Automatic Control 2

Sampling

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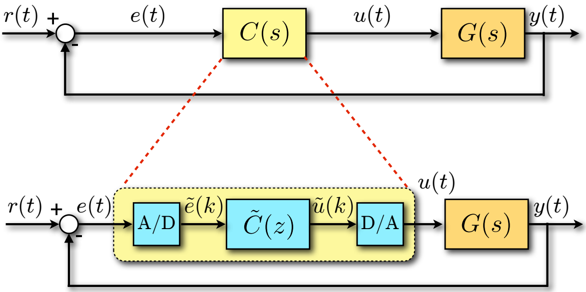
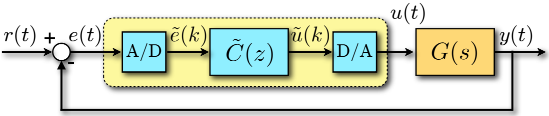
Time-discretization of continuous-time controllers

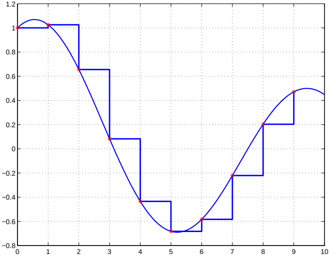


- We have designed an analog controller $C(s)$ in the *continuous-time* domain (by loop shaping, pole-placement, etc.)
- We want to implement $C(s)$ in digital form (for example, in a microcontroller). We need to convert the design to *discrete-time*

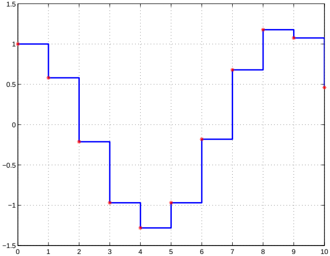
Relations between continuous/discrete-time signals



- Let the digital controller operate with sampling time T

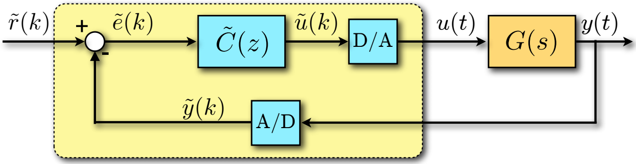


tracking error $\tilde{e}(k) \triangleq e(kT)$
for $k \in \{0, 1, \dots\}$



control input $u(t) = \tilde{u}(k) \triangleq u(kT)$
for $t \in [kT, (k+1)T)$

Time-discretization of continuous-time controllers



- Alternative scheme: more often the reference signal $r(t)$ is already given in digital form by its samples $\tilde{r}(k)$, $\tilde{r}(k) = r(kT)$, $k = 0, 1, \ldots$, T=sampling time

Problem

How to synthesize a *discrete-time* control law $\tilde{C}(z)$ that in closed-loop behaves as the given *continuous-time* controller $C(s)$?

Time-discretization

- Consider the continuous-time controller $C(s)$

$$u(t) = C(s)e(t) = \frac{N(s)}{D(s)}e(t) = \frac{b_{n-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}e(t)$$

- We can look at the control law as at a differential equation linking $u(t)$ to $e(t)$

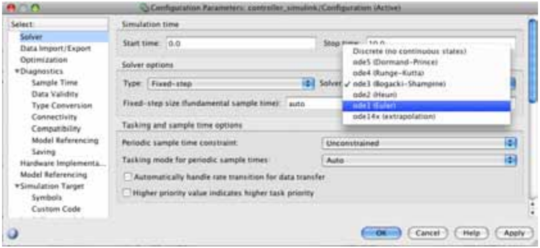
$$\frac{d^n}{dt^n}u(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}u(t) + \ldots + a_0 u(t) = b_{n-1}\frac{d^{n-1}}{dt^{n-1}}e(t) + \ldots + b_0 e(t)$$

- A *time-discretization* of $u(t) = C(s)e(t)$ is nothing else than a *numerical approximation* (with constant *integration-step* T)

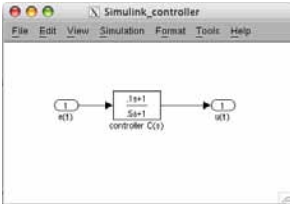
$$\tilde{u}(k) = \tilde{C}(z)\tilde{e}(k)$$

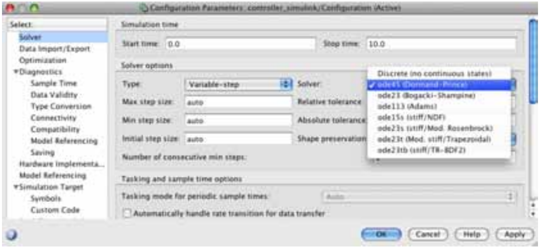
  linking the input samples $\tilde{u}(k) \triangleq u(kT)$ to the error samples $\tilde{e}(k) \triangleq e(kT)$
- There are many numerical methods to integrate a differential equation (constant step, variable step, n-th order approximations, etc.)

Numerical integration in Simulink



Fixed-step integration panel





Variable-step integration panel

Example of time-discretization

- Consider the continuous-time controller

$$u(t) = \frac{1}{s+2}e(t) \quad \Rightarrow \quad \frac{d}{dt}u(t) + 2u(t) = e(t)$$

- Use *Euler method* to approximate the derivative

$$\frac{d}{dt}u(t) \approx \frac{u((k+1)T) - u(kT)}{T} = \frac{1}{T}(\tilde{u}(k+1) - \tilde{u}(k))$$

- Recall that z represents the unit-shift operator $z\tilde{u}(k) = \tilde{u}(k+1)$

$$\tilde{u}(k) = \frac{1}{\left(\frac{z-1}{T}\right) + 2}\tilde{e}(k) \triangleq \tilde{C}(z)\tilde{e}(k)$$

- Note that formally this is equivalent to replace $s = \left(\frac{z-1}{T}\right)$ in $C(s)$ to get $\tilde{C}(z)$

Finite-difference approximation of the controller

- Consider a state-space realization of $C(s)$

$$\begin{cases} \frac{dx_c}{dt} & = & A_c x_c + B_c e \\ u & = & C_c x_c + D_c e \end{cases}$$

$$C(s) = C_c(sI - A_c)^{-1}B_c + D_c$$

- Integrate between time kT and $(k+1)T$

$$x_c((k+1)T) - x_c(kT) = \int_{kT}^{(k+1)T} \frac{dx_c(\tau)}{dt} d\tau$$

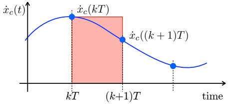
gives

$$x_c((k+1)T) - x_c(kT) = A_c \int_{kT}^{(k+1)T} x_c(\tau)d\tau + B_c \int_{kT}^{(k+1)T} e(\tau)d\tau$$

- Unfortunately, in general both $x(\tau)$ and $e(\tau)$ are not constant between consecutive sampling instants, so we can't use *exact* discretization (i.e., the exponential matrix $e^{A_c T}$)

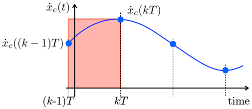
Approximation of the integral of $\dot{x}_c(\tau)$

- *(forward) Euler method*



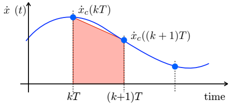
$$\underbrace{x_c((k+1)T) - x_c(kT)}_{(z-1)x_c(kT)} = \int_{kT}^{(k+1)T} \dot{x}_c(\tau)d\tau \approx \underbrace{T\dot{x}_c(kT)}_{Tsx_c(kT)} \quad \rightarrow \quad s = \frac{z-1}{T}$$

- *backward Euler method*



$$\underbrace{x_c(kT) - x_c((k-1)T)}_{(1-z^{-1})x_c(kT)} = \int_{(k-1)T}^{kT} \dot{x}_c(\tau)d\tau \approx \underbrace{T\dot{x}_c(kT)}_{Tsx_c(kT)} \quad \rightarrow \quad s = \frac{1-z^{-1}}{T}$$

- *Trapezoidal rule*



$$\underbrace{x_c((k+1)T) - x_c(kT)}_{(z-1)x_c(kT)} = \int_{kT}^{(k+1)T} \dot{x}_c(\tau)d\tau \approx \underbrace{\frac{T[\dot{x}_c((k+1)T) + \dot{x}_c(kT)]}{2}}_{\frac{T}{2}(z+1)sx_c(kT)}$$

$$\rightarrow \quad s = \frac{2(z-1)}{T(z+1)}$$

Finite-difference approximation of the controller

$$
\begin{aligned}
(z-1)x_c &= T(A_c x_c + B_c e) \longrightarrow \left[\left(\frac{z-1}{T}\right)I - A_c\right]x_c = B_c e \\
\tilde{C}(z) &= \frac{U(z)}{E(z)} = C_c\left[\left(\frac{z-1}{T}\right)I - A_c\right]^{-1}B_c + D_c = C\left(\frac{z-1}{T}\right)
\end{aligned}
$$

forward Euler method

$$
\begin{aligned}
(1-z^{-1})x_c &= T(A_c x_c + B_c e) \longrightarrow \left[\left(\frac{z-1}{zT}\right)I - A_c\right]x_c = B_c e \\
\tilde{C}(z) &= \frac{U(z)}{E(z)} = C_c\left[\left(\frac{1-z^{-1}}{T}\right)I - A_c\right]^{-1}B_c + D_c = C\left(\frac{1-z^{-1}}{T}\right)
\end{aligned}
$$

Backward Euler method

$$
\tilde{C}(z) = C_c\left[\left(\frac{2(z-1)}{T(z+1)}\right)I - A_c\right]^{-1}B_c + D_c = C\left(\frac{2(z-1)}{T(z+1)}\right)
$$

Tustin's method

Finite-difference approximation of the controller

- Formally, we just replace s in $C(s)$ with the corresponding function of z:
  - forward Euler method

$$s = \frac{z-1}{T} \quad \longrightarrow \quad \dot{x}_c(kT) \approx \frac{x_c((k+1)T) - x_c(kT)}{T}$$

  - backward Euler method

$$s = \frac{1-z^{-1}}{T} \quad \longrightarrow \quad \dot{x}_c(kT) \approx \frac{x_c(kT) - x_c((k-1)T)}{T}$$

  - Trapezoidal rule (*Tustin's method*)

$$s = \frac{2(z-1)}{T(z+1)} \quad \text{(bilinear transformation)}$$

- Note that all three methods preserve the DC gain: $z = 1 \rightarrow s = 0$ (no approximation error exists in constant steady-state !)
- Compare to the *exact discretization method*: $\tilde{A}_c = e^{TA_c}$, $\tilde{B}_c = \int_0^T e^{tA_c} dt B_c$, $\tilde{C}_c = C_c$, $\tilde{D}_c = D_c$

Relations between poles in s and z

- What is the relation between the poles s_i of $C(s)$ and the poles z_i of $\tilde{C}(z)$?
  - forward Euler differences:

$$\frac{z_i - 1}{T} = s_i \quad \Rightarrow z_i = 1 + Ts_i$$

  - backward Euler differences:

$$\frac{z_i - 1}{Tz_i} = s_i \quad \Rightarrow z_i = \frac{1}{1 - s_i T}$$

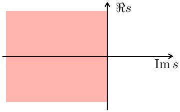
  - Tustin's method:

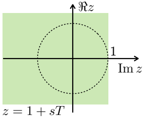
$$\frac{2(z_i - 1)}{T(z_i + 1)} = s_i \quad \Rightarrow z_i = \frac{1 + s_i T/2}{1 - s_i T/2}$$

  - Exact method: $z_i = e^{s_i T}$
- Note that the three (approximate) integration methods approximate the function $z = e^{sT}$ by a rational function

Relations between poles in s and z

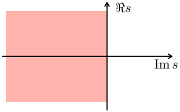
forward Euler differences

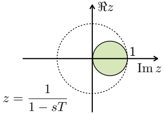




stable poles in s may be mapped to unstable poles in z

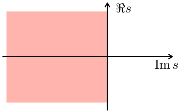
backward Euler differences

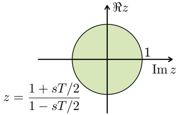




stable poles in s are also stable poles in z, marginally stable poles may become asymptotically stable

Tustin's rule





stable poles in s are also stable poles in z

Time-discretization in MATLAB

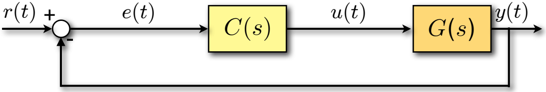
MATLAB

```
C2D Conversion of continuous-time models to discrete time.

    SYSD = C2D(SYSC,TS,METHOD) converts the continuous-time LTI
    model SYSC to a discrete-time model SYSD with sample time TS.
    The string METHOD selects the discretization method among the
    following:
       'zoh'       Zero-order hold on the inputs.
       'foh'       Linear interpolation of inputs (triangle appx.)
       'tustin'    Bilinear (Tustin) approximation.
       'prewarp'   Tustin approximation with frequency prewarping.
                   The critical frequency Wc is specified as fourth
                   input by C2D(SYSC,TS,'prewarp',Wc).
       'matched'   Matched pole-zero method (for SISO systems only).
    The default is 'zoh' when METHOD is omitted.
```

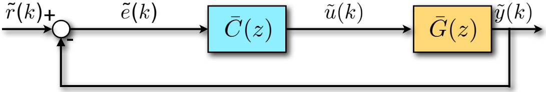
Sampled-data systems

- continuous-time control system (Laplace transform/frequency analysis)



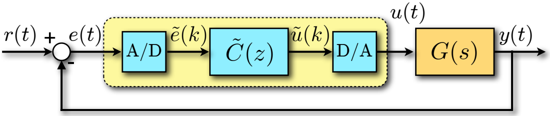
t = continuous time

- discrete-time control system (Z-transform)



k = discrete-time step counter

- *sampled-data system*

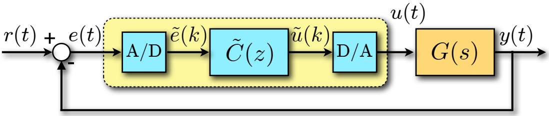


continuous-time and discrete-time signals

Why analyzing sampled-data systems

- The model $G(s)$ of the *process* to be controlled is (very often) given in continuous time
  - differential equations
  - actuator signals (electrical voltages to motors, etc.) vary continuously in time
  - output variables (temperature, pressure, position, etc.) vary continuously in time
- On the other hand, the *controller* is (almost always) implemented in digital form, $\tilde{C}(z)$:
  - cheaper to implement (computer code)
  - easier to reconfigure
  - can exploit time-sharing (multiple controllers on the same hardware)
  - much more versatile (arbitrary nonlinear control laws)
- Hence the need to analyze sampled-data systems, namely a continuous process in closed loop with a digital controller

Ways to analyze sampled-data systems



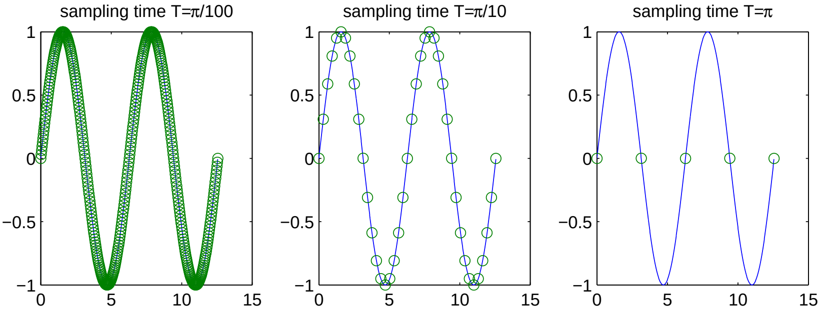
1. Convert the system $G(s)$ to its discrete-time equivalent $\bar{G}(z)$ (use for instance exact sampling), ignoring its inter-sampling behavior (controller's point of view or "*stroboscopic model*") $\Longrightarrow$ discrete-time analysis
2. Model the digital controller in continuous time

$$U(s) \simeq \frac{1-e^{-sT}}{sT}\tilde{C}(e^{sT})E(s)$$ (this can be shown ...)

   (process' point of view) $\Longrightarrow$ continuous-time analysis
3. Use numerical simulation (e.g., Simulink) (only provides an answer for a certain finite set of initial states)

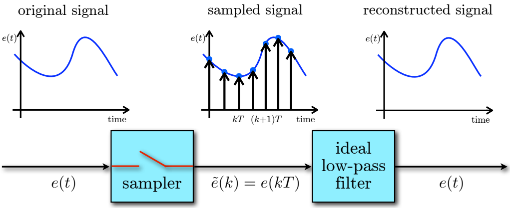
Choosing the sampling time

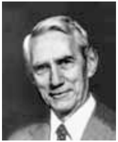
- How to choose the sampling time T ?
- Example: sampling of $\sin(t)$



- How can we say that a sampling time is “good” ?

Nyquist-Shannon sampling theorem





Claude Elwood Shannon

(1916–2001)

Sampling theorem

Let $e(t)$ be a signal and $E(j\omega) = \int_{-\infty}^{+\infty} e(\tau)e^{-j\omega\tau}d\tau$ its *Fourier transform* $\mathscr{F}[e]$.

Let $E(j\omega) = 0$ for $|\omega| \geq \omega_{\max}$. For all T such that $\omega_N \triangleq \frac{\pi}{T} > \omega_{\max}$

$$e(t) = \sum_{k=-\infty}^{+\infty} e(kT)\frac{\sin(\omega_N(t-kT))}{\omega_N(t-kT)}$$

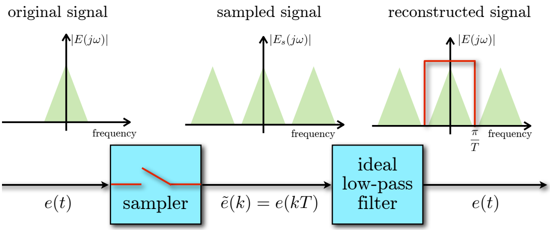
ω_N is called the *Nyquist frequency*, equal to half the sampling frequency $\omega_s = \frac{2\pi}{T}$

Shannon's reconstruction

- We can look at sampling as at the modulation of a *Dirac comb* $\sum_{k=-\infty}^{\infty} \delta(t-kT)$
- Let's go to Fourier transforms

$$\underbrace{\mathscr{F}\left[\sum_{k=-\infty}^{\infty} e(kT)\delta(t-kT)\right]}_{\text{modulated Dirac comb}} = \sum_{k=-\infty}^{\infty} e(kT)e^{-jkT\omega} = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(j(\omega+k\omega_s)) \triangleq \frac{1}{T}E_s(j\omega)$$

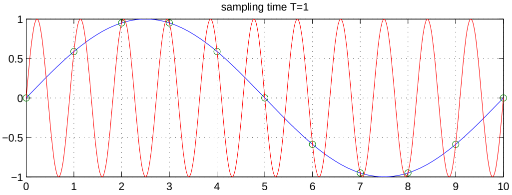
- $E_s(j\omega)$ is the periodic repetition of $E(j\omega)$ with period ω_s
- The ideal low-pass filter can only reconstruct $e(t)$ if $E(j\omega)=0$ for $|\omega|\geq\omega_{\max}$



Aliasing

$$E_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} E(j(\omega + k\omega_s))$$

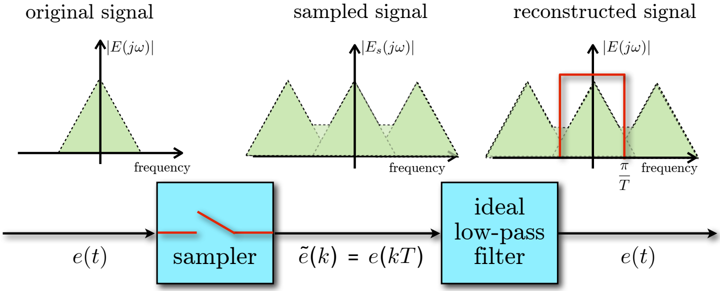
- The value $E_s(j\omega_1)$ at a certain frequency ω_1 not only depends on $E(j\omega_1)$, but also on all the (infinite) contributions $E(j(\omega_1 \pm k\omega_s))$
- The frequency $\omega_1 \pm k\omega_s$ is called an *alias* of ω_1
- Example: consider the signals $\sin(\omega_1 t)$ and $\sin(\omega_2 t)$ and let the sampling time $T = 1$ s



$$\omega_1 = \frac{2\pi}{10}$$
$$\omega_2 = \frac{2\pi}{T} + \frac{2\pi}{10} = \frac{22\pi}{10}$$

Aliasing

- What if $e(t)$ has spectral contributions $E(j\omega) \neq 0$ for $\omega > \omega_N = \frac{\pi}{T}$?



- *Aliasing* is the phenomenon for which frequencies $\omega > \omega_N$ make contributions in the frequency range $[-\omega_N, \omega_N]$
- Under aliasing conditions it is impossible to reconstruct the original signal
- An *anti-aliasing filter* is a low-pass filter that removes from $e(t)$ its high-frequency contents before sampling it. It partially mitigates the problem

Selecting the sampling time

- In *signal processing* one is interested in making the difference between the original and the reconstructed signal as small as possible (=high fidelity)
- In *control systems* one is interested that the closed-loop system behaves according to specs, not much in carefully reconstructing $e(t) = y(t) - r(t)$!
- For control purposes, the sampling time is mainly related to the closed-loop bandwidth / settling-time
- The sampling time used in control is typically larger than the one used signal processing
- That's why in control applications we are often ok with micro-controllers and don't need DSPs (digital signal processors)

Selecting the sampling time

- Let ω_c be the desired bandwidth of the closed-loop system
- To avoid aliasing effects (and satisfy the sampling theorem) we must set

$$\frac{\pi}{T} > \omega_c$$

- A good choice is

$$5\omega_c \leq \frac{2\pi}{T} \leq 100\omega_c$$

  so that the (non-ideal) anti-aliasing filter has cutoff frequency ω_f in between ω_c and ω_N
- By recalling the approximate relation $t_s\omega_c \approx \frac{5}{\zeta}$, where t_s is the settling time (1%) and ζ is the damping factor, we get

$$\frac{t_s}{100} \leq T \leq \frac{t_s}{5}$$

- A related good choice is also to let $T = \frac{t_r}{10}$, where t_r is the rise time of the open-loop system

Numerical errors

- Recall the relation between poles s_i and z_i

$$z_i = e^{s_i T}$$

- For $T \to 0$, we have $z_i \to 1$ whatever the poles s_i are !
- This can make troubles when working in finite precision (both in control design and in controller implementation)
- Consider the following example: we have two poles $s_1 = -1$ e $s_2 = -10$
  - Sample with $T = 1\ ms$ and get $z_1 = e^{-0.001} \approx 0.9990$, $z_2 = e^{-0.01} \approx 0.9900$. If we truncate after two digits, we get $z_1 = z_2$, and then $s_1 = \frac{1}{T}\ln z_1 = \frac{1}{T}\ln z_2 = s_2 \approx -10.05$
  - Sample with $T = 100\ ms$ and truncate, we get $z_1 = 0.90$, $z_2 = 0.36$, and then $s_1 \approx -1.05$, $s_2 \approx -10.21$

Final remarks on choice of sampling time

- Make the sampling time T small enough to reproduce the open-loop time response enough precisely ($T = \frac{t_r}{10}$), and to avoid aliasing effects (Nyquist frequency $\frac{\pi}{T}$ larger than closed-loop bandwidth)
- Make the sampling time T small enough to react enough readily to disturbances affecting the system
- Make the sampling time large enough to avoid numerical issues
- Make the sampling time large enough to avoid fast and expensive control hardware

Digital PID control

Any of the time-discretization we have seen so far can be used to convert a PID design in digital form. The following is most used:

- proportional part

$$P(t) = K_p(br(t) - y(t))$$ static relation, no need to approximate it !

- integral part

$$I(t) = \frac{K_p}{T_i}\int_0^t e(\tau)\,d\tau$$

is approximated by forward Euler as

$$I((k+1)T) = I(kT) + \frac{K_p T}{T_i} e(kT)$$

Digital PID

- derivative part:

$$\frac{T_d}{N}\frac{dD(t)}{dt}+D(t)=-K_pT_d\frac{dy(t)}{dt}$$

  is approximated by backward Euler as

$$D(kT)=\frac{T_d}{T_d+NT}D((k-1)T)-\frac{K_pT_dN}{T_d+NT}\big(y(kT)-y((k-1)T)\big)$$

  Note that the discrete pole $z=\frac{T_d}{T_d+NT}$ is inside the unit circle

- The complete control signal is

$$u(kT)=P(kT)+I(kT)+D(kT)$$

This type of approximation allows one to calculate P, I, and D actions separately

Digital PID in incremental form

- The digital PID form we described provides the full signal $u(kT)$ (*positional algorithm*)
- An alternative form is the so called *incremental form* (*velocity algorithm*), where the PID controller computes instead the input increments

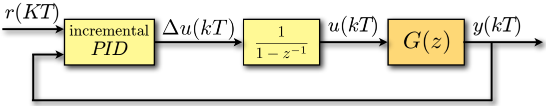
$$\Delta u(kT) = u(kT) - u((k-1)T) = \Delta P(kT) + \Delta I(kT) + \Delta D(kT)$$

where

$$\begin{aligned} \Delta P(kT) &= K_p(br(kT) - br((k-1)T) - y(kT) + y((k-1)T) \\ \Delta I(kT) &= \alpha_1(r(kT) - y(kT)) + \alpha_2(r((k-1)T) - y((k-1)T)) \\ \Delta D(kT) &= \beta_1 \Delta D((k-1)T) - \beta_2(y(kT) - 2y((k-1)T) + y((k-2)T)) \end{aligned}$$

- Advantage: increased numerical precision in the presence of finite word-length (signal *quantization*)
- This form cannot be used for P and PD controllers

Digital PID in incremental form



- An integrator is needed to reconstruct the input signal $u(kT)$ from the incremental PID form

$$u(kT) = u((k-1)T) + \Delta u(kT) \quad \longrightarrow \quad u(kT) = \frac{1}{1-z^{-1}}\Delta u(kT)$$

English-Italian Vocabulary

	
forward Euler method	*metodo di Eulero in avanti*
backward Euler method	*metodo di Eulero all'indietro*
trapezoidal rule	*regola dei trapezi*
sampled-data system	*sistema a dati campionati*
Nyquist-Shannon sampling theorem	*Teorema di Shannon*
Dirac comb	*pettine di Dirac*
aliasing	*aliasing*
cutoff frequency	*frequenza di taglio*

Translation is obvious otherwise.