University of Trento

Prof. Alberto Bemporad February 17, 2011

Automatic Control 1: Solutions

Exercise 1 (14 points)

The physical equations of the system are

$$
\begin{aligned}
C_1\dot{T}_1(t) &= -k_{01}(T_1(t) - T_0) + k_{12}(T_2(t) - T_1(t))\\
C_2\dot{T}_2(t) &= -k_{12}(T_2(t) - T_1(t)) + k_{23}(T_3(t) - T_2(t))\\
C_3\dot{T}_3(t) &= -k_{23}(T_3(t) - T_2(t)) + u(t)
\end{aligned}
$$

Introducing the state as $x = [x_1\ x_2\ x_3]'$, with $x_1 = T_1 - T_0$, $x_2 = T_2 - T_0$, $x_3 = T_3 - T_0$, for the given input u and output $y = x_1$ we obtain the state-space representation

$$
\begin{aligned}
\dot{x}_1(t) &= -\frac{k_{01}+k_{12}}{C_1}x_1(t) + \frac{k_{12}}{C_1}x_2(t)\\
\dot{x}_2(t) &= \frac{k_{12}}{C_2}x_1(t) - \frac{k_{12}+k_{23}}{C_2}x_2(t) + \frac{k_{23}}{C_2}x_3(t)\\
\dot{x}_3(t) &= \frac{k_{23}}{C_3}x_2(t) - \frac{k_{23}}{C_3}x_3(t) + \frac{1}{C_3}u(t)
\end{aligned}
$$

from which we easily define the system matrices

$$
A = \begin{bmatrix} -\frac{k_{01}+k_{12}}{C_1} & \frac{k_{12}}{C_1} & 0\\ \frac{k_{12}}{C_2} & -\frac{k_{12}+k_{23}}{C_2} & \frac{k_{23}}{C_2}\\ 0 & \frac{k_{23}}{C_3} & -\frac{k_{23}}{C_3} \end{bmatrix}, \quad B = \begin{bmatrix} 0\\ 0\\ \frac{1}{C_3} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0
$$

For the given values of the parameters, matrix A is

$$
A = \begin{bmatrix} -2 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 0 \end{bmatrix}
$$

which has eigenvalues in 0 and $-\frac{3}{2} \pm \frac{\sqrt{5}}{2} < 0$. Then, the system is marginally stable.

The observability matrix of the system for the given parameters is

$$
\Theta = \begin{bmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 5 & -\frac{k_{23}}{10} - 3 & \frac{k_{23}}{10} \end{bmatrix}
$$

It yields $\det(\Theta) = k_{23}/10$, and then the system is completely observable for $k_{23} \in (0, 10]$. In case $k_{23} = 0$, the observability matrix is

$$
\Theta = \begin{bmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 5 & -3 & 0 \end{bmatrix}
$$

whose rank is equal to 2, and then $\ker(\Theta) = \{x : x_1 = 0,\ x_2 = 0\} = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}$, $\forall \alpha \in \mathbb{R}$. The transformation matrix can be chosen as $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and noticing that $T^{-1} = T$, one has $\tilde{A} = T^{-1}AT = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 1 & -2 \end{array}\right]$. Considering that $\tilde{A}_{uo} = 0$, the system is non-reconstructable and non-detectable.

Exercise 2 (14 points)

The reachability matrix is $R = [B\ AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is full rank. Therefore, the system is completely reachable. The desired polynomial is $p_d(\lambda) = (\lambda - 0.5)^2 = \lambda^2 - \lambda + \frac{1}{4}$ and the characteristic polynomial of the closed-loop system $A + BK$, with $K = [k_1\ k_2]$, is $p_c(\lambda) = \det(\lambda I - A - BK) = \lambda^2 + (1 - k_2)\lambda - 1 - k_1 - k_2$. It yields $k_1 = -\frac{13}{4}$ and $k_2 = 2$. The control law is

$$u(k) = -\frac{13}{4}x_1(k) + 2x_2(k)$$

The transfer function is calculated as

$$G(z) = C(zI - A)^{-1}B + D = \frac{1}{z^2 + z - 1} + 1 = \frac{z^2 + z}{z^2 + z - 1}$$

The zeros are in 0 and 1, while the poles are in $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. The DC gain is not defined, because the system is unstable.

Exercise 3 (4 points)

In open-loop control the manipulated input variable is generated without measuring the output variable, while in closed-loop control the measurements of the output variables are fed back to the process through the controller (block schemes from the corse slides are welcome).