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Automatic Control 1: Solutions

Exercise 1 (15 points)

The system equations are

$$\begin{cases} A_1\dot{h}_1(t) &=& -(a_{12}+a_{13})\sqrt{2gh_1(t)}+\varphi(t) \\ A_2\dot{h}_2(t) &=& -a_{23}\sqrt{2gh_2(t)}+a_{12}\sqrt{2gh_1(t)} \end{cases}$$

Introducing the state and output variables as $x_1 = h_1$, $x_2 = h_2$, $u = \varphi = u$, $y = h_2$, we obtain a state-space representation with

$$\begin{cases} \dot{x}_1(t) &=& -\frac{a_{12}+a_{13}}{A_1}\sqrt{2gx_1(t)}+\frac{u(t)}{A_1} \\ \dot{x}_2(t) &=& -\frac{a_{23}}{A_2}\sqrt{2gx_2(t)}+\frac{a_{12}}{A_2}\sqrt{2gx_1(t)} \\ y(t) &=& x_2(t) \end{cases}$$

Substituting $a_{12} = a_{23} = a_{13} = 1\ m^2$, $A_1 = 200\ m^2$, $A_2 = 100\ m^2$, $g = 10$, it yields

$$\begin{cases} \dot{x}_1(t) &=& -\frac{\sqrt{20}}{100}\sqrt{x_1(t)}+\frac{u(t)}{200} \\ \dot{x}_2(t) &=& -\frac{\sqrt{20}}{100}\sqrt{x_2(t)}+\frac{\sqrt{20}}{100}\sqrt{x_1(t)} \end{cases}$$

Solving $\dot{x}_1 = \dot{x}_2 = 0$ with $\bar{u} = 10$, it yields

$$\bar{x}_1 = \bar{x}_2 = 5/4$$

Linearizing around these values, we obtain

$$A = \begin{bmatrix} -\frac{g(a_{12}+a_{13})^2}{\bar{u}A_1} & 0 \\ \frac{ga_{12}(a_{12}+a_{13})}{\bar{u}A_2} & -\frac{ga_{13}a_{23}(a_{12}+a_{13})}{a_{12}\bar{u}A_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{50} & 0 \\ \frac{1}{50} & -\frac{1}{50} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{200} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

The eigenvalues of matrix A are the elements on the diagonal, which have a strictly negative real part. Therefore, the linearized system is asymptotically stable. Finally, the observability matrix is

$$R = \begin{bmatrix} \frac{1}{200} & -\frac{1}{10000} \\ 0 & \frac{1}{10000} \end{bmatrix}$$

which is full rank. Therefore the system is completely reachable and, of consequence, controllable and stabilizable.

Exercise 2 (13 points)

As for the first request, the reachability matrix is $R = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$ which is full rank, making it possible to to place the poles of the closed-loop system $(A + BK)$ at any desired position. Given the controller gain $K = [k_1\ k_2]$, the characteristic polynomial is

$$p_c(\lambda) = \det(I - (A + BK)) = \det\left(\begin{bmatrix} \lambda - 2 & -1 \\ 1 - k_1 & \lambda - 3 - k_2 \end{bmatrix}\right) = \lambda^2 + (-k_2 - 5)\lambda - k_1 + 2k_2 + 7$$

The desired characteristic polynomial is $p_d(\lambda) = (\lambda+2)^2 = \lambda^2 + 4\lambda + 4$. Forcing the equalities between the coefficients of the two polynomials, it yields $k_1 = -15$ and $k_2 = -9$. The controller is then $u(t) = -9x_1(t) - 15x_2(t)$.

As for the second request, the observability matrix is $\Theta = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ which is full rank, and then it is possible to place the poles of the observer at will. Given an observer gain $L = [l_1 \; l_2]'$, the characteristic polynomial can be expressed as

$$p_c(\lambda) = \det(I - (A - LC)) = \det\left(\begin{bmatrix} \lambda - 2 + l_1 & l_1 - 1 \\ 1 + l_2 & \lambda - 3 + l_2 \end{bmatrix}\right) = \lambda^2 + (l_1 + l_2 - 5)\lambda - 4l_1 - l_2 + 7$$

The desired characteristic polynomial is $p_d(\lambda) = (\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$. Forcing the equalities between the coefficients of the two polynomials, it yields $l_1 = -1$ and $l_2 = 9$. The resulting Luenberger observer is

$$\hat{x}(t) = \begin{bmatrix} 3 & 2 \\ -10 & -6 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 9 \end{bmatrix} y(t)$$

Placing the poles at $(0.5, 0.5)$ is possible, because the system is completely observable. This would imply an unstable dynamics of the observation error, which would diverge to infinity, making the observer useless.

<u>**Exercise**</u> 3 (5 points)

For the Laplace transform $F(s)$ of a continuous-time function $f(t)$:

- Initial value theorem: $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
- Final value theorem: $\lim_{t\to+\infty} f(t) = \lim_{s\to 0} sF(s)$

For the Zeta transform $F(z)$ of a function $f(k)$:

- Initial value theorem: $f(0) = \lim_{z\to\infty} F(z)$
- Final value theorem: $\lim_{k\to+\infty} f(k) = \lim_{z\to 1}(z-1)F(z)$