Automatic Control 1

Integral action in state feedback control

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Reference tracking

- Assume the open-loop system completely reachable and observable
- We know state feedback we can bring the output $y(k)$ to zero asymptotically
- How to make the output $y(k)$ track a generic constant set-point $r(k) \equiv r$?
- Solution: set $u(k) = Kx(k) + v(k)$

$$v(k) = Fr(k)$$

- We need to choose gain $F$ properly to ensure reference tracking

\[
x(k + 1) = (A + BK)x(k) + BFr(k)\]
\[
y(k) = Cx(k)\]
Reference tracking

- To have $y(k) \rightarrow r$ we need a unit DC-gain from $r$ to $y$
  \[ C(I - (A + BK))^{-1}BF = I \]

- Assume we have as many inputs as outputs (example: $u, y \in \mathbb{R}$)
- Assume the DC-gain from $u$ to $y$ is invertible, that is $C \text{Adj}(I - A)B$ invertible
- Since state feedback doesn’t change the zeros in closed-loop
  \[ C \text{Adj}(I - A - BK)B = C \text{Adj}(I - A)B \]
  then $C \text{Adj}(I - A - BK)B$ is also invertible
- Set
  \[ F = (C(I - (A + BK))^{-1}B)^{-1} \]
Example

Poles placed in $(0.8 \pm 0.2j, 0.3)$. Resulting closed-loop:

\[
\begin{align*}
    x(k+1) &= \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
    y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \\
    u(k) &= \begin{bmatrix} -0.13 & -0.3 \end{bmatrix} x(k) + 0.08r(k)
\end{align*}
\]

The transfer function $G(z)$ from $r$ to $y$ is

\[
G(z) = \frac{2}{25z^2 - 40z + 17}, \text{ and } G(1) = 1
\]

Unit step response of the closed-loop system (evolution of the system from initial condition $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and reference $r(k) \equiv 1$, $\forall k \geq 0$)
Reference tracking

- Problem: we have no direct feedback on the tracking error \( e(k) = y(k) - r(k) \)
- Will this solution be \textit{robust} with respect to model uncertainties and exogenous disturbances?
- Consider an \textit{input disturbance} \( d(k) \) (modeling for instance a non-ideal actuator, or an unmeasurable disturbance)
Example (cont’d)

- Let the input disturbance $d(k) = 0.01$, $\forall k = 0, 1, \ldots$

- The reference is not tracked!
- The unmeasurable disturbance $d(k)$ has modified the nominal conditions for which we designed our controller
Integral action for disturbance rejection

- Consider the problem of regulating the output $y(k)$ to $r(k) \equiv 0$ under the action of the input disturbance $d(k)$
- Let’s augment the open-loop system with the integral of the output vector

$$q(k + 1) = q(k) + y(k)$$

- The augmented system is

$$
\begin{bmatrix}
    x(k + 1) \\
    q(k + 1)
\end{bmatrix} =
\begin{bmatrix}
    A & 0 \\
    C & I
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    q(k)
\end{bmatrix} +
\begin{bmatrix}
    B \\
    0
\end{bmatrix} u(k) +
\begin{bmatrix}
    B \\
    0
\end{bmatrix} d(k)
$$

- $y(k) = \begin{bmatrix}
    C & 0
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    q(k)
\end{bmatrix}$

- Design a stabilizing feedback controller for the augmented system

$$u(k) = \begin{bmatrix}
    K & H
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    q(k)
\end{bmatrix}$$
Rejection of constant disturbances

Theorem

Assume a stabilizing gain \([H \ K]\) can be designed for the system augmented with integral action. Then \(\lim_{k \to +\infty} y(k) = 0\) for all constant disturbances \(d(k) \equiv d\).
Rejection of constant disturbances

\[ \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K \\ H \end{bmatrix} \]

\textbf{Proof:}

- The state-update matrix of the closed-loop system is
- The matrix has asymptotically stable eigenvalues by construction
- For a constant excitation \( d(k) \) the extended state \( \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} \) converges to a steady-state value, in particular \( \lim_{k \to \infty} q(k) = \bar{q} \)
- Hence, \( \lim_{k \to \infty} y(k) = \lim_{k \to \infty} q(k + 1) - q(k) = \bar{q} - \bar{q} = 0 \)
Example (cont’d) – Now with integral action

Poles placed in \((0.8 \pm 0.2j, 0.3)\) for the augmented system. Resulting closed-loop:

\[
\begin{align*}
    x(k+1) &= \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(k) + d(k)) \\
    q(k+1) &= q(k) + y(k) \\
    y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \\
    u(k) &= \begin{bmatrix} -0.48 & -1 \end{bmatrix} x(k) - 0.056q(k)
\end{align*}
\]

Closed-loop simulation for \(x(0) = [0 0]'\), \(d(k) \equiv 1\):
Integral action for set-point tracking

Idea: Use the same feedback gains \((K, H)\) designed earlier, but instead of feeding back the integral of the output, feed back the integral of the tracking error

\[
q(k + 1) = q(k) + (y(k) - r(k))
\]
Example (cont’d)

\[
x(k+1) = \begin{bmatrix} 1.1 & 1 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(k) + d(k))
\]

\[
q(k+1) = q(k) + (y(k) - r(k))
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)
\]

\[
u(k) = \begin{bmatrix} -0.48 & -1 \end{bmatrix} x(k) - 0.056q(k)
\]

Response for \(x(0) = [0 0]’\), \(d(k) \equiv 1, r(k) \equiv 1\)

Looks like it’s working … but why?
Theorem

Assume a stabilizing gain \([H \: K]\) can be designed for the system augmented with integral action. Then \(\lim_{k \to +\infty} y(k) = r\) for all constant disturbances \(d(k) \equiv d\) and set-points \(r(k) \equiv r\).

Proof:

The closed-loop system

\[
\begin{bmatrix}
  x(k+1) \\
  q(k+1)
\end{bmatrix}
= \begin{bmatrix}
  A + BK & BH \\
  C & I
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  q(k)
\end{bmatrix}
+ \begin{bmatrix}
  B & 0 \\
  0 & -I
\end{bmatrix}
\begin{bmatrix}
  d(k) \\
  r(k)
\end{bmatrix}
\]

\[y(k) = \begin{bmatrix}
  C & 0
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  q(k)
\end{bmatrix}\]

has input \(\begin{bmatrix}
  d(k) \\
  r(k)
\end{bmatrix}\) and is asymptotically stable by construction.

For a constant excitation \(\begin{bmatrix}
  d(k) \\
  r(k)
\end{bmatrix}\) the extended state \(\begin{bmatrix}
  x(k) \\
  q(k)
\end{bmatrix}\) converges to a steady-state value, in particular \(\lim_{k \to \infty} q(k) = \bar{q}\).

Hence, \(\lim_{k \to \infty} y(k) - r(k) = \lim_{k \to \infty} q(k + 1) - q(k) = \bar{q} - \bar{q} = 0\) \(\Box\)
Integral action for continuous-time systems

- The same reasoning can be applied to continuous-time systems
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \\
  y(t) = Cx(t)
  \]

- Augment the system with the integral of the output \( q(t) = \int_0^t y(\tau)d\tau \), i.e.,
  \[
  \dot{q}(t) = y(t) = Cx(t)
  \]

- The augmented system is
  \[
  \frac{d}{dt} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\
  y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}
  \]

- Design a stabilizing controller \([KH]\) for the augmented system
- Implement
  \[
  u(t) = Kx(t) + H \int_0^t (y(\tau) - r(\tau))d\tau
  \]
### English-Italian Vocabulary

<table>
<thead>
<tr>
<th>English</th>
<th>Italian</th>
</tr>
</thead>
<tbody>
<tr>
<td>reference tracking</td>
<td>inseguimento del riferimento</td>
</tr>
<tr>
<td>steady state</td>
<td>regime stazionario</td>
</tr>
<tr>
<td>set point</td>
<td>livello di riferimento</td>
</tr>
</tbody>
</table>

Translation is obvious otherwise.