Prof. Alberto Bemporad July 19, 2010

Automatic Control 1: Solutions

Exercise 1 (13 points)

The system equation are

$$\begin{aligned} L\frac{di(t)}{dt} &= u(t) - Ri(t) - K\omega(t) \\ J\frac{d\omega(t)}{dt} &= Ki(t) - \beta\omega(t) \\ \frac{d\theta(t)}{dt} &= \omega(t) \end{aligned}$$

Posing $x_1 = \theta$, $x_2 = \omega$, $x_3 = i$, it yields

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{\beta}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \end{aligned}$$

With the given parameters, the observability matrix is obtained as

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 10K \end{bmatrix}$$

As $\det(\Theta) = 10K$, for $K > 0$ the system is completely observable. If $K = 0$, $\mathrm{rank}(\Theta) = 2$, and the system has a non-observable part of dimension equal to 1. To define the matrix T, we find the kernel of Θ as the span of $[0\ 0\ 1]'$, leading (for instance) to

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\tilde{A} = T^{-1}AT = \left[\begin{array}{c|cc} -2 & 0 & 0 \\ \hline 0 & -10 & 0 \\ 0 & 1 & 0 \end{array}\right], \quad \tilde{C} = CT = \left[\begin{array}{c|cc} 0 & 0 & 1 \end{array}\right]$$

In conclusion, for $K = 0$ the system is detectable. As for the last request, since the eigenvalues of A, for $K = 0$, are $(0, -10, -2)$, the system is marginally stable.

Exercise 2 (13 points)

The system matrices are

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1+\alpha \\ 1 \end{bmatrix}$$

and then the reachability matrix is

$$R = \begin{bmatrix} 0 & 1 & 2 \\ \alpha+1 & -\alpha-1 & \alpha+1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow \det(R) = 3(\alpha+1)$$

Then, for $\alpha \neq -1$ the system is completely reachable. If $\alpha = -1$, the rank of R is equal to 2. In this case, the image of R is can be generated (for instance) by the two vectors $[0\ 0\ 1]'$ and $[1\ 0\ 1]'$, and then a possible choice for T is

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \tilde{A} = T^{-1}AT = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right], \quad \tilde{B} = T^{-1}B = \left[\begin{array}{c} 0 \\ \hline 1 \\ 0 \end{array}\right]$$

Since the only eigenvalue of the non-reachable part of the system is equal to -1, the system in this case is not even stabilizable.

For $\alpha = 0$ the system is completely reachable. Defining $K = [k_1\ k_2\ k_3]$, the characteristic polynomial of the closed-loop system $(A + BK)$ is $p_c(\lambda) = \lambda^3 - (k_2 + k_3 + 1)\lambda^2 + (2k_2 - k_1 - 2)\lambda + k_3 - 1$, while the desired one is $p_d(\lambda) = \lambda^3$. Therefore, it yields $K = [-4/3\ 1/3\ \ -4/3]$. The controller is then

$$u(k) = -\frac{4}{3}x_1(k) + \frac{1}{3}x_2(k) - \frac{4}{3}x_3(k)$$

<u>**Exercise**</u> <u>3</u> (7 points)

- The transfer function of a continuous-time linear system (A, B, C, D) is the ratio

  $$G(s) = C(sI - A)^{-1}B + D$$

  between the Laplace transform $Y(s)$ of output and the Laplace transform $U(s)$ of the input signals, for the initial state $x_0 = 0$.

- The strategy consists first of all in augmenting the open-loop system with the integral of the output vector $q(k+1) = q(k) + y(k)$. The augmented system is

  $$\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k)$$

  $$y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix}$$

  Then, one has to design a stabilizing feedback controller as

  $$u(k) = \begin{bmatrix} K & H \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix}$$

  The output $y(k)$ is steered to zero asymptotically for any constant $d(k) = \bar{d}$.