Automatic Control 1

Pole placement by dynamic output feedback

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Output feedback control

- We know how to arbitrarily place the closed-loop poles by *state feedback*
- However, we may not want to directly measure the entire state vector $x$!
- Can we still place the closed-loop poles arbitrarily even if we only measure the *output* $y$?

Open-loop model:

$$ \begin{cases} 
    x(k+1) = Ax(k) + Bu(k) \\
    y(k) = Cx(k) 
\end{cases} $$

**state-space model**

$$ Y(z) = \frac{N(z)}{D(z)} U(z) $$

**transfer function**
Static output feedback (and “root locus”)

- Simple static feedback law: $u(k) = -Ky(k)$
- Closed-loop poles can be only placed on the *root locus* by changing the gain $K$
- Examples:

Root locus of a system with two asymptotically stable open-loop poles. The system is closed-loop asymptotically stable $\forall K > 0$

Root locus of a system with two asymptotically stable poles and an unstable open-loop pole. The system is closed-loop unstable $\forall K > 0$

MATLAB

```
rlocus(sys)
```

(Walter R. Evans, “Graphical analysis of control systems”, 1948)
State feedback control (review)

- Assume the system is completely reachable
- State feedback control law \( u(k) = Kx(k) + v(k) \)
- Closed-loop system

\[
\begin{cases}
  x(k+1) = (A + BK)x(k) + Bv(k) \\
  y(k) = Cx(k)
\end{cases}
\]

where

\[
\frac{N_K(z)}{D_K(z)} = C(zI - A - BK)^{-1}B, \quad \frac{N_K(z)}{D_K(z)} \triangleq C \text{Adj}(zI - A - BK)B \quad \text{and} \quad \frac{N_K(z)}{D_K(z)} \triangleq \text{det}(zI - A - BK)
\]

We can assign the roots of \( D_K(z) \) arbitrarily in the complex plane by properly choosing the state gain \( K \in \mathbb{R}^n \) (complex poles must have their conjugate)
State-feedback control (review)

- Assume \((A, B)\) in canonical reachability form

\[
A = \begin{bmatrix}
0 & & & \\
& & I_{n-1} & \\
& \vdots & & \\
0 & -a_0 & -a_1 & \cdots & -a_{n-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

- Let \(K = [k_1 \ \cdots \ k_n]\)

- The closed-loop matrix

\[
A + BK = \begin{bmatrix}
0 & & & \\
& & I_{n-1} & \\
& \vdots & & \\
0 & -(a_0 - k_1) & -(a_1 - k_2) & \cdots & -(a_{n-1} - k_n)
\end{bmatrix}
\]

is also in canonical form, so by choosing \(K\) we can decide its eigenvalues arbitrarily.
Zeros of closed-loop system

Fact

Linear state feedback does not change the zeros of the system: \( N_K(z) = N(z) \)

Example for \( x \in \mathbb{R}^3 \):

- Change the coordinates to canonical reachability form

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_3 & -a_2 & -a_1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad K = \begin{bmatrix}
k_3 \\
k_2 \\
k_1
\end{bmatrix}
\]

- Compute \( N(z) \)

\[
\text{Adj}(zI - A)B = \begin{bmatrix}
z^2 + a_1z + a_2 & z + a_1 & 1 \\
-a_3 & z(z + a_1) & z \\
-a_3z & -a_2z - a_3 & z^2
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
z \\
z^2
\end{bmatrix}
\]

- \( \text{Adj}(zI - A)B \) does not depend on the coefficients \( a_1, a_2, a_3 \)!
- Then \( \text{Adj}(zI - A - BK)B \) also does not depends on \( a_1 - k_1, a_2 - k_2, a_3 - k_3 \)!
- Hence \( N(z) = C\text{Adj}(zI - A)B = C\text{Adj}(zI - A - BK)B = N_K(z), \forall K' \in \mathbb{R}^n \)

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Potential issues in state feedback control

Measuring the entire state vector may be

- too expensive (many sensors)
- even impossible (high temperature, high pressure, inaccessible environment)

Can we use the estimate \( \hat{x}(k) \) instead of \( x(k) \) to close the loop?
Assume the open-loop system is completely observable (besides being reachable)

Construct the linear state observer

\[
\hat{x}(k + 1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))
\]

Set \(u(k) = K\hat{x}(k) + v(k)\)

The dynamics of the error estimate \(\tilde{x}(k) = x(k) - \hat{x}(k)\) is

\[
\tilde{x}(k + 1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) + L(Cx(k) - C\hat{x}(k)) = (A - LC)\tilde{x}(k)
\]

The error estimate does not depend on the feedback gain \(K\)!
Closed-loop dynamics

- Let’s combine the dynamics of the system, observer, and feedback gain

\[
\begin{align*}
    x(k + 1) &= Ax(k) + Bu(k) \\
    \hat{x}(k + 1) &= A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k)) \\
    u(k) &= K\hat{x}(k) + v(k) \\
    y(k) &= Cx(k)
\end{align*}
\]

- Take \(x(k), \hat{x}(k)\) as state components of the closed-loop system

\[
\begin{bmatrix}
    x(k) \\
    \hat{x}(k)
\end{bmatrix} = 
\begin{bmatrix}
    I & 0 \\
    I & -I
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    \hat{x}(k)
\end{bmatrix} \quad \text{(it is indeed a change of coordinates)}
\]

- The closed-loop dynamics is

\[
\begin{align*}
    \begin{bmatrix}
        x(k + 1) \\
        \hat{x}(k + 1)
    \end{bmatrix} &= 
    \begin{bmatrix}
        A + BK & -BK \\
        0 & A - LC
    \end{bmatrix}
    \begin{bmatrix}
        x(k) \\
        \hat{x}(k)
    \end{bmatrix} + 
    \begin{bmatrix}
        B \\
        0
    \end{bmatrix} v(k) \\
    y(k) &= 
    \begin{bmatrix}
        C & 0
    \end{bmatrix}
    \begin{bmatrix}
        x(k) \\
        \hat{x}(k)
    \end{bmatrix}
\end{align*}
\]
Closed-loop dynamics

- The transfer function from \( v(k) \) to \( y(k) \) is

\[
G(z) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} zI - A - BK & BK \\ 0 & zI - A + LC \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\
= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} (zI - A - BK)^{-1} & \ast \\ 0 & (zI - A + LC)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \\
= C(zI - A - BK)^{-1}B = \frac{N(z)}{D_K(z)}
\]

- Even if we substituted \( x(k) \) with \( \hat{x}(k) \), the input-output behavior of the closed-loop system didn’t change!

The closed-loop poles can be assigned arbitrarily using **dynamic** output feedback, as in the state feedback case.

The closed-loop transfer function does not depend on the observer gain \( L \).
Separation principle

The design of the control gain $K$ and of the observer gain $L$ can be done independently.

- Watch out! $G(z) = C(zI - A - BK)^{-1}B$ only represents the I/O (=input/output) behavior of the closed-loop system.
- The complete set of poles of the closed-loop system are given by
  \[
  \det(zI - \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix}) = \det(zI - A - BK) \det(zI - A + LC) = D_K(z)D_L(z)
  \]
- A zero/pole cancellation of the observer poles has occurred:
  \[
  G(z) = \begin{bmatrix} C & 0 \end{bmatrix} (zI - \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix})^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = \frac{N(z)D_L(z)}{D_K(z)D_L(z)}
  \]
Transient effects of the estimator gain

- The estimator gain $L$ seems irrelevant ...

- However, consider the effect of a nonzero initial condition $\begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix}$ for $v(k) \equiv 0$

\[ y(0) = Cx(0) \]

\[ y(1) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A+ BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = C(A + BK)x(0) - CBK\hat{x}(0) \]

\[ y(2) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A+ BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(1) \\ \hat{x}(1) \end{bmatrix} = C(A + BK)x(1) - CBK\hat{x}(1) \]

\[ = C(A + BK)^2x(0) - C(A + BK)BK\hat{x}(0) - CBK(A - LC)\hat{x}(0) \]

- The observer gain $L$ has an effect on the transient!
Choosing the estimator gain

- Intuitively, if $\hat{x}(k)$ is a poor estimate of $x(k)$ then the control action will also be poor

**Rule of thumb:** place the observer poles $\approx 10$ times faster than the controller poles

- Optimal methods exist to choose the observer poles (Kalman filter)
- Fact: The choice of $L$ is very important for determining the sensitivity of the closed-loop system with respect to input and output noise
Zero/pole cancellations

- We have zero/pole cancellations, the system has uncontrollable and/or unobservable modes

- Intuitively:
  - $\tilde{x}$ does not depend on $v \Rightarrow \tilde{x}$ is not controllable
  - $y$ depends on $\tilde{x}$ during transient $\Rightarrow \tilde{x}$ observable

- The reachability matrix $R$ is

$$
R = \begin{bmatrix}
\begin{bmatrix} B & 0 \end{bmatrix} & \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} & \begin{bmatrix} B \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}^{2n-1} & \begin{bmatrix} B \\ 0 \end{bmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
B & (A+BK)B & \cdots & (A+BK)^{2n-1}B \\
0 & 0 & \cdots & 0
\end{bmatrix}
$$

- Since $(A,B)$ is reachable, $\text{rank}(R) = n < 2n \Rightarrow$ uncontrollable modes
- The observability matrix $\Theta$ doesn’t have a similar structure
The state-space equations of the dynamic compensator are

\[
\begin{cases}
    \hat{x}(k+1) = (A + BK - LC)\hat{x}(k) + Bv(k) + Ly(k) \\
u(k) = K\hat{x}(k) + v(k)
\end{cases}
\]

Equivalently, its transfer function is given by (superposition of effects)

\[
U(z) = (K(zI - A - BK + LC)^{-1}B + I)V(z) + K(zI - A - BK + LC)^{-1}LY(z)
\]
Example: Control of a DC Motor

\[ G(s) = \frac{K}{s^3 + \beta s^2 + \alpha s} \]

**MATLAB**

```matlab
K=1; beta=.3; alpha=1;
G=tf(K,[1 beta alpha 0]);

% Controller
polesK=[-1,-0.5+0.6*j,-0.5-0.6*j];
polesKd=exp(ts*polesK);
K=-place(A,B,polesKd);

% Observer
polesL=[-10, -9, -8];
polesLd=exp(ts*polesL);
L=place(A',C',polesLd)';

% Closed-loop system, state=[x;xhat]
bigA=[A,B*K;L*C,A+B*K-L*C];
bigB=[B;B];
bigC=[C,zeros(1,3)];
bigD=0;
clsys=ss(bigA,bigB,bigC,bigD,ts);

x0=[1 1 1]'; % Initial state
xhat0=[0 0 0]'; % Initial estimate
T=20;
initial(clsys, [x0;xhat0],T);
pause

t=(0:ts:T)';
v=ones(size(t));
lsim(clsys,v);
```
Example: Control of a DC Motor

\[ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v(k) \equiv 0 \]

\[ x(0) = \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v(k) \equiv 1 \]
## English-Italian Vocabulary

<table>
<thead>
<tr>
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<th>Italian</th>
</tr>
</thead>
<tbody>
<tr>
<td>root locus</td>
<td>luogo delle radici</td>
</tr>
<tr>
<td>separation principle</td>
<td>principio di separazione</td>
</tr>
<tr>
<td>dynamic compensator</td>
<td>compensatore dinamico</td>
</tr>
</tbody>
</table>

Translation is obvious otherwise.