University of Trento

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Automatic Control 1: Solutions

Exercise 1 (9 points)

Let x_1, x_2, x_3 denote the quantities of apples in the three locations #1, #2, #3, respectively. The state space representation of the supply chain is described by the discrete-time linear model

$$\begin{cases} x(k+1) &= \begin{bmatrix} 0.1 & 0 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0 & 0.7 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k) \end{cases}$$

Asymptotic stability is achieved if and only if the eigenvalues $\lambda_{1,2,3}$ of the state-update matrix are such that $|\lambda_{1,2,3}| < 1$.
The eigenvalues are $\lambda_1 = \frac{1}{10}$, $\lambda_{2,3} = \frac{3}{20} \pm \frac{\sqrt{29}}{20}$ (whose moduli are strictly less than 1), so the system is asymptotically stable.

Exercise 2 (13 points)

Imposing $\dot{x}_1(t) = \dot{x}_2(t) = 0$, one equilibrium point is obtained as $(0,0)$. The matrices (A,B,C,D) of the linearized system are

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$$

Matrix A has eigenvalues in $-\frac{1}{2} \pm \frac{\sqrt{13}}{2}$, which are both real, one positive and one negative. Then the system linearized in $(0,0)$ is unstable.

To regulate (with state feedback) the output y on a generic constant reference r with zero steady-state error even in presence of the constant and unmeasurable disturbance $\bar{d}$ it is necessary to use an integral action. Then, a new state component is added as $q(t)$ such that $\dot{q}(t) = y(t)$. Defining $x_e = \begin{bmatrix} x_1 \\ x_2 \\ q \end{bmatrix}$, the matrices A_e and B_e for the extended state-space model are

$$A_e = \begin{bmatrix} 0 & 1 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

from which one obtains the reachability matrix $R_e = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Since the rank of R_e is 3, the system is completely reachable and the eigenvalues of the closed-loop system can be arbitrarily assigned. Given $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, the characteristic polynomial is $p_c(\lambda) = det(\lambda I - A_e - B_e K) = \lambda^3 + (1-k_2)\lambda^2 +$

$(-3-k_1)\lambda - k_3$, while the desired polynomial is $p_d(\lambda) = (\lambda+1)^3 = \lambda^3+3\lambda^2+3\lambda+1$. Then, the components of the feedback gain K are $k_1 = -6$, $k_2 = -2$, $k_3 = -1$. The controller can be expressed as

$$\begin{cases} \dot{q}(t) &= y(t) - r \\ u(t) &= -6x_1(t) - 2x_2(t) - q(t) \end{cases}$$

<u>**Exercise** 3</u> (9 points)

The system matrices are

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ b \end{bmatrix}, \ C = \begin{bmatrix} 0 & 2 \end{bmatrix}, \ D = 0$$

The observability matrix is $\Theta = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, which has rank 1. Hence the system is not observable with an unobservable subspace of dimension 1. The system is already in observability canonical form, and the only eigenvalue of the unobservable part is a. Therefore, the system is reconstructable if $a = 0$, detectable if $|a| < 1$, and not reconstructable nor detectable if $|a| \geq 1$. The observability properties of the system are not affected by b.

The transfer function of the system is

$$G(z) = C(zI - A)^{-1}B = \frac{2b(z-a)}{(z-1)(z-a)} = \frac{2b}{z-1}$$

In general, the non-observability of the system implies zero/poles cancellations in the transfer function, since the poles of the unobservable part do not affect the input-output characteristics of the system. This is exactly what happens in the given system, where the pole in a is cancelled.