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Automatic Control 1: Solutions

Exercise 1 (13 points)

The physical equations of the system are

$$m\ddot{y}(t) = u(t) - ky(t) - \beta\dot{y}(t) + d(t)$$

Introducing the state variablesa $x_1 = y$ and $x_2 = \dot{y}$, for the given input u and output y we obtain the state-space representation

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \frac{1}{m}\left(u(t) - kx_1(t) - \beta x_2(t) + d(t)\right)\end{aligned}$$

from which we easily define the system matrices

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

The reachability matrix is

$$R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{\beta}{m^2} \end{bmatrix}$$

Therefore, $det(R) = -1/m^2 \neq 0$ for all $m > 0$ (and, implicitly, for all $k \geq 0$ and $\beta \geq 0$). As a consequence, the system is completely reachable (and then, also controllable and stabilizable) for all the values of the parameters in the given intervals.

To regulate (with state feedback) the position $y(t)$ on a generic constant reference r with zero steady-state error even in presence of the constant and unmeasurable disturbance $d(t)$ it is necessary to use an integral action. Then, a new state component is added as $q(t)$ such that $\dot{q}(t) = y(t)$. Defining $x_e = \begin{bmatrix} x_1 \\ x_2 \\ q \end{bmatrix}$, the matrices A_e and B_e for the extended state-space model substituting the numerical values are

$$A_e = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

from which one obtains the reachability matrix $R_e = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{3}{8} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$. Since the rank of R_e is 3, the system is completely reachable and the eigenvalues of the closed-loop system can be arbitrarily assigned. Given $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, one has that the characteristic polynomial is $p_c(\lambda) = det(\lambda I - A_e - B_e K) = \lambda^3 + \left(\frac{1}{2} - \frac{k_2}{2}\right)\lambda^2 + \left(1 - \frac{k_1}{2}\right)\lambda - \frac{k_3}{2}$, while the desired polynomial is $p_d(\lambda) = (\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$. Then, the components of K are $k_1 = -4$, $k_2 = -5$, $k_3 = -2$. The controller can be expressed as

$$\left\{\begin{array}{rcl} \dot{q}(t) & = & y(t) - r \\ u(t) & = & -4y(t) - 5\dot{y}(t) - 2q(t) \end{array}\right.$$

Exercise 2 (11 points)

By setting $\dot{x}_1 = \dot{x}_2 = 0$ for $\bar{u} = 0$ we obtain the equilibrium state $\bar{x}_1 = 0$, $\bar{x}_2 = 1$. Linearizing around these values, the required matrices are

$$A = \begin{bmatrix} -e^{-\bar{x}_1} & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

The eigenvalues of matrix A can be calculated as the roots of $\det(\lambda I - A) = \lambda^2 + \lambda + 1$, obtaining $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$.
Therefore, the linearized system is asymptotically stable.
Finally, the transfer function is

$$G(s) = C(sI - A)^{-1}B + D = \frac{s+1}{s^2 + s + 1}$$

and has DC gain equal to 1, one zero in -1, and two poles in $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$.

Exercise 3 (6 points)

- The system is called (completely) observable if no pair of states are indistinguishable from the output. The observability matrix is defined as

$$\Theta = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

  and the system is completely observable if and only if the rank of Θ is equal to n.

- It is possible to determine the current state $x(k)$ even if the system is not completely observable when the system is *reconstructable*.