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Automatic Control 1: Solutions

Exercise 1 (13 points)

Let the state $x = [\,F \ W \ R\,]'$, the input $u = [\,T_{FW} \ T_{WR}\,]'$ and the disturbance vector $d = [\,P \ S\,]'$.

1. The state space representation of the system is the following

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} d(k) \\
y(k) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k)
\end{aligned}
$$

2. If only $R(k)$ is measurable, the output matrix is $C = [\,0 \ 0 \ 1\,]$, leading to the following observability matrix

$$
\Theta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.9 \\ 0 & 0 & 0.81 \end{bmatrix}
$$

As $\det(\Theta) = 0$, the system is not observable. Note that as $\text{rank}(\Theta) = 1$, the system has a non-observable part of dimension equal to 2 and the state and output matrices are already in canonical form. In conclusion, the system is not detectable nor reconstructable.

3. The eigenvalues of A are $(1, 1, 0.9)$. The eigenvalue in 1 has algebraic moltiplicity 2. The geometric multiplicity of the eigenvalue in 1 is the dimension of its autospace, that is, the solution space of the linear system $Av = v$ or, equivalently, $(A - I)v = 0$:

$$
\begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & -0.1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The generic solution $v = [v_1 \ v_2 \ 0]'$ spans a two-dimensional space. Therefore, the system is marginally stable.

Exercise 2 (10 points)

The system matrices are

$$
A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

and the reachability matrix is

$$R = [B\ AB\ A^2B] = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow \mathrm{rank}(R) = 2.$$

The system is not completely reachable (and not controllable also, since we are considering a continuous-time system). To compute a canonical decomposition of the system, let us take the vectors $v_1 = [0\ 0\ 1]'$ and $v_2 = [1\ 0\ 1]'$ as the basis of $\mathrm{Im}(R)$, for which a possible completion is $w_1 = [0\ 1\ 0]'$. Accordingly, define the transformation matrix T as

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \tilde{A} = T^{-1}AT = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right], \quad \tilde{B} = T^{-1}B = \left[\begin{array}{c} 0 \\ \hline 1 \\ 0 \end{array}\right]$$

Since the only eigenvalue of the unreachable part of the system is equal to -1, the system is stabilizable.

Through state feedback it is possible to place the other two eigenvalues in -1. We act on the reachable part

$$\tilde{A}_c = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{B}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and find the vector $K_c = [\tilde{k}_1\ \tilde{k}_2]$. The desired polynomial is$p_d(\lambda) = (\lambda+1)^2$ and the characteristic polynomial of the closed-loop system $\tilde{A}_c + \tilde{B}_c K_c$ is $p_c(\lambda) = \lambda^2 + (2 - k_1)\lambda - k_2$. By equaling the polynomials, we get $\tilde{k}_1 = 0$ and $\tilde{k}_2 = -1$. Any feedback $\tilde{K} = [K_{uc}\ 0\ -1]$ places the eigenvalues of the system in reachability canonical form in -1, we can take for instance $K_{uc} = 0$. Finally, for the original system we get the feedback gain

$$K = \tilde{K}T^{-1} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$

and the resulting state-feedback controller $u(t) = -x_1(t)$.

<u>Exercise 3</u> (7 points)

- Given a transfer function $G(s) = C(sI - A)^{-1}B + D$ of a continuous-time dynamical system, it is possible to represent it as a ratio of two polynomials: $G(s) = N(s)/D(s)$. The zeros are the roots of $N(s)$, while the poles are the roots of $D(s)$.
- The transfer function is easily obtained as

  $$G(s) = 1(s-0)^{-1}\sqrt{2} + 0 = \frac{\sqrt{2}}{s}$$

  Therefore, the system has no zeros, a pole in zero, and is marginally stable.