Automatic Control 2

Model reduction

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Systems reduction

- The complexity of the control law often depends on the order of the system (for example in state-space methods like dynamic compensation)
- For control design purposes, can we approximate the model with another model of *reduced order* that preserves the original transfer function as much as possible ?
- We already know that uncontrollable and unobservable modes do not affect the transfer function. They can be eliminated by operating a canonical decomposition
- Can we try to eliminate other modes that are *weakly* uncontrollable and/or *weakly* observable, and get a *numerically well-conditioned* lower-order state-space realization ?

Model reduction and balanced transformations answer the above questions

Unbalanced realizations and scaling

- Consider the linear system

$$\begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 10^{-6} \\ 10^{6} \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 10^{6} & 10^{-6} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{cases}$$

- The state x_1 is “weakly” reachable, but “very”observable
- The state x_2 is “very” reachable, but “weakly” observable
- Let’s rescale the system by operating the change of coordinates

$$z = \begin{bmatrix} 10^{6} & 0 \\ 0 & 10^{-6} \end{bmatrix} x$$

- The system expressed in new coordinates is numerically balanced

$$\begin{cases} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \end{cases}$$

Grammians

- Consider the discrete-time linear system

$$\begin{cases} x(k+1) = Ax(k)+Bu(k) \\ y(k) = Cx(k)+Du(k) \end{cases}$$

- Since now on we assume that matrix A is asymptotically stable
- The *controllability Grammian* for discrete-time systems is the matrix

$$W_c \triangleq \sum_{j=0}^{\infty} A^j BB'(A')^j$$

MATLAB
```
Wc = gram(sys,'c')
```

- The *observability Grammian* for discrete-time systems is the matrix

$$W_o \triangleq \sum_{j=0}^{\infty} (A')^j C'CA^j$$

MATLAB
```
Wc = gram(sys,'o')
```

- Similar definitions exist for continuous-time systems

For discrete-time systems the controllability Grammian is related to the cost of minimum energy control: $\min \sum_{j=0}^{\infty} u^2(j) = x(0)'W_c^{-1}x(0)$. The observability Grammian to the output energy of the free response: $\sum_{j=0}^{\infty} y^2(j) = x(0)'W_o x(0)$. The Grammians solve the Lyapunov equations $W_c = AW_cA' + BB'$ and $W_o = A'W_oA + C'C$, respectively

Balanced state-space realizations

Definition

A state-space realization is called *balanced* if the Grammians W_c and W_o are equal and diagonal

$$W_c = W_o = \Sigma, \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_n \end{bmatrix}$$

- A procedure to derive the transformation matrix T such that the equivalent state-space form $\tilde{A} = T^{-1}AT,\ \tilde{B} = T^{-1}B,\ \tilde{C} = CT,\ \tilde{D} = D$ is balanced is described in [1]
- the procedure is implemented in the MATLAB function `balreal`

MATLAB
```
[sysb,σ,T^-1,T] = balreal(sys)
```
where $\sigma = [\sigma_1^2 \ \dots \ \sigma_n^2]'$

[1] A.J. Laub, M.T. Heath, C.C. Paige, R.C. Ward, "Computation of System Balancing Transformations and Other Applications of Simultaneous Diagonalization Algorithms," IEEE Trans. Automatic Control, vol. 32, pp. 115-122, 1987

Model reduction

- Once the system is in balanced form we can easily reduce the order of the model by eliminating the states associated with small σ_i's

$$\tilde{W}_c = \tilde{W}_o = \begin{bmatrix} \Sigma_1 & \cancel{0} \\ \cancel{0} & \cancel{\Sigma_2} \end{bmatrix} \qquad \Sigma_2 \ll \Sigma_1$$

$$\begin{aligned} z(k+1) &= \begin{bmatrix} \tilde{A}_{11} & \cancel{\tilde{A}_{12}} \\ \cancel{\tilde{A}_{21}} & \cancel{\tilde{A}_{22}} \end{bmatrix} z(k) + \begin{bmatrix} \tilde{B}_1 \\ \cancel{\tilde{B}_2} \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} \tilde{C}_1 & \cancel{\tilde{C}_2} \end{bmatrix} z(k) \end{aligned}$$

$$\begin{aligned} z_1(k+1) &= \tilde{A}_{11} z_1(k) + \tilde{B}_1 u(k) \\ y(k) &= \tilde{C}_1 z_1(k) \end{aligned}$$

MATLAB

```
rsys = modred(sys,elim,'del')
```

`elim` = indices of the states to be eliminated

- A similar idea applies to continuous-time systems

Example

- Transfer function: $G(s) = \frac{s^3+11s^2+36s+26}{s^4+14.6s^3+74.96s^2+153.7s+99.65}$
- State-space realization in canonical reachability form:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -99.65 & -153.7 & -74.96 & -14.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 26 & 36 & 11 & 1 \end{bmatrix}, \quad D = 0$$

- Controllability and observability Grammians:

$$W_c = \begin{bmatrix} 0.0000 & 0.0000 & -0.0001 & 0.0000 \\ 0.0000 & 0.0001 & -0.0000 & -0.0006 \\ -0.0001 & -0.0000 & 0.0006 & 0.0000 \\ 0.0000 & -0.0006 & 0.0000 & 0.0408 \end{bmatrix} \qquad W_o = \begin{bmatrix} 84.6757 & 122.1890 & 37.4774 & 3.3919 \\ 122.1890 & 179.5744 & 55.2945 & 5.0110 \\ 37.4774 & 55.2945 & 17.0406 & 1.5448 \\ 3.3919 & 5.0110 & 1.5448 & 0.1401 \end{bmatrix}$$

- After balancing, we get

$$\tilde{W}_c = \tilde{W}_o = \begin{bmatrix} 0.1394 & 0 & 0 & 0 \\ 0 & 0.0095 & 0 & 0 \\ 0 & 0 & 0.0006 & 0 \\ 0 & 0 & 0 & 0.0000 \end{bmatrix}$$

MATLAB

```
» sys=ss(A,B,C,D);
» [sysb,sigma,Tinv,T] = balreal(sys);
» Wc=diag(sigma);
» Wo=Wc
```

Example

- After applying the transformation matrix T we get

$$\tilde{A} = \begin{bmatrix} -3.601 & 0.8212 & -0.6163 & -0.05831 \\ -0.8212 & -0.593 & 1.027 & 0.09033 \\ -0.6163 & -1.027 & -5.914 & -1.127 \\ 0.05831 & 0.09033 & 1.127 & -4.492 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} -1.002 \\ -0.1064 \\ -0.08612 \\ 0.008112 \end{bmatrix}$$

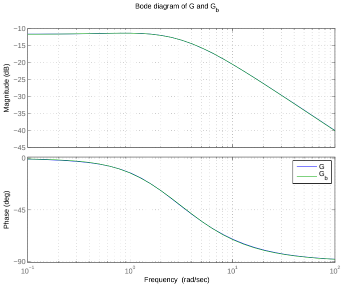
$$\tilde{C} = \begin{bmatrix} -1.002 & 0.1064 & -0.08612 & -0.008112 \end{bmatrix}, \quad \tilde{D} = 0$$

- Let's eliminate states z_3, z_4 and get a model of reduced order 2
- The transfer function of the reduced-order model is

$$G_b(s) = \frac{0.9926s + 0.7297}{s^2 + 4.194s + 2.81}$$

Example (cont'd)

- Let's compare the frequency responses of G and G_b: they are almost indistinguishable !



In general, the smaller are the removed singular values σ_i with respected to the ones we keep, the more similar are the responses of the original and reduced-order models

- Note that after balancing the states completely loose their physical meaning
- The original state x can be recovered (approximately) from the reduced state z_1 using the transformation matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$

$$x = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} z_1 \quad \leftarrow x \text{ is treated here as an output of the reduced-order model}$$

- The reduced state z_1 can be estimated by a state observer (the pair $\tilde{A}_1, \tilde{C}_1$ is observable by construction)

Matched DC gain method for model reduction

- Consider again the complete model

$$\begin{aligned}
x_1(k+1) &= A_{11}x_1(k)+A_{12}x_2(k)+B_1u(k)\\
x_2(k+1) &= A_{21}x_1(k)+A_{22}x_2(k)+B_2u(k)\\
y(k) &= C_1x_1(k)+C_2x_2(k)
\end{aligned}$$

- Assume the dynamics of x_2 are "infinitely fast": $x_2(k+1)\approx x_2(k)$
- We can eliminate the states contained in x_2 by substituting

$$x_2(k)=(I-A_{22})^{-1}(A_{21}x_1(k)+B_2u(k))$$

therefore obtaining

$$\begin{aligned}
x_1(k+1) &= (A_{11}+A_{12}(I-A_{22})^{-1}A_{21})x_1(k)+(B_1+A_{12}(I-A_{22})^{-1}B_2)u(k)\\
y(k) &= (C_1+C_2(I-A_{22})^{-1}A_{21})x_1(k)+C_2(I-A_{22})^{-1}B_2u(k)
\end{aligned}$$

- A similar idea can be applied in continuous-time, by setting $\dot{x}_2(t)\approx 0$

$$x_2(t)=A_{22}^{-1}(A_{21}x_1(t)+B_2u(t))$$

Matched DC gain method for model reduction

Property

The matched DC-gain method preserves the DC gain of the original full-order model

Proof:

Simply observe that for both the original and the reduced-order model in steady-state x_1, x_2 depend on u in the same way □

MATLAB

```
rsys=modred(sys,elim,'mdc')
```

`elim` is the vector of state indices to eliminate

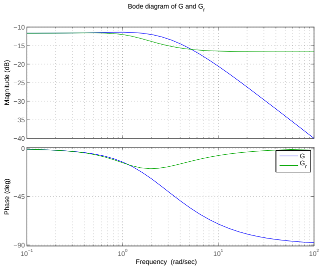
Example (cont'd)

- Consider again the state-space realization

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -99.65 & -153.7 & -74.96 & -14.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 26 & 36 & 11 & 1 \end{bmatrix}, \quad D = 0$$

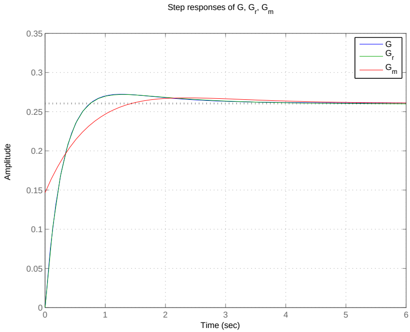
- Let's eliminate the states x_3 and x_4 using the matched DC-gain method
- We get a 2$^{\text{nd}}$ order model whose transfer function is

$$G_m(s) = \frac{0.1467s^2 + 0.4803s + 0.3469}{s^2 + 2.05s + 1.329}$$



- Emphasis here is on matching at low frequencies (DC gain in particular!)
- DC gains: $G(0) = G_m(0) = 0.2609$

Example (cont'd)



- Original DC gain: $G(0) = 0.2609$
- DC gain of $G_m(s)$: $G_m(0) = 0.2609$
- DC gain of $G_b(s)$: $G_b(0) = 0.2597$

- The matched DC-gain method is only good to capture the DC gain exactly
- Model reduction via balanced transformation provides best match

Comments on model reduction

- How good is the reduced-order model should be judged on the performance of the original system in closed-loop with a controller based on the reduced model
- A good reduced-order model provides very good closed-loop performance and a low-order dynamic control law at the same time
- Let's see an example ...

Example (cont'd)

- Consider the LQR performance index

$$\min \sum_{k=0}^{\infty} y^2(k) + \rho u^2(k)$$

- LQR controller based on complete model (A, B, C, D):

$$u(t) = Kx(t) + Hr(t), \qquad H = \frac{1}{C(-A-BK)^{-1}B+D}$$

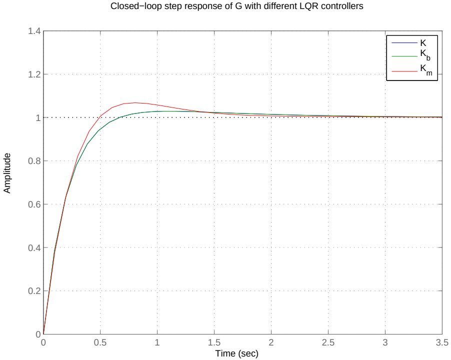
- LQR controller based on reduced model (A_1, B_1, C_1, D_1):

$$\begin{aligned} \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] &= T^{-1}x \\ u(t) &= K_b z_1(t) + H_b r(t), \qquad H_b = \frac{1}{C(-A-BK_b[I\ \ 0]T^{-1})^{-1}B+D} \end{aligned}$$

- LQR controller based on the model reduced by the matched DC gain method:

$$\begin{aligned} x_1 &= \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right] x \\ u(t) &= K_m x_1(t) + H_m r(t), \qquad H_m = \frac{1}{C(-A-BK_m[I\ \ 0])^{-1}B+D} \end{aligned}$$

Example (cont'd)



$$\begin{aligned} u &= Kx \\ &= \begin{bmatrix} -29.5403 & -43.0578 & -13.2412 & -1.2001 \end{bmatrix} x \end{aligned}$$

$$\begin{aligned} u &= \begin{bmatrix} K_b & 0 \end{bmatrix} T^{-1} x \\ &= \begin{bmatrix} -29.5145 & -43.0825 & -13.2468 & -1.1999 \end{bmatrix} x \end{aligned}$$

$$\begin{aligned} u &= K_m \begin{bmatrix} I & 0 \end{bmatrix} x \\ &= \begin{bmatrix} -17.5373 & -21.4324 & 0 & 0 \end{bmatrix} x \end{aligned}$$

- Note the similarity between controllers K and $\begin{bmatrix} K_b & 0 \end{bmatrix} T^{-1}$

English-Italian Vocabulary

	
model reduction	*riduzione dell'ordine del modello*
Grammian	*gramiano*

Translation is obvious otherwise.