University of Trento

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Automatic Control 2: Solutions

Exercise 1 (10 points)

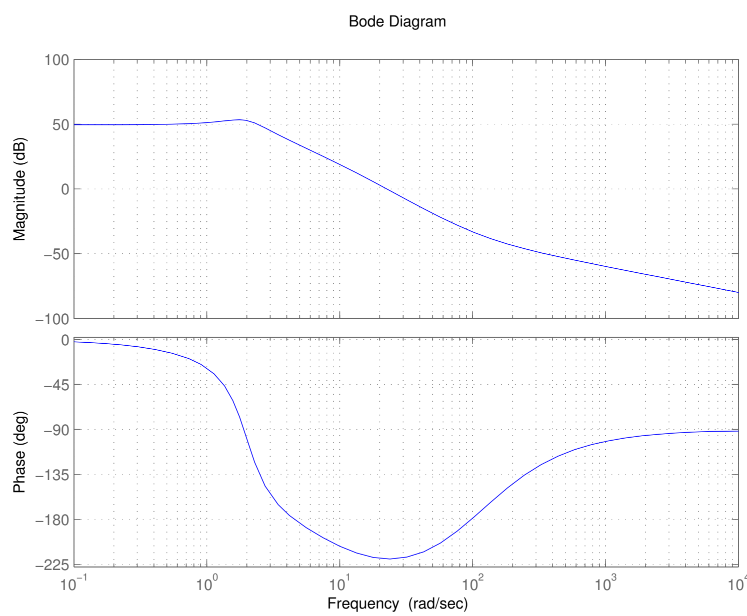
The Bode form of P(s) is the following

$$P(s) = 300\frac{(s/100+1)(s/120+1)}{(s/10+1)(s^2/4+1/3s+1)}$$

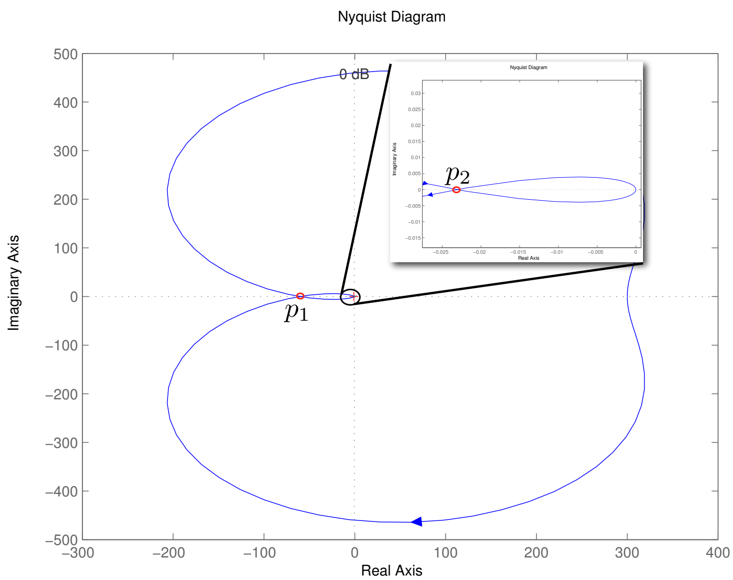
The second order polynomial is a complex and conjugate pair of poles, then, for the polynomial identity principle, $\omega_n = 2$ and $\zeta = 1/3$.
For the sake of simplicity, a table of the magnitude and phase contribution of the basic components is depicted in the following

	Magnitude dB	Phase deg
K_B	$20\ \log_{10}(300) = 49.5$	0
$(s/100+1)$	$+20$ dB/dec, $\omega_{z1} = 100$	$+\pi/2$
$(s/120+1)$	$+20$ dB/dec, $\omega_{z2} = 120$	$+\pi/2$
$(s/10+1)$	-20 dB/dec, $\omega_{z1} = 10$	$-\pi/2$
$(s^2/4+1/3s+1)$	-40 dB/dec, $\omega_n = 2$	$-\pi$

The real Bode diagram is depicted in the following Figure



Denoting with p_1 and p_2 the two suggested points on the $-\pi$ axis, i.e. the purely real negative semiaxis, the Nyquist diagram is shown in the following Figure



As the nominal plant is asymptotically stable, exploiting the Nyquist criterion, the stability of the closed loop is ensured iff there are no (clockwise) encirclements of the point -1, while moving the point $-1/K$. We distinguish three cases as follows

- $-\frac{1}{K} < p_1$. The closed loop is stable (no encirclements)
- $p_1 < -\frac{1}{K} < p_2$. The closed loop is unstable (2 clockwise encirclements)
- $p_2 < -\frac{1}{K} < 0$. The closed loop is stable (1 clockwise and 1 counter-clockwise encirclement)

As a result, if $K = 0.01$, the closed loop is stable.

<u>**Exercise 2**</u> (10 points)

The design specifications must be translated as follows.

- $\hat{s} \leq 20\%$
  By recovering the approximate formulae $\hat{s} = 0.85M_r - 1$ and $M_p = \frac{2.3 - M_r}{1.25}$, the desired phase margin is $M_p \simeq 41$ deg
- $t_r \leq 0.3$ s
  Since $t_r B_3 \simeq 3$ and $\omega_c = [0.5 \ldots 0.8]\, B_3$, assuming $t_r = 3$, the desired bandwidth is $B_3 = 10$ dB and $\omega_c = [5 \ldots 8]$. A good choice is $\omega_c = 6$ rad/s

In order to satisfy the steady state specification, since the $P(s)$ is of *type* 0, there is no need to add integrator. Hence K_c can be chosen as $K_c = 1/(K_B e_d)$, where K_B is the dc-gain of $P(s)$. In this case $K_c = 0.5$.
By looking at the phase and the magnitude of $P(j\omega_c)$, the gap to compensate can be calculated as follows

$$\angle(K_c G(j6)) = -210.64 \text{ deg}, \;\; |K_c G(j6)|_{\text{dB}} = -31.73 \text{ dB}$$

$$\Delta M = 0 - 31.73 \simeq 32 \text{ dB}, \;\; \Delta\phi = 210.64 - (180 - M_p) \simeq 72 \text{ deg}$$

In order to satisfy the phase specification, three *lead* networks with $\alpha_1 = 0.4$, centered in w_c for which $\tau_1 = \frac{1}{w_c\sqrt{\alpha_1}} = 0.2635$. These networks will increase the phase for about 75 deg and the magnitude for about

12 dB.

The phase specification is satisfied, while the new magnitude specification is $\Delta M' = \Delta M - 12 = 20$ dB. In order to shift the magnitude diagram while preserving the phase diagram at $\omega_c = 6$ rad/s, two *lead* networks are placed at $\omega_{low} = 1/100$ rad/s, with $\alpha_2 = 0.3$ and $\tau_2 = \frac{1}{w_{low}\sqrt{\alpha_2}} = 182.5742$.

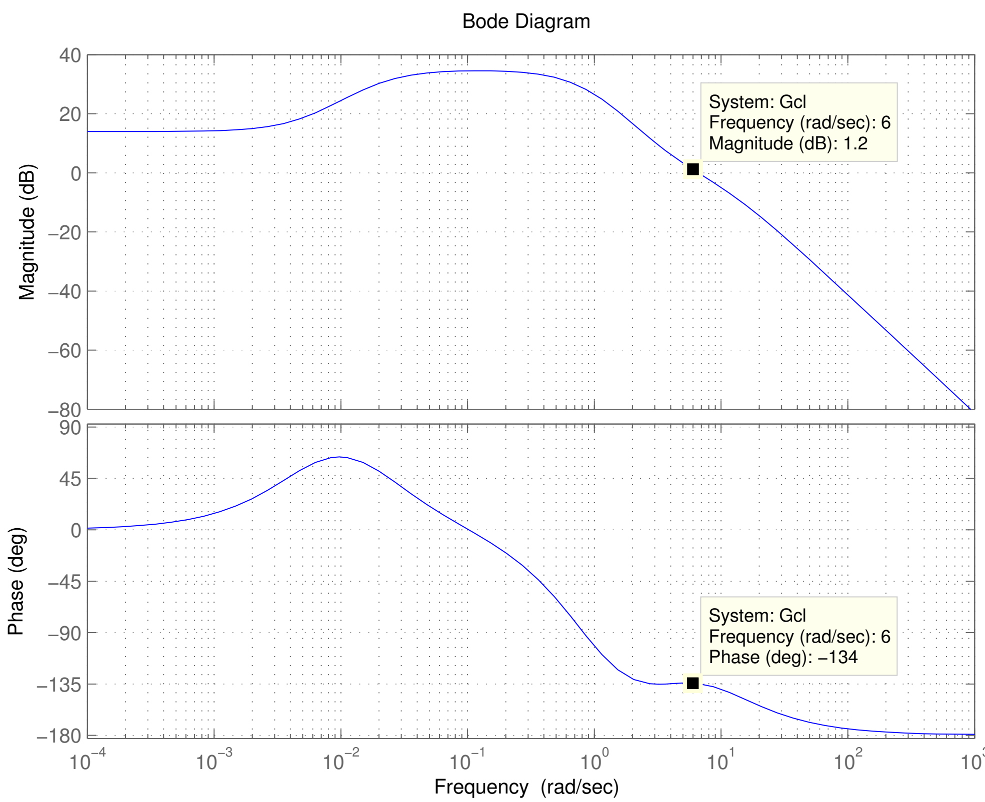
The corresponding lead network functions have the form

$$C_{1,2}(s) = \frac{1 + \tau_{1,2}s}{1 + \alpha_{1,2}\tau_{1,2}s}$$

At this point, the design specifications are verified for the closed loop transfer function

$$F(s) = \frac{L(s)}{1 + L(s)}, \text{ where } L(s) = K_c C_1^3(s) C_2^2(s) G(s)$$

In the following Figure is shown the Bode diagram of $L(s)$



<u>Exercise</u> 3 (6 points)

The auxiliary control variable can be chosen as $v(t) = 2u(t) + \cos(x_1(t) + x_2(t))$, leading to a linearized system with matrices $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $v(t)$ as control variable. The system is completely reachable, and then it is possible to fulfill the requirements. Following the usual procedure for pole-placement, it yields $p_c(\lambda) = \lambda^2 + (-2 - k_2)\lambda - 2 - k_1 + 3k_2$, $p_d(\lambda) = \lambda^2 + 3\lambda + 2$, leading to $K = [k_1\ k_2] = [-19\ \ -5]$ for the closed-loop system $(A + BK)$. The overall control law is then

$$u(t) = \frac{-19_1(t) - 5x_2(t) - \cos(x_1(t) + x_2(t))}{2}$$

The same result could be obtained choosing $v(t) = -x_1(t) - x_2(t) + \cos(x_1(t) + x_2(t)) + 2u(t)$.

Exercise 4 (7 points)

- Closed-loop stability is ensured if (A, B) is stabilizable, R is symmetric positive definite, Q is symmetric semi-positive definite, and (A, C_q) is detectable, where C_q is the Cholesky factor of Q, i.e. an upper-triangular matrix such that $C_q' C_q = Q$.
- Fixing Q, to larger values of R will correspond an increasing weight for the control variable in the cost function. As a consequence, the controller will use less energy for the control variable, and the state will converge more slowly. For this reason, $R_1 \to c$, $R_2 \to a$, $R_3 \to b$.