Automatic Control 1

Integral action in state feedback control

Prof. Alberto Bemporad

University of Trento

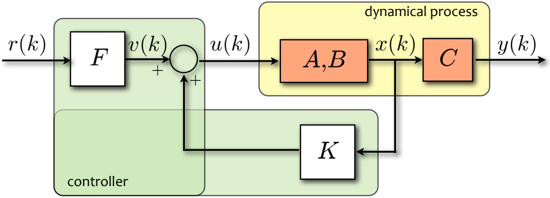
Academic year 2010-2011

Reference tracking

- Assume the open-loop system completely reachable and observable
- We know state feedback we can bring the output $y(k)$ to zero asymptotically
- How to make the output $y(k)$ *track* a generic constant *set-point* $r(k) \equiv r$?
- Solution: set $u(k) = Kx(k) + v(k)$

$$v(k) = Fr(k)$$

- We need to choose gain F properly to ensure reference tracking



$$\begin{aligned} x(k+1) &= (A+BK)x(k) + BFr(k) \\ y(k) &= Cx(k) \end{aligned}$$

Reference tracking

- To have $y(k) \to r$ we need a unit DC-gain from r to y

$$C(I-(A+BK))^{-1}BF = I$$

- Assume we have as many inputs as outputs (example: $u, y \in \mathbb{R}$)
- Assume the DC-gain from u to y is invertible, that is $C\,\mathrm{Adj}(I-A)B$ invertible
- Since state feedback doesn't change the zeros in closed-loop

$$C\,\mathrm{Adj}(I-A-BK)B = C\,\mathrm{Adj}(I-A)B$$

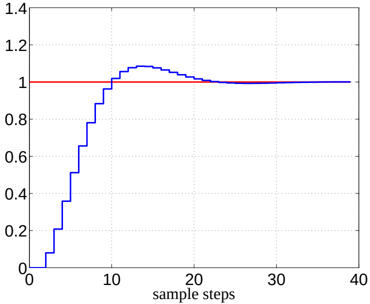
  then $C\,\mathrm{Adj}(I-A-BK)B$ is also invertible
- Set

$$F = (C(I-(A+BK))^{-1}B)^{-1}$$

Example

Poles placed in $(0.8 \pm 0.2j, 0.3)$. Resulting closed-loop:

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \\
u(k) &= \begin{bmatrix} -0.13 & -0.3 \end{bmatrix} x(k) + 0.08 r(k)
\end{aligned}
$$

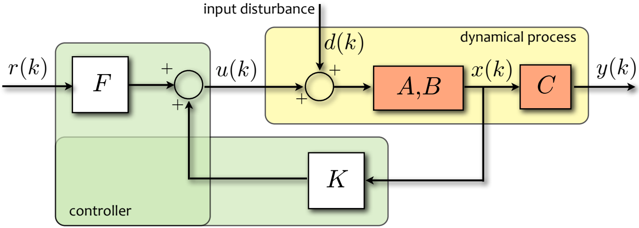


The transfer function $G(z)$ from r to y is $G(z) = \frac{2}{25z^2 - 40z + 17}$, and $G(1) = 1$

Unit step response of the closed-loop system (=evolution of the system from initial condition $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and reference $r(k) \equiv 1$, $\forall k \geq 0$)

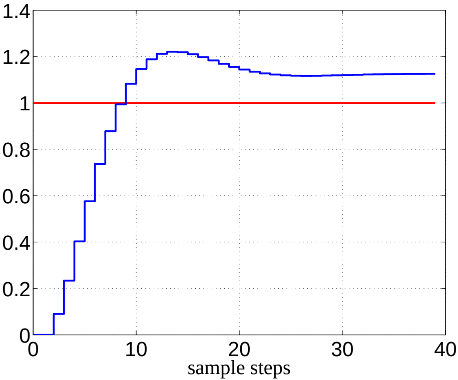
Reference tracking

- Problem: we have no direct feedback on the tracking error $e(k) = y(k) - r(k)$
- Will this solution be *robust* with respect to model uncertainties and exogenous disturbances ?
- Consider an *input disturbance* $d(k)$ (modeling for instance a non-ideal actuator, or an unmeasurable disturbance)



Example (cont'd)

- Let the input disturbance $d(k) = 0.01, \forall k = 0, 1, \ldots$



- The reference is not tracked !
- The unmeasurable disturbance $d(k)$ has modified the nominal conditions for which we designed our controller

Integral action for disturbance rejection

- Consider the problem of regulating the output $y(k)$ to $r(k) \equiv 0$ under the action of the input disturbance $d(k)$
- Let's augment the open-loop system with the integral of the output vector

$$\underbrace{q(k+1) = q(k) + y(k)}_{\text{integral action}}$$

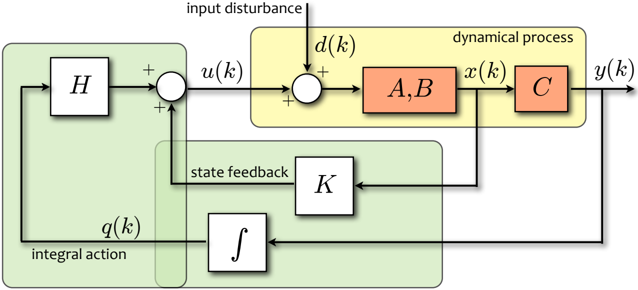
- The augmented system is

$$\begin{aligned}
\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} d(k) \\
y(k) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix}
\end{aligned}$$

- Design a stabilizing feedback controller for the augmented system

$$u(k) = \begin{bmatrix} K & H \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix}$$

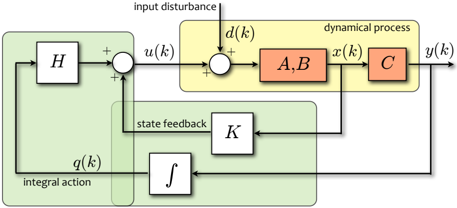
Rejection of constant disturbances



Theorem

Assume a stabilizing gain $[H\ K]$ can be designed for the system augmented with integral action. Then $\lim_{k\to+\infty} y(k) = 0$ for all constant disturbances $d(k) \equiv d$

Rejection of constant disturbances



Proof:

- The state-update matrix of the closed-loop system is

$$\left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & H \end{bmatrix}\right)$$

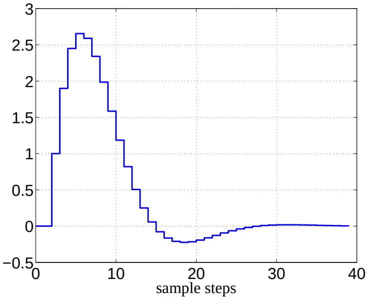
- The matrix has asymptotically stable eigenvalues by construction
- For a constant excitation $d(k)$ the extended state $\begin{bmatrix} x(k) \\ q(k) \end{bmatrix}$ converges to a steady-state value, in particular $\lim_{k\to\infty} q(k) = \bar{q}$
- Hence, $\lim_{k\to\infty} y(k) = \lim_{k\to\infty} q(k+1) - q(k) = \bar{q} - \bar{q} = 0$ □

Example (cont'd) – Now with integral action

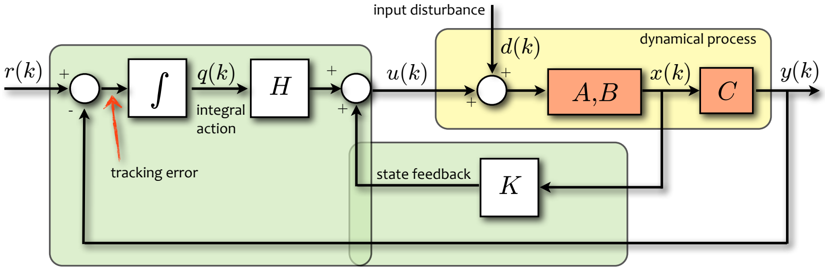
Poles placed in $(0.8 \pm 0.2j, 0.3)$ for the augmented system. Resulting closed-loop:

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(k)+d(k)) \\
q(k+1) &= q(k)+y(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \\
u(k) &= \begin{bmatrix} -0.48 & -1 \end{bmatrix} x(k) - 0.056q(k)
\end{aligned}
$$

Closed-loop simulation for $x(0) = [0\ 0]'$, $d(k) \equiv 1$:



Integral action for set-point tracking



Idea: Use the same feedback gains (K,H) designed earlier, but instead of feeding back the integral of the output, feed back the integral of the tracking error

$$\underbrace{q(k+1) = q(k) + (y(k) - r(k))}_{\text{integral action}}$$

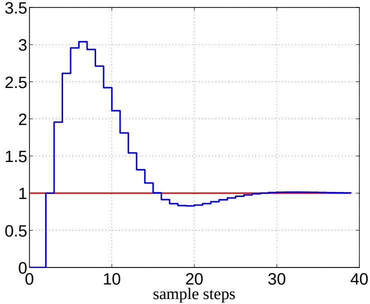
Example (cont'd)

$$x(k+1) = \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(k)+d(k))$$

$$q(k+1) = q(k) + \underbrace{(y(k)-r(k))}_{\text{tracking error}}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$u(k) = \begin{bmatrix} -0.48 & -1 \end{bmatrix} x(k) - 0.056 q(k)$$



Response for $x(0) = [0\ 0]'$, $d(k) \equiv 1$, $r(k) \equiv 1$

Looks like it's working ... but why ?

Tracking & rejection of constant disturbances/set-points

Theorem

Assume a stabilizing gain $[H\ K]$ can be designed for the system augmented with integral action. Then $\lim_{k\to+\infty} y(k) = r$ for all constant disturbances $d(k) \equiv d$ and set-points $r(k) \equiv r$

Proof:

- The closed-loop system

$$\begin{aligned}\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} &= \begin{bmatrix} A+BK & BH \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} d(k) \\ r(k) \end{bmatrix} \\ y(k) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix}\end{aligned}$$

  has input $\begin{bmatrix} d(k) \\ r(k) \end{bmatrix}$ and is asymptotically stable by construction

- For a constant excitation $\begin{bmatrix} d(k) \\ r(k) \end{bmatrix}$ the extended state $\begin{bmatrix} x(k) \\ q(k) \end{bmatrix}$ converges to a steady-state value, in particular $\lim_{k\to\infty} q(k) = \bar{q}$
- Hence, $\lim_{k\to\infty} y(k) - r(k) = \lim_{k\to\infty} q(k+1) - q(k) = \bar{q} - \bar{q} = 0$ □

Integral action for continuous-time systems

- The same reasoning can be applied to continuous-time systems

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

- Augment the system with the integral of the output $q(t)=\int_0^t y(\tau)d\tau$, i.e.,

$$\underbrace{\dot{q}(t)=y(t)=Cx(t)}_{\text{integral action}}$$

- The augmented system is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x(t) \\ q(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ q(t) \end{bmatrix}+\begin{bmatrix} B \\ 0 \end{bmatrix}u(t) \\ y(t) &= \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ q(t) \end{bmatrix} \end{aligned}$$

- Design a stabilizing controller $[K\ H]$ for the augmented system
- Implement

$$u(t)=Kx(t)+H\int_0^t (y(\tau)-r(\tau))d\tau$$

English-Italian Vocabulary

	
reference tracking	*inseguimento del riferimento*
steady state	*regime stazionario*
set point	*livello di riferimento*

Translation is obvious otherwise.