Prof. Alberto Bemporad April 21, 2011

Automatic Control 1: Solutions

Exercise 1 (8 points)

The physical equations of the system are

$$\begin{aligned} J_1\dot{\omega}_1(t) &= -\,k(\theta_1(t)-\theta_2(t)) - \beta(\omega_1(t)-\omega_2(t)) + u(t) \\ J_2\dot{\omega}_2(t) &= -\,k(\theta_2(t)-\theta_1(t)) - \beta(\omega_2(t)-\omega_1(t)) \end{aligned}$$

Introducing, as required, the state variables as $x_1 = \theta_2 - \theta_1$, $x_2 = \omega_1$ and $x_3 = \omega_2$, for the given input u and output $y = x_3$, we obtain the state-space representation

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) - x_2(t) \\ \dot{x}_2(t) &= \frac{k}{J_1}x_1(t) - \frac{\beta}{J_1}(x_2(t)-x_3(t)) + \frac{1}{J_1}u(t) \\ \dot{x}_3(t) &= -\frac{k}{J_2}x_1(t) - \frac{\beta}{J_2}(x_3(t)-x_2(t)) \end{aligned}$$

from which we define the system matrices

$$A = \begin{bmatrix} 0 & -1 & 1 \\ \frac{k}{J_1} & -\frac{\beta}{J_1} & \frac{\beta}{J_1} \\ -\frac{k}{J_2} & \frac{\beta}{J_2} & -\frac{\beta}{J_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = 0$$

Substituting the numerical values, matrix A becomes

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

which has eigenvalues in 0 and $-1 \pm \sqrt{3}i$. Having only one eigenvalue in 0, its algebraic and geometric multiplicity coincide. Therefore, since the other two eigenvalues have strictly negative real part, the system is marginally stable.

Exercise 2 (9 points)

Solving $\dot{x}_1 = \dot{x}_2 = 0$ with $\bar{u} = -1$, one obtains $\bar{x}_1 = -1$, $\bar{x}_2 = 0$. Linearizing around these values, the required matrices are

$$A = \begin{bmatrix} 3\bar{x}_1^2 & -1 \\ 0 & e^{\bar{x}_2} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad D = 1$$

The transfer function is

$$G(s) = C(sI-A)^{-1}B + D = \frac{(s-2)^2}{(s-1)(s-3)}$$

which has two zeros in 2, and two poles in 1 and 3, respectively. The eigenvalues of matrix A coincide with the poles of the transfer function, which have both strictly positive real part. Therefore, the linearized system is unstable.

Exercise 3 (10 points)

Given the system matrices $A = \begin{bmatrix} 1 & 0 \\ 2 & a \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the reachability matrix is

$$R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & a+2 \end{bmatrix}$$

Since $det(R) = 1 + a$, the system is completely reachable for all $a \neq -1$. If $a = -1$, we have $R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. A base for $Im(R)$ is $v_1 = [1\ 1]'$, and we choose , for instance, $w_1 = [1\ 0]'$ as a completion. We obtain $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The matrices in the canonical reachability form are $\tilde{A} = T^{-1}AT = \left[\begin{array}{c|c} -1 & 0 \\ \hline 2 & 1 \end{array}\right]$, and $\tilde{B} = T^{-1}B = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$. Since $A_{uc} = -1$, if $a = -1$ the system is not controllable, nor stabilizable.

For $a = 0$, it is possible to steer the state of the system from an initial condition $x(0)$ to any other state value in 2 time steps, because the system is completely reachable. Given $x(0)$, the state at time $k = 2$ is $x(2) = A^2x(0) + ABu(0) + Bu(1)$. Then, we solve

$$x(2) - A^2x(0) = R \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

leading to $u(0) = -1$ and $u(1) = 2$.

To design the state-feedback control law $u = Kx$, with $K = [k_1\ k_2]$, we first compute the desired polynomial $p_d(\lambda) = (\lambda - \frac{1}{2})^2 = \lambda^2 - \lambda + \frac{1}{4}$. The characteristic polynomial is $p_c(\lambda) = det(\lambda I - A - BK) = \lambda^2 - (1 + k_1 + k_2)\lambda - k_2$, which leads to $k_1 = \frac{1}{4}$ and $k_2 = -\frac{1}{4}$. The control law is then

$$u(k) = \frac{1}{4}x_1(k) - \frac{1}{4}x_2(k)$$

Exercise 4 (4 points)

If $n_o = n$, the system is completely observable. The system is reconstructable if all the eigenvalues of A_{uo} are equal to 0, while it is detectable if all the eigenvalues of A_{uo} are in absolute value strictly smaller than 1.