Automatic Control 2

Frequency domain analysis

Prof. Alberto Bemporad

University of Trento

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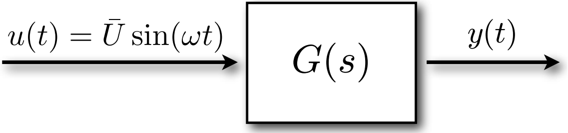
Frequency response

Definition

The *frequency response* of a linear dynamical system with transfer function $G(s)$ is the complex function $G(j\omega)$ of the real angular frequency $\omega \geq 0$



Theorem

If $G(s)$ is asymptotically stable (poles with negative real part), for $u(t) = \bar{U}\sin(\omega t)$ in steady-state conditions $\lim_{t\to\infty} y(t) - y_{ss}(t) = 0$, where

$$y_{ss}(t) = \bar{U}|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

The frequency response $G(j\omega)$ of a system allows us to analyze the response of the system to sinusoidal excitations at different frequencies ω

Frequency response – Proof

- Set $\bar{U} = 1$ for simplicity. The input $u(t) = \sin\omega t\, 1\!\!1(t)$ has Laplace transform

$$U(s) = \frac{\omega}{s^2 + \omega^2}$$

- As we are interested in the steady-state response, we can neglect the natural response $y_\ell(t)$ of the output. In fact, as $G(s)$ is asymptotically stable, $y_\ell(t) \to 0$ for $t \to \infty$
- The forced response of the output $y(t)$ has Laplace transform

$$Y_f(s) = G(s)\frac{\omega}{s^2 + \omega^2} = \frac{G(s)\omega}{(s - j\omega)(s + j\omega)}$$

- Let's compute the partial fraction decomposition of $Y_f(s)$

$$Y_f(s) = \underbrace{\frac{\omega G(j\omega)}{2j\omega(s - j\omega)} + \frac{\omega G(-j\omega)}{-2j\omega(s + j\omega)}}_{Y_{ss}(s)\text{ = steady-state response}} + \underbrace{\sum_{i=1}^{n} \frac{R_i}{s - p_i}}_{\text{transient forced response}^1}$$

[1] R_i=residue of $G(s)\frac{\omega}{s^2+\omega^2}$ in $s = p_i$. We are assuming here that $G(s)$ has distinct poles

Frequency response – Proof (cont'd)

- As $G(s)$ is asymptotically stable, the inverse Laplace transform $\mathcal{L}^{-1}\left[\sum_{i=1}^{n}\frac{R_i}{s-p_i}\right]=\sum_{i=1}^{n}R_ie^{p_it}$ tends to zero asymptotically
- The remaining steady-state output response is therefore

$$
\begin{aligned}
y_{ss}(t) &= \mathcal{L}^{-1}\left[\frac{G(j\omega)}{2j(s-j\omega)}+\frac{G(-j\omega)}{-2j(s+j\omega)}\right]\\
&= \frac{-j}{2}G(j\omega)e^{j\omega t}+\frac{j}{2}G(-j\omega)e^{-j\omega t}\\
&= \frac{-j}{2}G(j\omega)e^{j\omega t}+\overline{\left(\frac{-j}{2}G(j\omega)e^{j\omega t}\right)}\\
&= 2\Re\left[\frac{-j}{2}G(j\omega)e^{j\omega t}\right]= \underbrace{\operatorname{Im}\left[|G(j\omega)|e^{j\angle G(j\omega)+j\omega t)}\right]}_{Re(-j(a+jb))=Re(-ja-j^2b)=Re(b-ja)=b}\\
&= |G(j\omega)|\sin(\omega t+\angle G(j\omega))
\end{aligned}
$$

- We can repeat the proof for $\bar{U}\neq 1$ and get the general result. □

Example

$$u(t) = 5\sin 10t \longrightarrow \boxed{G(s) = \frac{10}{(1+0.1s)^2}} \longrightarrow y(t)$$

- The poles are $p_1 = p_2 = -10$, the system is asymptotically stable
- The steady-state response is

$$y_{ss}(t) = 5|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

  where

$$G(j\omega) = \frac{10}{(1+0.1j\omega)^2} = \frac{10}{1+0.2j\omega - 0.01\omega^2}$$

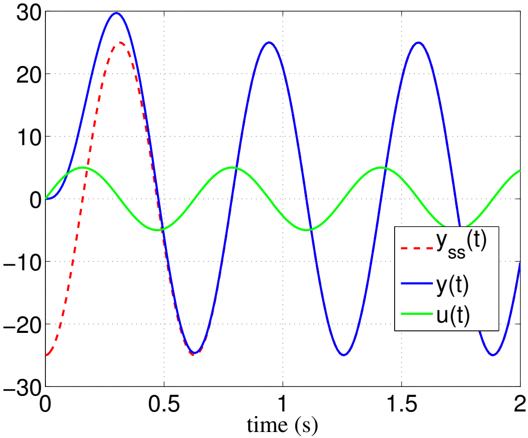
- For $\omega = 10$ rad/s,

$$G(10j) = \frac{10}{1+2j-1} = \frac{5}{j} = -5j$$

- Finally, we get

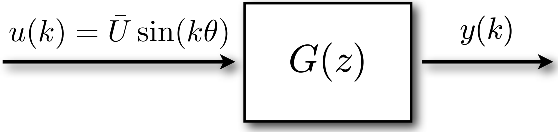
$$y_{ss}(t) = 5 \cdot 5\sin(10t - \frac{\pi}{2}) = 25\sin(10t - \frac{\pi}{2})$$

Example (cont'd)



$$u(t) = 5\sin(10t) \;\to\; y_{ss}(t) = 25\sin(10t - \frac{\pi}{2})$$

Frequency response of discrete-time systems



A similar result holds for discrete-time systems:

Theorem

If $G(z)$ is asymptotically stable (poles inside the unit circle), for $u(k) = \bar{U}\sin(k\theta)$ in steady-state conditions $\lim_{k\to\infty} y(k) - y_{ss}(k) = 0$, where

$$y_{ss}(k) = \bar{U}|G(e^{j\theta})|\sin(k\theta + \angle G(e^{j\theta}))$$

Bode plot

- The *Bode plot* is a graph of the module $|G(j\omega)|$ and phase $\angle G(j\omega)$ of a transfer function $G(s)$, evaluated in $s = j\omega$

- The Bode plot shows the system's frequency response as a function of ω, for all $\omega > 0$

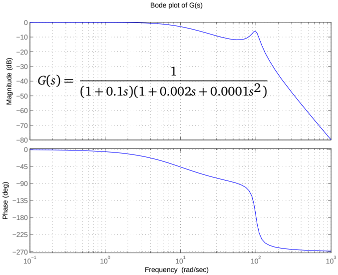
Hendrik Wade Bode

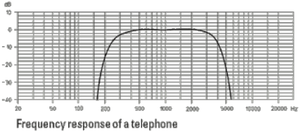
(1905–1982)

Bode plot

Bode magnitude plot

Bode phase plot





Example: frequency response of a telephone, approx. 300–3,400 Hz, good enough for speech transmission

- The frequency ω axis is in *logarithmic scale*
- The module $|G(j\omega)|$ is expressed as *decibel (dB)*

$$|G(j\omega)|_{\text{db}} = 20 \log_{10} |G(j\omega)|$$

- Example: the DC-gain $G(0) = 1$, $|G(j0)|_{\text{db}} = 20 \log_{10} 1 = 0$

MATLAB

```
»bode(G)

»evalfr(G,w)
```

Bode form

- To study the frequency response of the system is useful to rewrite the transfer function $G(s)$ in *Bode form*

$$G(s) = \frac{K}{s^h} \frac{\Pi_i(1+s\tau_i)}{\Pi_j(1+sT_j)} \frac{\Pi_i\left(1+\frac{2\zeta_i'}{\omega_{ni}'}s+\frac{1}{\omega'^2_{ni}}s^2\right)}{\Pi_j\left(1+\frac{2\zeta_j}{\omega_{nj}}s+\frac{1}{\omega^2_{nj}}s^2\right)}$$

- K is the *Bode gain*
- h is the *type* of the system, that is the number of poles in $s=0$
- T_j (for real $T_j>0$) is said a *time constant* of the system
- ζ_j is a *damping ratio* of the system, $-1<\zeta_j<1$
- ω_{nj} is a *natural frequency* of the system

Bode magnitude plot

$$|G(j\omega)|_{\rm dB} = 20\log_{10}\left|\frac{K}{(j\omega)^h}\frac{\Pi_i(1+j\omega\tau_i)}{\Pi_j(1+j\omega T_j)}\frac{\Pi_i\left(1+\frac{2j\zeta'_i\omega}{\omega'_{ni}}-\frac{\omega^2}{\omega'^2_{ni}}\right)}{\Pi_j\left(1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega^2_{nj}}\right)}\right|$$

- Recall the following properties of logarithms:

$$\log\alpha\beta = \log\alpha+\log\beta, \quad \log\frac{\alpha}{\beta} = \log\alpha-\log\beta, \quad \log\alpha^\beta = \beta\log\alpha$$

- Thus we get

$$\begin{aligned}|G(j\omega)|_{\rm dB} \quad &= \quad 20\log_{10}|K| - h\cdot 20\log_{10}\omega \\ &+\sum_i 20\log_{10}|1+j\omega\tau_i| - \sum_j 20\log_{10}|1+j\omega T_j| \\ &+\sum_i 20\log_{10}\left|1+\frac{2j\zeta'_i\omega}{\omega'_{ni}}-\frac{\omega^2}{\omega'^2_{ni}}\right| - \sum_j 20\log_{10}\left|1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega^2_{nj}}\right|\end{aligned}$$

Bode magnitude plot

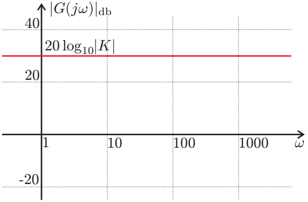
- We can restrict our attention to four basic components only:

$$
\begin{aligned}
|G(j\omega)|_{\text{dB}} &= \underbrace{20\log_{10}|K|}_{\#1} - h\cdot\underbrace{20\log_{10}\omega}_{\#2} \\
&+ \sum_i 20\log_{10}|1+j\omega\tau_i| - \sum_j \underbrace{20\log_{10}|1+j\omega T_j|}_{\#3} \\
&+ \sum_i 20\log_{10}\left|1+\frac{2j\zeta_i'\omega}{\omega_{ni}'}-\frac{\omega^2}{\omega_{ni}'^2}\right| - \sum_j \underbrace{20\log_{10}\left|1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega_{nj}^2}\right|}_{\#4}
\end{aligned}
$$

Bode magnitude plot

- Basic component #1

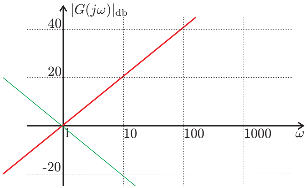
$$20\log_{10}|K|$$



- Basic component #2

$$20\log_{10}|\omega|$$

$$-20\log_{10}|\omega|$$



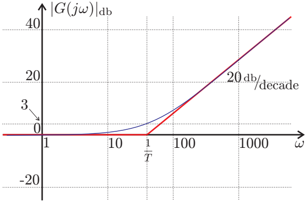
Bode magnitude plot

- Basic component #3

$$20\log_{10}|1+j\omega T|$$

$|G(j\omega)| \approx 1$ for $\omega \ll \frac{1}{T}$

$|G(j\omega)| \approx \omega T$ for $\omega \gg \frac{1}{T}$

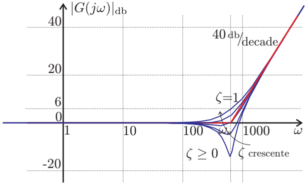


- Basic component #4

$$20\log_{10}\left|1+2j\zeta\frac{\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right|$$

$|G(j\omega)| \approx 1$ for $\omega \ll \omega_n$

$|G(j\omega)| \approx \frac{\omega^2}{\omega_n^2}$ for $\omega \gg \omega_n$

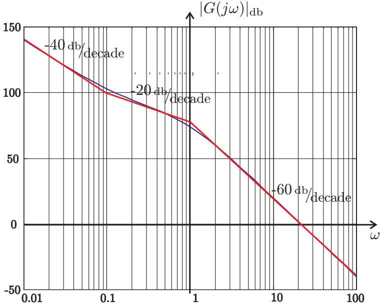


For $\zeta < \frac{1}{\sqrt{2}}$ the peak of the response is obtained at the *resonant frequency* $\omega_{\text{peak}} = \omega_n\sqrt{1-2\zeta^2}$

Example

Draw the Bode plot of the transfer function

$$G(s) = \frac{1000(1+10s)}{s^2(1+s)^2}$$



- For $\omega \ll 0.1$: $|G(j\omega)| \approx \frac{1000}{\omega^2}$
- For $\omega = 0.1$: $20\log_{10}\frac{1000}{\omega^2} = 20\log_{10} 10^5 = 100$
- For $0.1 < \omega < 1$: effect of zero $s = -0.1$, increase by 20 dB/decade
- For $\omega > 1$: effect of double pole $s = -1$, decrease by 40 dB/decade

Bode phase plot

$$\angle G(j\omega) = \angle\left(\frac{K}{(j\omega)^h}\frac{\Pi_i(1+j\omega\tau_i)}{\Pi_j(1+j\omega T_j)}\frac{\Pi_i\left(1+\frac{2j\zeta_i'\omega}{\omega_{ni}'}-\frac{\omega^2}{\omega_{ni}'^2}\right)}{\Pi_j\left(1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega_{nj}^2}\right)}\right)$$

- Because of the following properties of exponentials

$$\begin{aligned}
&\angle(\rho e^{j\theta}) = \theta \\
&\angle(\alpha\beta) = \angle(\rho_\alpha e^{j\theta_\alpha}\rho_\beta e^{j\theta_\beta}) = \angle(\rho_\alpha\rho_\beta e^{j(\theta_\alpha+\theta_\beta)}) = \theta_\alpha+\theta_\beta = \angle\alpha+\angle\beta \\
&\angle\frac{\alpha}{\beta} = \angle\frac{\rho_\alpha e^{j\theta_\alpha}}{\rho_\beta e^{j\theta_\beta}} = \angle\left(\frac{\rho_\alpha}{\rho_\beta}e^{j(\theta_\alpha-\theta_\beta)}\right) = \theta_\alpha-\theta_\beta = \angle\alpha-\angle\beta
\end{aligned}$$

- we get

$$\begin{aligned}
\angle G(j\omega) \quad = \quad & \angle K - \angle\left((j\omega)^h\right) \\
& + \sum_i \angle(1+j\omega\tau_i) - \sum_j \angle(1+j\omega T_j) \\
& + \sum_i \angle\left(1+\frac{2j\zeta_i'\omega}{\omega_{ni}'}-\frac{\omega^2}{\omega_{ni}'^2}\right) - \sum_j \angle\left(1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega_{nj}^2}\right)
\end{aligned}$$

Bode phase plot

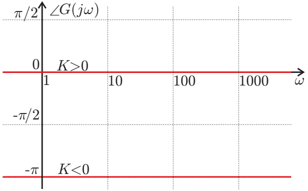
- We can again restrict our attention to four basic components only:

$$
\begin{aligned}
\angle G(j\omega) \quad = \quad & \underbrace{\angle K}_{\#1} - \underbrace{\angle\left((j\omega)^h\right)}_{\#2} \\
& + \sum_i \angle(1 + j\omega\tau_i) - \sum_j \underbrace{\angle(1 + j\omega T_j)}_{\#3} \\
& + \sum_i \angle\left(1 + \frac{2j\zeta_i'\omega}{\omega_{ni}'} - \frac{\omega^2}{\omega_{ni}'^2}\right) - \sum_j \underbrace{\angle\left(1 + \frac{2j\zeta_j\omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2}\right)}_{\#4}
\end{aligned}
$$

Bode phase plot

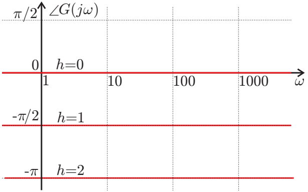
- Basic component #1

$$\angle K = \begin{cases} 0 & \text{for} \quad K > 0 \\ -\pi & \text{for} \quad K < 0 \end{cases}$$



- Basic component #2

$$\angle\left((j\omega)^h\right) = \frac{h\pi}{2}$$



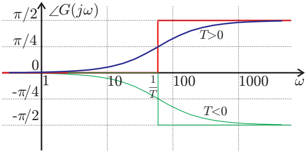
Bode phase plot

- Basic component #3

$$\angle(1+j\omega T) = \text{atan}(\omega T)$$

$\angle G(j\omega) \approx 0$ for $\omega \ll \frac{1}{T}$
$\angle G(j\omega) \approx \frac{\pi}{2}$ for $\omega \gg \frac{1}{T}$, $T > 0$

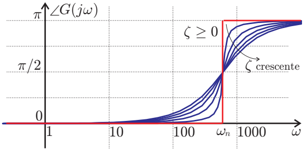
$$\angle(1-j\omega T) = -\text{atan}(\omega T)$$



- Basic component #4

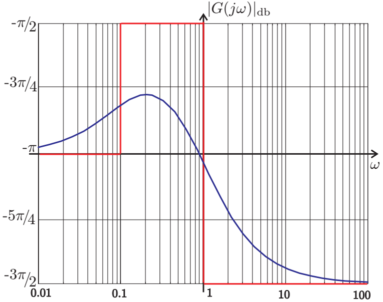
$$\angle\left(1+2j\zeta\frac{\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right)$$

$\angle G(j\omega) \approx 0$ for $\omega \ll \omega_n$
For $\zeta \geq 0$, $\angle G(j\omega) = \frac{\pi}{2}$ for $\omega = \omega_n$,
$\angle G(j\omega) \approx \angle\left(-\frac{\omega^2}{\omega_n^2}\right) = \pi$ for $\omega \gg \omega_n$



Example (cont'd)

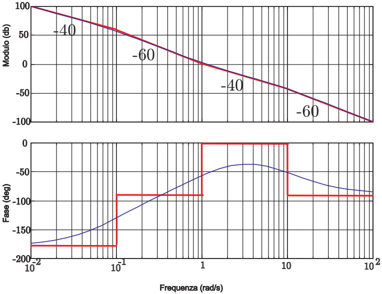
$$G(s) = \frac{1000(1+10s)}{s^2(1+s)^2}$$



- For $\omega \ll 0.1$: $\angle G(j\omega) \approx -\pi$
- For $0.1 < \omega < 1$: effect of zero in $s = -0.1$, add $\frac{\pi}{2}$
- Per $\omega > 1$: effect of double pole in $s = -1$, subtract $2\frac{\pi}{2} = \pi$

Example

$$G(s) = \frac{10(1+s)}{s^2(1-10s)(1+0.1s)}$$



- For $\omega \ll 0.1$: $|G(j\omega)| \approx \frac{10}{\omega^2}$ (slope=-40 dB/dec), $\angle G(j\omega) \approx -\pi$
- For $\omega = 0.1$: $20\log_{10}\frac{10}{\omega^2} = 60$ dB
- For $0.1 < \omega < 1$: effect of unstable pole $s = 0.1$, decrease module by 20 dB/dec, increase phase by $\frac{\pi}{2}$
- For $1 < \omega < 10$: effect of zero $s = -1$, +20 dB/dec module, $+\frac{\pi}{2}$ phase
- For $\omega > 10$: effect of pole $s = -10$, -20 dB/dec module, -$\frac{\pi}{2}$ phase

Summary table for drawing asymptotic Bode plots

		magnitude	phase
stable real pole	$T > 0$	-20 dB/dec	$-\pi/2$
unstable real pole	$T < 0$	-20 dB/dec	$\pi/2$
stable real zero	$\tau > 0$	$+20$ dB/dec	$\pi/2$
unstable real zero	$\tau < 0$	$+20$ dB/dec	$-\pi/2$
pair of stable complex poles	$\zeta > 0$	-40 dB/dec	$-\pi$
pair of unstable complex poles	$\zeta < 0$	-40 dB/dec	π
pair of stable complex zeros	$\zeta' > 0$	$+40$ dB/dec	π
pair of unstable complex zeros	$\zeta' < 0$	$+40$ dB/dec	$-\pi$

More on damped oscillatory modes

- Let $s = a \pm jb$ be complex poles, $b \neq 0$
- Since $(s-(a-jb))(s-(a+jb)) = (s-a)^2 + b^2 = s^2 - 2as + (a^2+b^2) = (a^2+b^2)(1 - \frac{2a}{a^2+b^2}s + \frac{1}{a^2+b^2}s^2)$, we get

$$\omega_n = \sqrt{a^2+b^2} = |a \pm jb|, \quad \zeta = -\frac{a}{\sqrt{a^2+b^2}} = -\cos\angle(a \pm jb)$$

- Note that $\zeta > 0$ if and only if $a < 0$
- Vice versa, $a = -\zeta\omega_n$, and $b = \omega_n\sqrt{1-\zeta^2}$
- The natural response is

$$Me^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)$$

  where M, ϕ depend on the initial conditions
- The frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called *damped natural frequency*

Zero/pole/gain form and Bode form

- Sometimes the transfer function $G(s)$ is given in *zero/pole/gain form* (ZPK)

$$G(s) = \frac{K'}{s^h} \frac{\Pi_i(s - z_i)}{\Pi_j(s - p_j)} \frac{\Pi_i\left(s^2 + 2\zeta'_i\omega'_{ni}s + {\omega'}^2_{ni}\right)}{\Pi_j\left(s^2 + 2\zeta_j\omega_{nj}s + \omega^2_{nj}\right)}$$

- The relations between Bode form and ZPK form are

$$z_i = -\frac{1}{\tau_i},\ \ p_j = -\frac{1}{T_i},\ \ K = K' \frac{\Pi_i(-z_i)\Pi_i{\omega'}^2_{ni}}{\Pi_j(-p_j)\Pi_i\omega^2_{ni}}$$

MATLAB

```
»zpk(G)
```

Nyquist (or polar) plot

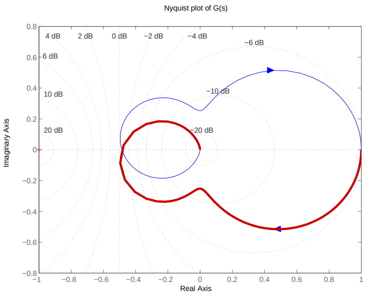
- Let $G(s)$ be a transfer function of a linear time-invariant dynamical system
- The *Nyquist plot* (or *polar plot*) is the graph in **polar** coordinates of $G(j\omega)$ for $\omega \in [0, +\infty)$ in the complex plane
- $G(j\omega) = \rho(\omega)e^{j\phi(\omega)}$, where $\rho(\omega) = |G(j\omega)|$ and $\phi(\omega) = \angle G(j\omega)$
- The Nyquist plot is one of the classical methods used in stability analysis of linear systems
- The Nyquist plot combines the Bode magnitude & phase plots in a single plot

Harry Nyquist

(1889–1976)

Nyquist plot

To draw a Nyquist plot of a transfer function $G(s)$, we can give some hints:



$$G(s) = \frac{1}{(1+0.1s)(1+0.002s+0.0001s^2)}$$

- For $\omega = 0$, the Nyquist plot equals the DC gain $G(0)$
- If $G(s)$ is strictly proper, $\lim_{\omega\to\infty} G(j\omega) = 0$
- In this case the angle of arrival equals $(n_z^- - n_z^+ - n_p^- + n_p^+)\frac{\pi}{2}$, where $n_{z[p]}^{+[-]}$ is the # of zeros [poles] with positive [negative] real part
- $G(-j\omega) = \overline{G(j\omega)}$

A system without any zero with positive real part is called *minimum phase*

Nyquist criterion

- Consider a transfer function $G(s) = N(s)/D(s)$ under unit static feedback $u(t) = -(y(t) - r(t))$
- As $y(t) = G(s)(-y(t) + r(t))$, the closed-loop transfer function from $r(t)$ to $y(t)$ is

$$W(s) = \frac{G(s)}{1 + G(s)} = \frac{N(s)}{D(s) + N(s)}$$

Nyquist stability criterion

The number N_W of closed-loop unstable poles of $W(s)$ is equal to the number N_R of clock-wise rotations of the Nyquist plot around $s = -1 + j0$ plus the number N_G of unstable poles of $G(s)$. $[N_W = N_R + N_G]$

Corollary: simplified Nyquist criterion

For open-loop asympt. stable systems $G(s)$ ($N_G = 0$), the closed-loop system $W(s)$ is asymptotically stable if and only if the Nyquist plot $G(j\omega)$ does not encircle clock-wise the critical point $-1 + j0$. $[N_W = N_R]$

Nyquist stability criterion

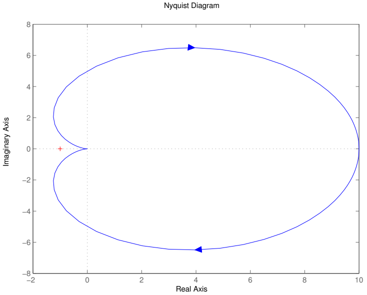
Proof:

- Follows by the *Argument principle*: "The polar diagram of $1 + G(s)$ has a number of clock-wise rotations around the origin equal to the number of zeros of $1 + G(s)$ (=roots of $N(s) + G(s)$) minus the number of poles of $1 + G(s)$ (=roots of $D(s)$) of $1 + G(s)$"
- The poles of $W(s)$ are the zeros of $1 + G(s)$ □

- Note: the number of *counter-clockwise* encirclements counts as a negative number of clockwise encirclements
- Use of Nyquist criterion for open-loop stable systems:
  1. draw the Nyquist plot of $G(s)$
  2. count the number of clockwise rotations around $-1 + j0$
  3. if $-1 + j0$ is not encircled, the closed-loop system $W(s) = G(s)/(1 + G(s))$ is stable
- The Nyquist criterion is limited to SISO systems

Example

Consider the transfer function $G(s) = \frac{10}{(s+1)^2} = \frac{10}{s^2+2s+1}$ under static output feedback $u(t) = -(y(t) - r(t))$



MATLAB

```
»G=tf(10,[1 2 1])
»nyquist(G)
```

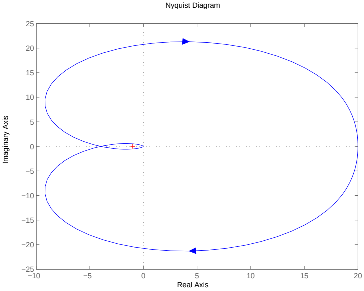
- The closed-loop transfer function is

$$W(s) = \frac{G(s)}{1+G(s)}$$

- The Nyquist plot of $G(s)$ does not encircle $-1+j0$
- By Nyquist criterion, $W(s)$ is asymptotically stable

Example

Consider the transfer function $G(s) = \dfrac{10}{s^3 + 3s^2 + 2s + 1}$ under unit output feedback $u(t) = -2(y(t) - r(t))$



MATLAB

```
»G=tf(10,[1 3 2 1]); K=2
»nyquist(K*G)
»L=feedback(G,K)
»pole(L)
  -3.8797
   0.4398 + 2.2846i
   0.4398 - 2.2846i
```

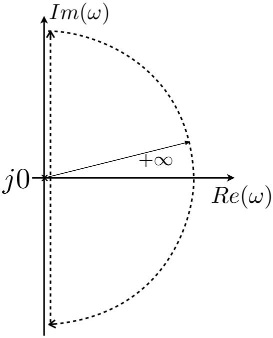
- The closed-loop transfer function is

$$W(s) = \frac{2G(s)}{1 + 2G(s)}$$

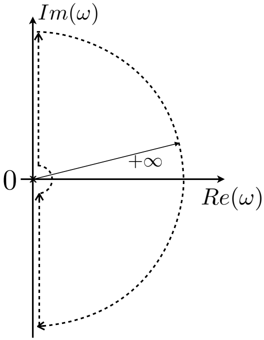
- The Nyquist plot of $2G(s)$ encircles $-1 + j0$ twice
- By Nyquist criterion, $W(s)$ is unstable

Dealing with imaginary poles

- The Nyquist plot is generated by a closed curve, the *Nyquist contour*, rotating clock-wise from $0-j\infty$ to $0+j\infty$ and back to $0-j\infty$ along a semi-circle of radius ∞, avoiding singularities of $G(j\omega)$
- We distinguish three types of curves, depending on the number of poles on the imaginary axis:

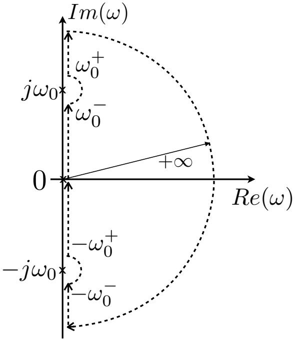


no poles on imaginary axis



pole(s) in $s=0$

(counted as stable)



pole(s) in $s=\pm j\omega$

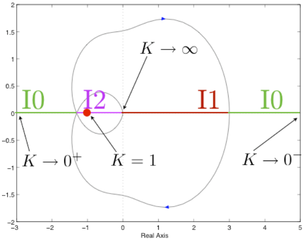
(counted as stable)

Stability analysis of static output feedback

- Under static output feedback $u(t) = -K(y(t) - r(t))$, the closed-loop transfer function from $r(t)$ to $y(t)$ is

$$W(s) = \frac{KG(s)}{1 + KG(s)}$$

- The number of encirclements of $-1 + j0$ of $KG(s)$ is equal to the number of encirclements of $-\frac{1}{K} + j0$ of $G(s)$
- To analyze closed-loop stability for different values of K is enough to draw Nyquist plot of $G(s)$ and move the point $-\frac{1}{K} + j0$ on the real axis



$$G(s) = \frac{60}{(s^2 + 2s + 20)(s + 1)}$$

I0=no unstable closed-loop poles

I2 =two unstable closed-loop poles

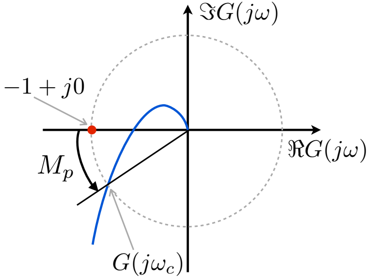
I1=an unstable closed-loop pole

Phase margin

- Assume $G(s)$ is open-loop asymptotically stable
- Let ω_c be the *gain crossover frequency*, that is $|G(j\omega_c)| = 1$
- To avoid encircling the point $-1 + j0$, we want the phase $\angle G(j\omega_c)$ as far away as possible from $-\pi$
- The *phase margin* is the quantity

$$M_p = \angle G(j\omega_c) - (-\pi)$$

- If $M_p > 0$, unit negative feedback control is asymptotically stabilizing
- For robustness of stability we would like a large positive phase margin

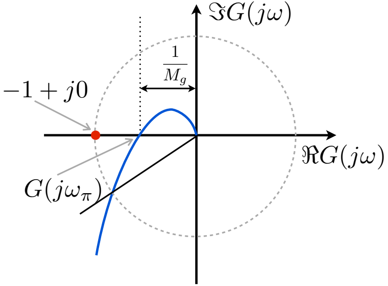


Gain margin

- Assume $G(s)$ is open-loop asymptotically stable
- Let ω_π be the *phase crossover frequency* such that $\angle G(j\omega_\pi) = -\pi$
- To avoid encircling the point $-1 + j0$, we want the point $G(j\omega_\pi)$ as far as possible away from $-1 + j0$.
- The *gain margin* is the inverse of $|G(j\omega_\pi)|$, expressed in dB

$$M_g = 20\log_{10}\frac{1}{|G(j\omega_\pi)|} = -|G(j\omega_\pi)|_{dB}$$

- For robustness of stability we like to have a large gain margin

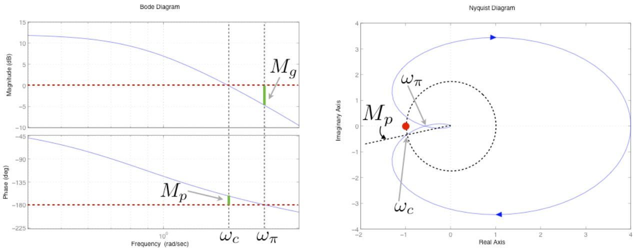


Sometimes the gain margin is defined as

$$M_g = \frac{1}{|G(j\omega_\pi)|}$$

and therefore $G(j\omega_\pi) = -\frac{1}{M_g} + j0$

Stability analysis using phase and gain margins

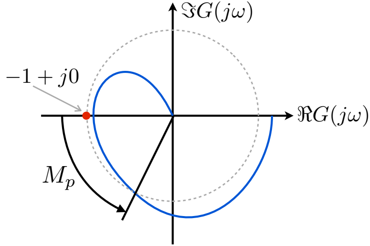


$$G(s) = \frac{8}{s^3 + 4s^2 + 4s + 2}$$

- The phase and gain margins help assessing the degree of robustness of a closed-loop system against uncertainties in the magnitude and phase of the process model
- They can be also applied to analyze the stability of dynamic output feedback laws $u(t) = C(s)(y(t) - r(t))$, by looking at the Bode plots of the loop function $C(j\omega)G(j\omega)$ (see next lecture on "loop shaping")

Stability analysis using phase and gain margins

- However in some cases phase and gain margins are not good indicators, see the following example



- The system is characterized by a large phase margin, but the polar plot is very close to the point $(-1, 0)$
- The system is not very robust to model uncertainties changing $G(s)$ from its nominal value
- In conclusion, phase and gain margins give good indications on how the loop function should be modified, but the complete Bode and Nyquist plots must be checked to conclude about closed-loop stability

English-Italian Vocabulary

	
frequency response	*risposta in frequenza*
angular frequency	*pulsazione*
steady-state	*regime permanente*
Bode plot	*diagramma di Bode*
Bode form	*forma di Bode*
Bode gain	*guadagno di Bode*
system's type	*tipo del sistema*
time constant	*costante di tempo*
damping ratio	*fattore di smorzamento*
natural frequency	*frequenza naturale*
zero/pole/gain form	*forma poli/zeri*
minimum phase system	*sistema a fase minima*
Nyquist plot	*diagramma di Nyquist*
crossover frequency	*frequenza di attraversamento*