Frequency response

**Definition**

The *frequency response* of a linear dynamical system with transfer function $G(s)$ is the complex function $G(j\omega)$ of the real angular frequency $\omega \geq 0$.

\[
\begin{align*}
    u(t) &= \bar{U} \sin(\omega t) \\
    y(t) &= G(s) u(t)
\end{align*}
\]

**Theorem**

If $G(s)$ is asymptotically stable (poles with negative real part), for $u(t) = \bar{U} \sin(\omega t)$ in steady-state conditions \(\lim_{t \to \infty} y(t) - y_{ss}(t) = 0\), where

\[
y_{ss}(t) = \bar{U}|G(j\omega)| \sin(\omega t + \angle G(j\omega))
\]

The frequency response $G(j\omega)$ of a system allows us to analyze the response of the system to sinusoidal excitations at different frequencies $\omega$. 
Frequency response – Proof

- Set $\bar{U} = 1$ for simplicity. The input $u(t) = \sin \omega t \Pi(t)$ has Laplace transform

$$U(s) = \frac{\omega}{s^2 + \omega^2}$$

- As we are interested in the steady-state response, we can neglect the natural response $y_\ell(t)$ of the output. In fact, as $G(s)$ is asymptotically stable, $y_\ell(t) \rightarrow 0$ for $t \rightarrow \infty$

- The forced response of the output $y(t)$ has Laplace transform

$$Y_f(s) = G(s) \frac{\omega}{s^2 + \omega^2} = \frac{G(s)\omega}{(s-j\omega)(s+j\omega)}$$

- Let’s compute the partial fraction decomposition of $Y_f(s)$

$$Y_f(s) = \frac{\omega G(j\omega)}{2j\omega(s-j\omega)} + \frac{\omega G(-j\omega)}{-2j\omega(s+j\omega)} + \sum_{i=1}^{n} \frac{R_i}{s-p_i}$$

$Y_{ss}(s) = \text{steady-state response}$

transient forced response

$\text{transient forced response}^{1}$

$^{1}R_i=\text{residue of } G(s)\frac{\omega}{s^2+\omega^2} \text{ in } s = p_i$. We are assuming here that $G(s)$ has distinct poles
Frequency response – Proof (cont’d)

As $G(s)$ is asymptotically stable, the inverse Laplace transform

$$\mathcal{L}^{-1}\left[ \sum_{i=1}^{n} \frac{R_i}{s-p_i} \right] = \sum_{i=1}^{n} R_i e^{p_i t}$$

tends to zero asymptotically.

The remaining steady-state output response is therefore

$$y_{ss}(t) = \mathcal{L}^{-1}\left[ \frac{G(j\omega)}{2j(s-j\omega)} + \frac{G(-j\omega)}{-2j(s+j\omega)} \right]$$

$$= \frac{-j}{2} G(j\omega) e^{j\omega t} + \frac{j}{2} G(-j\omega) e^{-j\omega t}$$

$$= \frac{-j}{2} G(j\omega) e^{j\omega t} + \left( \frac{-j}{2} G(j\omega) e^{j\omega t} \right)$$

$$= 2\Re \left[ \frac{-j}{2} G(j\omega) e^{j\omega t} \right] = \Im \left[ |G(j\omega)| e^{j\angle G(j\omega) + j\omega t} \right]$$

$$= |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

We can repeat the proof for $\bar{U} \neq 1$ and get the general result. □
Example

\[ u(t) = 5 \sin 10t \]

\[ G(s) = \frac{10}{(1 + 0.1s)^2} \]

\[ y(t) \]

- The poles are \( p_1 = p_2 = -10 \), the system is asymptotically stable
- The steady-state response is

\[ y_{ss}(t) = 5|G(j\omega)| \sin(\omega t + \angle G(j\omega)) \]

where

\[ G(j\omega) = \frac{10}{(1 + 0.1j\omega)^2} = \frac{10}{1 + 0.2j\omega - 0.01\omega^2} \]

- For \( \omega = 10 \text{ rad/s} \),

\[ G(10j) = \frac{10}{1 + 2j - 1} = \frac{5}{j} = -5j \]

- Finally, we get

\[ y_{ss}(t) = 5 \cdot 5 \sin(10t - \frac{\pi}{2}) = 25 \sin(10t - \frac{\pi}{2}) \]
Example (cont’d)

\[ u(t) = 5 \sin(10t) \rightarrow y_{ss}(t) = 25 \sin(10t - \frac{\pi}{2}) \]
Frequency response of discrete-time systems

A similar result holds for discrete-time systems:

**Theorem**

If $G(z)$ is asymptotically stable (poles inside the unit circle), for $u(k) = \bar{U} \sin(k\theta)$ in steady-state conditions $\lim_{k \to \infty} y(k) - y_{ss}(k) = 0$, where

$$y_{ss}(k) = \bar{U} |G(e^{i\theta})| \sin(k\theta + \angle G(e^{i\theta}))$$
The **Bode plot** is a graph of the module $|G(j\omega)|$ and phase $\angle G(j\omega)$ of a transfer function $G(s)$, evaluated in $s = j\omega$

The Bode plot shows the system’s frequency response as a function of $\omega$, for all $\omega > 0$

Hendrik Wade Bode (1905–1982)
Bode plot

Bode magnitude plot

\[ G(s) = \frac{1}{(1 + 0.1s)(1 + 0.002s + 0.0001s^2)} \]

Bode phase plot

The frequency \( \omega \) axis is in logarithmic scale

The module \( |G(j\omega)| \) is expressed as decibel (dB)

\[ |G(j\omega)|_{db} = 20 \log_{10} |G(j\omega)| \]

Example: the DC-gain \( G(0) = 1, |G(j0)|_{db} = 20 \log_{10} 1 = 0 \)

MATLAB

\[
\text{>> bode}(G) \\
\text{>> evalfr}(G, w)
\]
Bode form

To study the frequency response of the system is useful to rewrite the transfer function $G(s)$ in **Bode form**

$$G(s) = \frac{K}{s^h} \frac{\prod_i (1 + s \tau_i)}{\prod_j (1 + s T_j)} \frac{\prod_i \left(1 + \frac{2\zeta_i}{\omega_{ni}^\prime} s + \frac{1}{\omega_{ni}^\prime} s^2\right)}{\prod_j \left(1 + \frac{2\zeta_j}{\omega_{nj}^\prime} s + \frac{1}{\omega_{nj}^\prime} s^2\right)}$$

- $K$ is the **Bode gain**
- $h$ is the **type** of the system, that is the number of poles in $s = 0$
- $T_j$ (for real $T_j > 0$) is said a **time constant** of the system
- $\zeta_j$ is a **damping ratio** of the system, $-1 < \zeta_j < 1$
- $\omega_{nj}$ is a **natural frequency** of the system
Bode magnitude plot

\[ |G(j\omega)|_{dB} = 20 \log_{10} \left| \frac{K \prod_i (1 + j\omega \tau_i)}{(j\omega)^h \prod_j (1 + j\omega T_j)} \frac{\prod_i \left( 1 + \frac{2j\zeta_i \omega}{\omega'_{ni}} - \frac{\omega^2}{\omega'_{ni}^2} \right)}{\prod_j \left( 1 + \frac{2j\zeta_j \omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2} \right)} \right| \]

- Recall the following properties of logarithms:

\[
\log \alpha \beta = \log \alpha + \log \beta, \quad \log \frac{\alpha}{\beta} = \log \alpha - \log \beta, \quad \log \alpha^\beta = \beta \log \alpha
\]

- Thus we get

\[
|G(j\omega)|_{dB} = 20 \log_{10} |K| - h \cdot 20 \log_{10} \omega \\
+ \sum_i 20 \log_{10} |1 + j\omega \tau_i| - \sum_j 20 \log_{10} |1 + j\omega T_j| \\
+ \sum_i 20 \log_{10} \left| 1 + \frac{2j\zeta_i \omega}{\omega'_{ni}} - \frac{\omega^2}{\omega'_{ni}^2} \right| - \sum_j 20 \log_{10} \left| 1 + \frac{2j\zeta_j \omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2} \right|
\]
Bode magnitude plot

- We can restrict our attention to four basic components only:

\[
|G(j\omega)|_{dB} = 20 \log_{10} |K| - h \cdot 20 \log_{10} \omega
\]

\[
+ \sum_i 20 \log_{10} |1 + j\omega \tau_i| - \sum_j 20 \log_{10} |1 + j\omega T_j|
\]

\[
+ \sum_i 20 \log_{10} \left| 1 + \frac{2j\zeta_i \omega}{\omega_n^i} - \frac{\omega^2}{\omega_n^i} \right| - \sum_j 20 \log_{10} \left| 1 + \frac{2j\zeta_j \omega}{\omega_n^j} - \frac{\omega^2}{\omega_n^j} \right|
\]

\[\text{#1} \quad \text{#2} \quad \text{#3} \quad \text{#4}\]
**Bode magnitude plot**

- **Basic component #1**
  \[ 20 \log_{10} |K| \]

- **Basic component #2**
  \[ 20 \log_{10} |\omega| \]
  \[ -20 \log_{10} |\omega| \]
**Bode magnitude plot**

- **Basic component #3**

\[
20 \log_{10} |1 + j \omega T|
\]

\[|G(j\omega)| \approx 1 \text{ for } \omega \ll \frac{1}{T}\]

\[|G(j\omega)| \approx \omega T \text{ for } \omega \gg \frac{1}{T}\]

- **Basic component #4**

\[
20 \log_{10} \left| 1 + 2j \zeta \frac{\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2} \right|
\]

\[|G(j\omega)| \approx 1 \text{ for } \omega \ll \omega_n\]

\[|G(j\omega)| \approx \frac{\omega^2}{\omega_n^2} \text{ for } \omega \gg \omega_n\]

For \(\zeta < \frac{1}{\sqrt{2}}\) the peak of the response is obtained at the **resonant frequency** \(\omega_{\text{peak}} = \omega_n \sqrt{1 - 2\zeta^2}\)
Example

Draw the Bode plot of the transfer function

\[ G(s) = \frac{1000(1 + 10s)}{s^2(1 + s)^2} \]

- For \( \omega \ll 0.1 \): \(|G(j\omega)| \approx \frac{1000}{\omega^2}\)
- For \( \omega = 0.1 \): \(20 \log_{10} \frac{1000}{\omega^2} = 20 \log_{10} 10^5 = 100\)
- For \( 0.1 < \omega < 1 \): effect of zero \( s = -0.1 \), increase by 20 dB/decade
- For \( \omega > 1 \): effect of double pole \( s = -1 \), decrease by 40 dB/decade
Bode phase plot

\[ \angle G(j \omega) = \angle \left( \frac{K}{(j \omega)^h} \frac{\Pi_i (1 + j \omega \tau_i)}{\Pi_j (1 + j \omega T_j)} \frac{\Pi_i \left( 1 + \frac{2j \zeta_i' \omega}{\omega_{ni}^'} - \frac{\omega^2}{\omega_{ni}^2} \right)}{\Pi_j \left( 1 + \frac{2j \zeta_j \omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2} \right)} \right) \]

- Because of the following properties of exponentials

\[ \angle (\rho e^{j \theta}) = \theta \]
\[ \angle (\alpha \beta) = \angle (\rho_\alpha e^{j \theta_\alpha} \rho_\beta e^{j \theta_\beta}) = \angle (\rho_\alpha \rho_\beta e^{j (\theta_\alpha + \theta_\beta)}) = \theta_\alpha + \theta_\beta = \angle \alpha + \angle \beta \]
\[ \frac{\alpha}{\beta} = \angle \frac{\rho_\alpha e^{j \theta_\alpha}}{\rho_\beta e^{j \theta_\beta}} = \angle \left( \frac{\rho_\alpha}{\rho_\beta} e^{j (\theta_\alpha - \theta_\beta)} \right) = \theta_\alpha - \theta_\beta = \angle \alpha - \angle \beta \]

- We get

\[
\angle G(j \omega) = \angle K - \angle (j \omega)^h \\
+ \sum_i \angle (1 + j \omega \tau_i) - \sum_j \angle (1 + j \omega T_j) \\
+ \sum_i \angle \left( 1 + \frac{2j \zeta_i' \omega}{\omega_{ni}^'} - \frac{\omega^2}{\omega_{ni}^2} \right) - \sum_j \angle \left( 1 + \frac{2j \zeta_j \omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2} \right)
\]
Bode phase plot

- We can again restrict our attention to four basic components only:

\[
\angle G(j\omega) = \angle K - \angle (j\omega^h)\]

\[
+ \sum_i \angle (1 + j\omega \tau_i) - \sum_j \angle (1 + j\omega T_j)\]

\[
+ \sum_i \angle \left(1 + \frac{2j\zeta_i'\omega}{\omega_{ni}'} - \frac{\omega^2}{\omega_{ni}'^2}\right) - \sum_j \angle \left(1 + \frac{2j\zeta_j\omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2}\right)\]

\#1

\#2

\#3

\#4
Bode phase plot

- Basic component #1

\[ \angle K = \begin{cases} 
0 & \text{for } K > 0 \\
-\pi & \text{for } K < 0 
\end{cases} \]

- Basic component #2

\[ \angle (j\omega)^h = \frac{h\pi}{2} \]
Bode phase plot

- **Basic component #3**

\[ \angle (1 + j\omega T) = \tan(\omega T) \]

\[ \angle G(j\omega) \approx 0 \text{ for } \omega \ll \frac{1}{T} \]
\[ \angle G(j\omega) \approx \frac{\pi}{2} \text{ for } \omega \gg \frac{1}{T}, T > 0 \]

\[ \angle (1 - j\omega T) = -\tan(\omega T) \]

- **Basic component #4**

\[ \angle \left(1 + 2j\zeta \frac{\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right) \]

\[ \angle G(j\omega) \approx 0 \text{ for } \omega \ll \omega_n \]

For \( \zeta \geq 0 \), \( \angle G(j\omega) = \frac{\pi}{2} \text{ for } \omega = \omega_n \),
\[ \angle G(j\omega) \approx \angle \left(-\frac{\omega^2}{\omega_n^2}\right) = \pi \text{ for } \omega \gg \omega_n \]
Example (cont’d)

\[ G(s) = \frac{1000(1 + 10s)}{s^2(1 + s)^2} \]

- For \( \omega \ll 0.1 \): \( \angle G(j\omega) \approx -\pi \)
- For \( 0.1 < \omega < 1 \): effect of zero in \( s = -0.1 \), add \( \frac{\pi}{2} \)
- Per \( \omega > 1 \): effect of double pole in \( s = -1 \), subtract \( 2 \frac{\pi}{2} = \pi \)
Example

\[ G(s) = \frac{10(1 + s)}{s^2(1 - 10s)(1 + 0.1s)} \]

- For \( \omega \ll 0.1 \): \( |G(j\omega)| \approx \frac{10}{\omega^2} \) (slope=-40 dB/dec), \( \angle G(j\omega) \approx -\pi \)
- For \( \omega = 0.1 \): \( 20 \log_{10} \frac{10}{\omega^2} = 60 \) dB
- For \( 0.1 < \omega < 1 \): effect of unstable pole \( s = 0.1 \), decrease module by 20 dB/dec, increase phase by \( \frac{\pi}{2} \)
- For \( 1 < \omega < 10 \): effect of zero \( s = -1 \), +20 dB/dec module, +\( \frac{\pi}{2} \) phase
- For \( \omega > 10 \): effect of pole \( s = -10 \), -20 dB/dec module, -\( \frac{\pi}{2} \) phase
### Summary table for drawing asymptotic Bode plots

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable real pole</td>
<td>$T &gt; 0$</td>
<td>$-20 \text{ dB/dec}$</td>
<td>$-\pi/2$</td>
</tr>
<tr>
<td>Unstable real pole</td>
<td>$T &lt; 0$</td>
<td>$-20 \text{ dB/dec}$</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Stable real zero</td>
<td>$\tau &gt; 0$</td>
<td>$+20 \text{ dB/dec}$</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Unstable real zero</td>
<td>$\tau &lt; 0$</td>
<td>$+20 \text{ dB/dec}$</td>
<td>$-\pi/2$</td>
</tr>
<tr>
<td>Pair of stable complex poles</td>
<td>$\zeta &gt; 0$</td>
<td>$-40 \text{ dB/dec}$</td>
<td>$-\pi$</td>
</tr>
<tr>
<td>Pair of unstable complex poles</td>
<td>$\zeta &lt; 0$</td>
<td>$-40 \text{ dB/dec}$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Pair of stable complex zeros</td>
<td>$\zeta' &gt; 0$</td>
<td>$+40 \text{ dB/dec}$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Pair of unstable complex zeros</td>
<td>$\zeta' &lt; 0$</td>
<td>$+40 \text{ dB/dec}$</td>
<td>$-\pi$</td>
</tr>
</tbody>
</table>
More on damped oscillatory modes

- Let $s = a \pm jb$ be complex poles, $b \neq 0$

- Since $(s - (a - jb))(s - (a + jb)) = (s - a)^2 + b^2 = s^2 - 2as + (a^2 + b^2) = (a^2 + b^2)(1 - \frac{2a}{a^2+b^2}s + \frac{1}{a^2+b^2}s^2)$, we get

$$\omega_n = \sqrt{a^2 + b^2} = |a \pm jb|, \quad \zeta = -\frac{a}{\sqrt{a^2 + b^2}} = -\cos \angle(a \pm jb)$$

- Note that $\zeta > 0$ if and only if $a < 0$

- Vice versa, $a = -\zeta \omega_n$, and $b = \omega_n \sqrt{1 - \zeta^2}$

- The natural response is

$$Me^{-\zeta \omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t + \phi)$$

where $M, \phi$ depend on the initial conditions

- The frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ is called *damped natural frequency*
Zero/pole/gain form and Bode form

- Sometimes the transfer function $G(s)$ is given in **zero/pole/gain form** (ZPK)

$$G(s) = \frac{K'}{s^h} \prod_i (s - z_i) \prod_i \left( s^2 + 2\zeta_i \omega_{ni}s + \omega_{ni}^2 \right) \prod_j (s - p_j) \prod_j \left( s^2 + 2\zeta_j \omega_{nj}s + \omega_{nj}^2 \right)$$

- The relations between Bode form and ZPK form are

$$z_i = -\frac{1}{\tau_i}, \quad p_j = -\frac{1}{T_i}, \quad K = K' \frac{\prod_i (-z_i) \prod_i \omega_{ni}^2}{\prod_j (-p_j) \prod_i \omega_{nj}^2}$$

**MATLAB**

```matlab
>> zpk(G)
```
Nyquist (or polar) plot

- Let $G(s)$ be a transfer function of a linear time-invariant dynamical system.

- The **Nyquist plot** (or **polar plot**) is the graph in **polar** coordinates of $G(j\omega)$ for $\omega \in [0, +\infty)$ in the complex plane.

- $G(j\omega) = \rho(\omega)e^{j\phi(\omega)}$, where $\rho(\omega) = |G(j\omega)|$ and $\phi(\omega) = \angle G(j\omega)$

- The Nyquist plot is one of the classical methods used in stability analysis of linear systems.

- The Nyquist plot combines the Bode magnitude & phase plots in a single plot.

Harry Nyquist (1889–1976)
Nyquist plot

To draw a Nyquist plot of a transfer function $G(s)$, we can give some hints:

- For $\omega = 0$, the Nyquist plot equals the DC gain $G(0)$
- If $G(s)$ is strictly proper, $\lim_{\omega \to \infty} G(j\omega) = 0$
- In this case the angle of arrival equals $(n_z^- - n_z^+ - n_p^- + n_p^+) \frac{\pi}{2}$, where $n_{z[p]}^{[-]}$ is the # of zeros [poles] with positive [negative] real part
- $G(-j\omega) = G(j\omega)$

A system without any zero with positive real part is called *minimum phase*
Nyquist criterion

- Consider a transfer function \( G(s) = \frac{N(s)}{D(s)} \) under unit static feedback 
  \( u(t) = -(y(t) - r(t)) \)
- As \( y(t) = G(s)(-y(t) + r(t)) \), the closed-loop transfer function from \( r(t) \) to \( y(t) \) is 
  \[
  W(s) = \frac{G(s)}{1 + G(s)} = \frac{N(s)}{D(s) + N(s)}
  \]

Nyquist stability criterion

The number \( N_W \) of closed-loop unstable poles of \( W(s) \) is equal to the number \( N_R \) of clock-wise rotations of the Nyquist plot around \( s = -1 + j0 \) plus the number \( N_G \) of unstable poles of \( G(s) \). \([N_W = N_R + N_G]\)

Corollary: simplified Nyquist criterion

For open-loop asympt. stable systems \( G(s) \) \((N_G = 0)\), the closed-loop system \( W(s) \) is asymptotically stable if and only if the Nyquist plot \( G(j\omega) \) does not encircle clock-wise the critical point \(-1 + j0\). \([N_W = N_R]\)
Nyquist stability criterion

Proof:

- Follows by the *Argument principle*: “The polar diagram of $1 + G(s)$ has a number of clock-wise rotations around the origin equal to the number of zeros of $1 + G(s)$ (\(=\)roots of $N(s) + G(s)$) minus the number of poles of $1 + G(s)$ (\(=\)roots of $D(s)$) of $1 + G(s)$”
- The poles of $W(s)$ are the zeros of $1 + G(s)$

Note: the number of *counter-clockwise* encirclements counts as a negative number of clockwise encirclements

Use of Nyquist criterion for open-loop stable systems:

1. draw the Nyquist plot of $G(s)$
2. count the number of clockwise rotations around $-1 + j0$
3. if $-1 + j0$ is not encircled, the closed-loop system $W(s) = G(s)/(1 + G(s))$ is stable

The Nyquist criterion is limited to SISO systems
Example

Consider the transfer function

\[ G(s) = \frac{10}{(s + 1)^2} = \frac{10}{s^2 + 2s + 1} \]

under static output feedback 

\[ u(t) = -(y(t) - r(t)) \]

- The closed-loop transfer function is

\[ W(s) = \frac{G(s)}{1 + G(s)} \]

- The Nyquist plot of \( G(s) \) does not encircle \(-1 + j0\).
- By Nyquist criterion, \( W(s) \) is asymptotically stable.

MATLAB

```matlab
>> G = tf(10, [1 2 1])
>> nyquist(G)
```
Example

Consider the transfer function \( G(s) = \frac{10}{s^3 + 3s^2 + 2s + 1} \) under unit output feedback \( u(t) = -2(y(t) - r(t)) \)

The closed-loop transfer function is

\[
W(s) = \frac{2G(s)}{1 + 2G(s)}
\]

The Nyquist plot of \( 2G(s) \) encircles \(-1 + j0\) twice

By Nyquist criterion, \( W(s) \) is unstable

MATLAB

```matlab
G=tf(10,[1 3 2 1]); K=2
nyquist(K*G)
L=feedback(G,K)
pole(L)
  -3.8797
  0.4398 + 2.2846i
  0.4398 - 2.2846i
```
Dealing with imaginary poles

- The Nyquist plot is generated by a closed curve, the **Nyquist contour**, rotating clock-wise from $0 - j\infty$ to $0 + j\infty$ and back to $0 - j\infty$ along a semi-circle of radius $\infty$, avoiding singularities of $G(j\omega)$.

- We distinguish three types of curves, depending on the number of poles on the imaginary axis:

  - No poles on imaginary axis
  - Pole(s) in $s = 0$ (counted as stable)
  - Pole(s) in $s = \pm j\omega$ (counted as stable)
Stability analysis of static output feedback

Under static output feedback $u(t) = -K(y(t) - r(t))$, the closed-loop transfer function from $r(t)$ to $y(t)$ is

$$W(s) = \frac{KG(s)}{1 + KG(s)}$$

The number of encirclements of $-1 + j0$ of $KG(s)$ is equal to the number of encirclements of $-\frac{1}{K} + j0$ of $G(s)$.

To analyze closed-loop stability for different values of $K$ is enough to draw Nyquist plot of $G(s)$ and move the point $-\frac{1}{K} + j0$ on the real axis.

$$G(s) = \frac{60}{(s^2 + 2s + 20)(s + 1)}$$

$I_0$ = no unstable closed-loop poles
$I_2$ = two unstable closed-loop poles
$I_1$ = an unstable closed-loop pole
Phase margin

- Assume $G(s)$ is open-loop asymptotically stable
- Let $\omega_c$ be the *gain crossover frequency*, that is $|G(j\omega_c)| = 1$
- To avoid encircling the point $-1 + j0$, we want the phase $\angle G(j\omega_c)$ as far away as possible from $-\pi$
- The *phase margin* is the quantity

$$M_p = \angle G(j\omega_c) - (-\pi)$$

- If $M_p > 0$, unit negative feedback control is asymptotically stabilizing
- For robustness of stability we would like a large positive phase margin
Gain margin

- Assume $G(s)$ is open-loop asymptotically stable
- Let $\omega_\pi$ be the phase crossover frequency such that $\angle G(j\omega_\pi) = -\pi$
- To avoid encircling the point $-1 + j0$, we want the point $G(j\omega_\pi)$ as far as possible away from $-1 + j0$.
- The gain margin is the inverse of $|G(j\omega_\pi)|$, expressed in dB

$$M_g = 20 \log_{10} \frac{1}{|G(j\omega_\pi)|} = -|G(j\omega_\pi)|_{dB}$$

- For robustness of stability we like to have a large gain margin

Sometimes the gain margin is defined as

$$M_g = \frac{1}{|G(j\omega_\pi)|}$$

and therefore $G(j\omega_\pi) = -\frac{1}{M_g} + j0$
Stability analysis using phase and gain margins

- The phase and gain margins help assessing the degree of robustness of a closed-loop system against uncertainties in the magnitude and phase of the process model.
- They can be also applied to analyze the stability of dynamic output feedback laws \( u(t) = C(s)(y(t) - r(t)) \), by looking at the Bode plots of the loop function \( C(j\omega)G(j\omega) \) (see next lecture on “loop shaping”).

\[
G(s) = \frac{8}{s^3 + 4s^2 + 4s + 2}
\]
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- However in some cases phase and gain margins are not good indicators, see the following example

The system is characterized by a large phase margin, but the polar plot is very close to the point \((-1, 0)\)

- The system is not very robust to model uncertainties changing \(G(s)\) from its nominal value

- In conclusion, phase and gain margins give good indications on how the loop function should be modified, but the complete Bode and Nyquist plots must be checked to conclude about closed-loop stability
## English-Italian Vocabulary

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<td>frequency response</td>
<td>risposta in frequenza</td>
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<td>angular frequency</td>
<td>pulsazione</td>
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<td>steady-state</td>
<td>regime permanente</td>
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<td>Bode plot</td>
<td>diagramma di Bode</td>
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<td>Bode form</td>
<td>forma di Bode</td>
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<td>Bode gain</td>
<td>guadagno di Bode</td>
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<tr>
<td>system’s type</td>
<td>tipo del sistema</td>
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<td>time constant</td>
<td>costante di tempo</td>
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<td>damping ratio</td>
<td>fattore di smorzamento</td>
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<td>natural frequency</td>
<td>frequenza naturale</td>
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<td>zero/pole/gain form</td>
<td>forma poli/zeri</td>
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<td>minimum phase system</td>
<td>sistema a fase minima</td>
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<td>Nyquist plot</td>
<td>diagramma di Nyquist</td>
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<td>crossover frequency</td>
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