Automatic Control 2

Nonlinear systems

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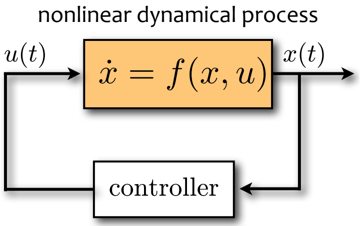
Academic year 2010-2011

Nonlinear dynamical systems



- Most existing processes in practical applications are described by *nonlinear dynamics* $\dot{x} = f(x,u)$
- Often the dynamics of the system can be *linearized* around an operating point and a *linear controller* designed for the linearized process
- Question #1: will the closed-loop system composed by the nonlinear process + linear controller be asymptotically stable ? (nonlinear stability analysis)
- Question #2: can we design a stabilizing *nonlinear controller* based on the nonlinear open-loop process ? (nonlinear control design)

This lecture is based on the book "Applied Nonlinear Control" by J.J.E. Slotine and W. Li, 1991

Positive definite functions

- Key idea: if the energy of a system dissipates over time, the system asymptotically reaches a minimum-energy configuration
- Assumptions: consider the autonomous nonlinear system $\dot{x} = f(x)$, with $f(\cdot)$ differentiable, and let $x = 0$ be an equilibrium ($f(0) = 0$)
- Some definitions of positive definiteness of a function $V : \mathbb{R}^n \mapsto \mathbb{R}$
  - V is called *locally positive definite* if $V(0) = 0$ and there exists a *ball* $B_\epsilon = \{x : \|x\|_2 \leq \epsilon\}$ around the origin such that $V(x) > 0 \ \forall x \in B_\epsilon \setminus 0$
  - V is called *globally positive definite* if $B_\epsilon = \mathbb{R}^n$ (i.e. $\epsilon \to \infty$)
  - V is called *negative definite* if $-V$ is positive definite
  - V is called *positive semi-definite* if $V(x) \geq 0 \ \forall x \in B_\epsilon, x \neq 0$
  - V is called *positive semi-negative* if $-V$ is positive semi-definite
- Example: let $x = [x_1 \ x_2]'$, $V : \mathbb{R}^2 \to \mathbb{R}$
  - $V(x) = x_1^2 + x_2^2$ is globally positive definite
  - $V(x) = x_1^2 + x_2^2 - x_1^3$ is locally positive definite
  - $V(x) = x_1^4 + \sin^2(x_2)$ is locally positive definite and globally positive semi-definite

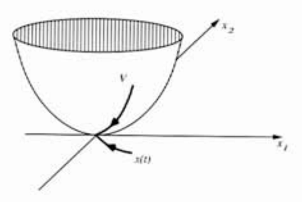
Lyapunov's direct method

Theorem

Given the nonlinear system $\dot{x} = f(x)$, $f(0) = 0$, let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be positive definite in a ball B_ϵ around the origin, $\epsilon > 0$, $V \in C^1(\mathbb{R})$. If the function

$$\dot{V}(x) = \nabla V(x)'\dot{x} = \nabla V(x)'f(x)$$

is negative definite on B_ϵ, then the origin is an asymptotically stable equilibrium point with *domain of attraction* B_ϵ ($\lim_{t\to+\infty} x(t) = 0$ for all $x(0) \in B_\epsilon$). If $\dot{V}(x)$ is only negative semi-definite on B_ϵ, then the the origin is a stable equilibrium point.



Such a function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is called a *Lyapunov function* for the system $\dot{x} = f(x)$

Example of Lyapunov's direct method

- Consider the following autonomous dynamical system

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$
$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)$$

- as $f_1(0,0) = f_2(0,0) = 0$, $x = 0$ is an equilibrium
- consider then the candidate Lyapunov function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

which is globally positive definite. Its time derivative $\dot{V}$ is

$$\dot{V}(x_1, x_2) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$$

- It is easy to check that $\dot{V}(x_1, x_2)$ is negative definite if $\|x\|_2^2 = x_1^2 + x_2^2 < 2$. Then for any B_ϵ with $0 < \epsilon < \sqrt{2}$ the hypotheses of Lyapunov's theorem are satisfied, and we can conclude that $x = 0$ is an asymptotically stable equilibrium
- Any B_ϵ with $0 < \epsilon < \sqrt{2}$ is a domain of attraction

Example of Lyapunov's direct method (cont'd)

- Cf. Lyapunov's indirect method: the linearization around $x = 0$ is

$$\frac{\partial f(0,0)}{\partial x} = \left.\begin{bmatrix} 3x_1^2 - 3x_2^2 - 2 & -6x_1x_2 \\ 10x_1x_2 & 5x_1^2 + 3x_2^2 - 2 \end{bmatrix}\right|_{x=0} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

  which is an asymptotically stable matrix
- Lyapunov's indirect method tells us that the origin is *locally* asymptotically stable
- Lyapunov's direct method also tells us that B_ϵ is a domain of attraction for all $0 < \epsilon < \sqrt{2}$

- Consider this other example: $\dot{x} = -x^3$. The origin as an equilibrium. But $\frac{\partial f(0,0)}{\partial x} = -3 \cdot 0^2 = 0$, so Lyapunov indirect method is useless.
- Lyapunov's direct method with $V = x^2$ provides $\dot{V} = -2x^4$, and therefore we can conclude that $x = 0$ is (globally) asymptotically stable

Case of continuous-time linear systems

Let's apply Lyapunov's direct method to linear autonomous systems $\dot{x} = Ax$

- Let $V(x) = x'Px$, with $P = P' \succ 0$ (P=positive definite and symmetric matrix)
- The derivative $\dot{V}(x) = \dot{x}'Px + x'P\dot{x} = x'(A'P + PA)x$
- $\dot{V}(x)$ is negative definite if and only if

$$A'P + PA = -Q$$

  for some $Q \succ 0$ (for example, $Q = I$)
- Given a matrix $Q \succ 0$, the matrix equation $A'P + PA = -Q$ is called *Lyapunov equation*

Theorem:

The autonomous linear system $\dot{x} = Ax$ is asymptotically stable $\Leftrightarrow$ $\forall Q \succ 0$ the Lyapunov equation $A'P + PA = -Q$ has one and only one solution $P \succ 0$

MATLAB

```
»P=lyap(A',Q)
```

⟵ Note the transposition of matrix A !

Case of discrete-time linear systems

- Lyapunov's direct method also applies to discrete-time nonlinear systems [1] $x(k+1) = f(x(k))$, considering positive definite functions $V(x)$ and the differences along the system trajectories

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k))$$

  instead of the derivative $\dot{V}(x)$

- Set again $V(x) = x'Px$, with $P \succ 0$, and impose ΔV is negative definite

$$\Delta V(x) = (Ax)'P(Ax) - x'Px = x'(A'PA - P)x = -x'Qx$$

- $\Delta V(x)$ is negative definite if and only if for some $Q \succ 0$

$$A'PA - P = -Q$$

MATLAB
`»P=dlyap(A',Q)`

- The matrix equation $A'PA - P = -Q$ is called *discrete-time Lyapunov equation*

[1] J.P. LaSalle, "Stability Theory for Difference Equations," in Studies in Ordinary Differential Equations, MAA studies in Mathematics, Jack Hale Ed., vol. 14, pp.1–31, 1997

Nonlinear control design

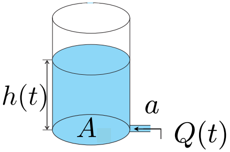
- In *nonlinear control design* a (usually nonlinear) feedback control law is designed based on the nonlinear dynamics $\dot{x} = f(x, u)$
- Most nonlinear control design techniques are based on simultaneously constructing a feedback control law $u(x)$ *and* a Lyapunov function V for $\dot{x} = f(x, u(x))$
- A simple nonlinear technique is *feedback linearization*, that is to algebraically transform the dynamics of the nonlinear system into a linear one and then apply linear control techniques to stabilize the transformed system
- Example:

$$
\begin{array}{rcl}
\dot{x}_1 & = & a_{11}x_1 + a_{12}x_2 \\
\dot{x}_2 & = & \underbrace{f_2(x_1, x_2) + g_2(x_1, x_2)u}_{v}
\end{array}
\longrightarrow
\begin{array}{l}
\dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\
\dot{x}_2 = v
\end{array}
$$

$$
v = Kx \text{ (stabilizing gain)} \longrightarrow u = \frac{Kx - f_2(x_1, x_2)}{g_2(x_1, x_2)}
$$

- Very successful in several control applications (robotics, aeronautics, ...)
- Note the difference between *feedback linearization* and conventional linearization $\dot{x} = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0)$ we've seen earlier !

Example of feedback linearization



- Consider the problem of regulating the fluid level h in a tank to a fixed set-point h_d
- The process is described by the nonlinear dynamics

$$A\dot{h}(t) = -a\sqrt{2gh(t)} + u(t)$$

- We don't want to linearize around $h = h_d$ and use linear control techniques, that would only ensure *local* stability (cf. Lyapunov linearization method)
- Define instead

$$u(t) = a\sqrt{2gh(t)} + Av(t)$$

where $v(t)$ is a new "equivalent input" to be defined next by a control law

Example of feedback linearization (cont'd)

- The resulting dynamics becomes

$$\dot{h}(t) = v(t)$$

- Choose $v(t) = -\alpha e(t)$, with $\alpha > 0$ and $e(t) = h(t) - h_d$. The resulting closed-loop error dynamics becomes

$$\dot{e}(t) = -\alpha e(t)$$

which is asymptotically stable ($h(t)$ tends asymptotically to h_d)

- The resulting nonlinear control law applied to the tank system is

$$u(t) = \underbrace{a\sqrt{2gh(t)}}_{\text{nonlinearity cancellation}} - A\alpha(\underbrace{h(t) - h_d}_{\text{linear error feedback}})$$

- Can we generalize this idea ?

Feedback linearization

- Consider for simplicity single-input nonlinear systems, $u \in \mathbb{R}$
- Let the dynamical system be in *nonlinear canonical controllability form*

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= x_3(t)\\
&\vdots\\
\dot{x}_n(t) &= f_n(x(t)) + g_n(x(t))u(t)
\end{aligned}$$

  and assume $g_n(x) \neq 0,\ \forall x \in \mathbb{R}^n$
- Define $u(t) = \frac{1}{g_n(x(t))}\left(v(t) - f_n(x(t))\right)$ to get the equivalent linear system

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= x_3(t)\\
&\vdots\\
\dot{x}_n(t) &= v(t)
\end{aligned}$$

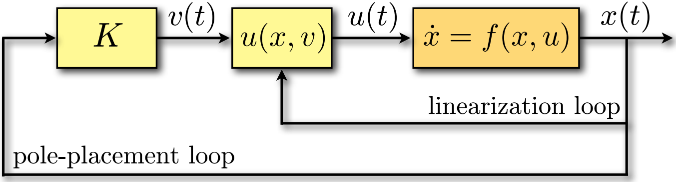
Feedback linearization (cont'd)

- The resulting equivalent linear system $\dot{x} = Ax + Bv$ is the cascade of n integrators

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

  and is completely reachable

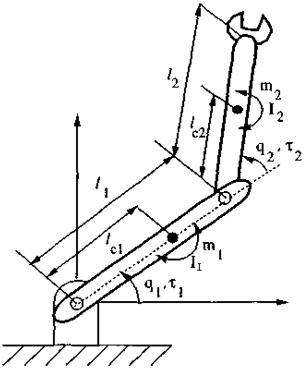
- To steer the state x asymptotically to the origin, we design a control law $v = Kx$ (by pole-placement, LQR, etc.)



- Note that the nonlinear model must be rather accurate for feedback linearization to work

Feedback linearization for robotic manipulation

- Consider a two-link robot
- Each joint equipped with a motor providing input torque τ_i, an encoder measuring the joint position q_i, and a tachometer measuring the joint velocity $\dot{q}_i$, $i = 1, 2$
- Objective of control design: make $q_1(t)$ and $q_2(t)$ follow desired position histories $q_{d1}(t)$ and $q_{d2}(t)$
- $q_{d1}(t)$ and $q_{d2}(t)$ are specified by the motion planning system of the robot



Feedback linearization for robotic manipulation

- Use Lagrangean equations to determine the dynamic equations of the robot

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

where

$$\begin{aligned} H_{11} &= m_1 l_{c_1}^2 + I_1 + m_2 \left(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cos q_2 \right) + I_2 \\ H_{22} &= m_2 l_{c_2}^2 + I_2 \\ H_{12} = H_{21} &= m_2 l_1 l_{c_2} \cos q_2 + m_2 l_{c_2}^2 + I_2 \\ h &= m_2 l_1 l_{c_2} \sin q_2 \\ g_1 &= m_1 l_{c_1} g \cos q_1 + m_2 g \left(l_{c_2} \cos(q_1 + q_2) + l_1 \cos q_1 \right) \\ g_2 &= m_2 l_{c_2} g \cos(q_1 + q_2) \end{aligned}$$

Feedback linearization for robotic manipulation

- The system dynamics can be compactly written as

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

- Multiply both sides by $H^{-1}(q)$ and obtain the second-order differential equation

$$\ddot{q} = -H^{-1}(q)C(q,\dot{q})\dot{q} - H^{-1}(q)g(q) + \tau$$

- Define the control input τ to feedback-linearize the robot dynamics

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

  where

$$v = \ddot{q}_d - 2\lambda\dot{e} - \lambda^2 e, \ \lambda > 0$$

  $v = [v_1 \ v_2]$ is the equivalent input, and $e = q - q_d$ being the tracking error on positions

Feedback linearization for robotic manipulation

- The resulting error dynamics is

$$\ddot{e} + 2\lambda\dot{e} + \lambda^2 e = 0,\ \lambda > 0$$

leading to the asymptotic convergence of the tracking error $e(t) = q(t) - q_d(t)$ and its derivative $\dot{e}(t) = \dot{q}(t) - \dot{q}(t)$ to zero

$$\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} = e^{-\lambda t} \begin{bmatrix} 1+\lambda t & t \\ -\lambda^2 t & 1-\lambda t \end{bmatrix} \begin{bmatrix} e(0) \\ \dot{e}(0) \end{bmatrix}$$

MATLAB

```
» syms lam t
» A=[0 1;-lam^2 -2*lam];
» factor(expm(A*t))
```

- In robotics, feedback linearization is also known as *computed torque*, and can be applied to robots with an arbitrary number of joints

English-Italian Vocabulary

	
nonlinear system	*sistema non lineare*
Lyapunov function	*funzione di Lyapunov*
feedback linearization	*feedback linearization*

Translation is obvious otherwise.