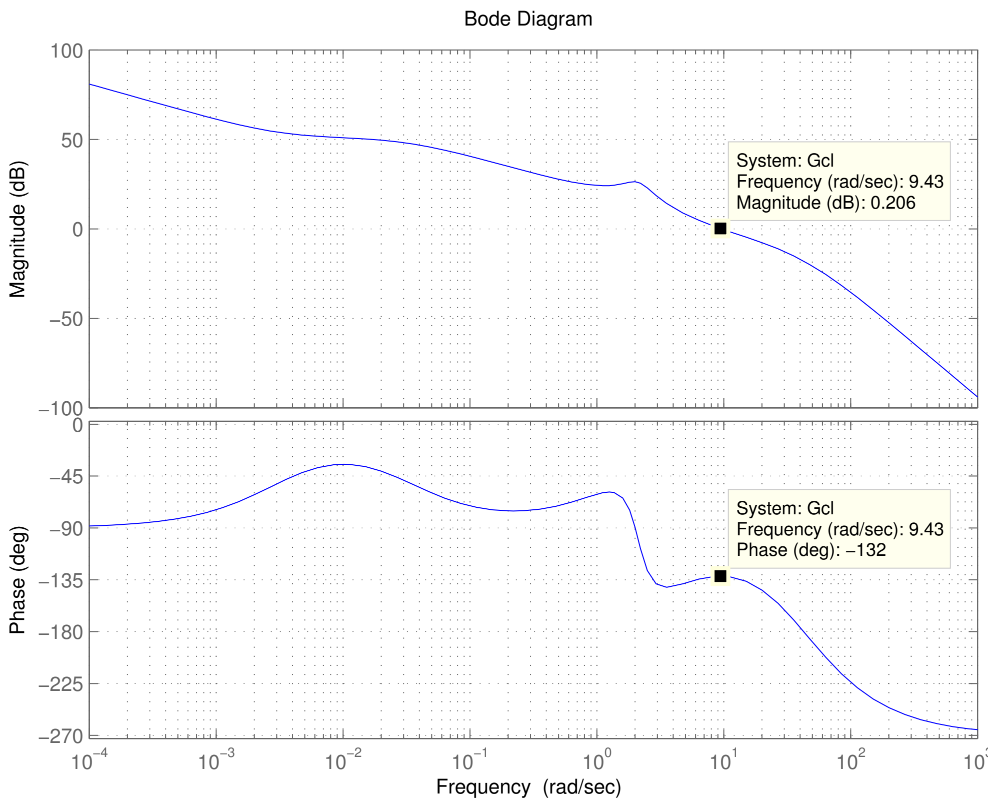
After satisfying the phase specification, the new specification on magnitude becomes $\Delta M' = \Delta M - 26 - 10 = 19$ dB. In order to shift the magnitude diagram while preserving the phase diagram at $\omega_c = 9$ rad/s, another lead network is selected with $\omega_{low} = 1/100$ rad/s, $\alpha_3 = 0.3$, and $\tau_3 = \frac{1}{w_{low}\sqrt{\alpha_2}} \simeq 316$ s.
The lead networks have transfer functions

$$C_i(s) = \frac{1 + \tau_i s}{1 + \alpha_i \tau_i s}, \ i = 1, 2, 3$$

The design specifications are verified for the closed-loop transfer function

$$W(s) = \frac{L(s)}{1 + L(s)}, \ \text{where } L(s) = C_1^2(s) C_2(s) C_3(s) G(s)$$

The following figure shows the Bode diagram of $L(s)$, showing that the crossover frequency is roughly 9 rad/s, and correspondingly the phase has increased from the original -317 deg to -132 deg, i.e., an increase of $317 - 132 = 185$ deg.



Exercise 3 (3 points)

- Both the functions $V_a(x)$ and $V_b(x)$ are $\mathbb{C}^1(\mathbb{R})$. $V_a(x)$ is not positive definite, hence cannot be a Lyapunov function. Regarding $V_b(x)$ we get

$$\dot{V}_b(x) = \frac{\partial V_b(x)}{\partial x} \dot{x} = x(-x^5) = -x^6$$

which is negative definite for all $x \in \mathbb{R}$. Since V_b is a positive definite function, V_b is also a Lyapunov function.

- Since the system is linear it is straightforward to note that it is unstable (one eigenvalue outside the unit circle). As a consequence, there exist no Lyapunov function for such a system.

Exercise 4 (7 points)

- In model predictive control an optimal control problem is solved at each time t over a finite future horizon of N steps. As a result, we obtain a sequence of N control moves $u^*(t)$, $u^*(t+1)$, ..., $u^*(t+N-1)$. The receding horizon policy consists in applying only the first move $u^*(t)$: at the next time instant the optimal control problem will be solved again after getting a new measurement, and so on.

- The "windup" effect is due to the presence of a saturation of the control variable when the controller has an integral action (e.g. PID controllers suffer such problem, while purely proportional controller do not). The reason is the "windup" of the integrator contained in the controller, which keeps integrating the tracking error even if the input is saturating.
  To solve the problem, some "anti-windup" techniques are available. In particular, one of the following should be described: incremental algorithm, back-calculation, conditional integration, or observer approach. The reader is referred to the course slides for their detailed description.