Automatic Control 1

Pole placement by dynamic output feedback

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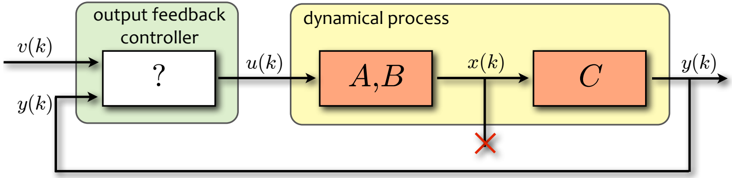
Output feedback control



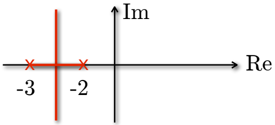
- We know how to arbitrarily place the closed-loop poles by *state feedback*
- However, we may not want to directly measure the entire state vector x !
- Can we still place the closed-loop poles arbitrarily even if we only measure the *output* y ?

Open-loop model:

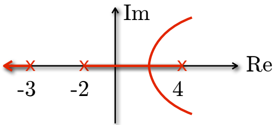
$$\underbrace{\begin{cases} x(k+1) = Ax(k)+Bu(k) \\ y(k) = Cx(k) \end{cases}}_{\text{state-space model}} \Longleftrightarrow \underbrace{Y(z) = \frac{N(z)}{D(z)}U(z)}_{\text{transfer function}}$$

Static output feedback (and "root locus")

- Simple static feedback law: $u(k) = -Ky(k)$
- Closed-loop poles can be only placed on the *root locus* by changing the gain K
- Examples:



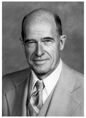
Root locus of a system with two asymptotically stable open-loop poles. The system is closed-loop asymptotically stable $\forall K > 0$



Root locus of a system with two asymptotically stable poles and an unstable open-loop pole. The system is closed-loop unstable $\forall K > 0$

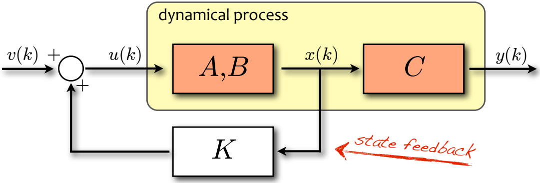
MATLAB
`»rlocus(sys)`

(Walter R. Evans, "Graphical analysis of control systems", 1948)



(1920-1999)

State feedback control (review)



- Assume the system is completely reachable
- State feedback control law $u(k) = Kx(k) + v(k)$
- Closed-loop system

$$\left\{ \begin{array}{rcl} x(k+1) & = & (A+BK)x(k) + Bv(k) \\ y(k) & = & Cx(k) \end{array} \right. \quad \Longleftrightarrow \quad Y(z) = \frac{N_K(z)}{D_K(z)} V(z)$$

where

$$\frac{N_K(z)}{D_K(z)} = C(zI - A - BK)^{-1}B, \quad \begin{array}{rcl} N_K(z) & \triangleq & C\,\mathrm{Adj}(zI - A - BK)B \\ D_K(z) & \triangleq & \det(zI - A - BK) \end{array}$$

We can assign the roots of $D_K(z)$ arbitrarily in the complex plane by properly choosing the state gain $K \in \mathbb{R}^n$ (complex poles must have their conjugate)

State-feedback control (review)

- Assume (A,B) in canonical reachability form

$$A = \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_{n-1} \\ -a_0 & \begin{matrix} -a_1 & \dots & -a_{n-1} \end{matrix} \end{array}\right], \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- Let $K = \begin{bmatrix} k_1 & \dots & k_n \end{bmatrix}$
- The closed-loop matrix

$$A + BK = \left[\begin{array}{cc} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_{n-1} \\ -(a_0 - k_1) & \begin{matrix} -(a_1 - k_2) & \dots & -(a_{n-1} - k_n) \end{matrix} \end{array}\right]$$

is also in canonical form, so by choosing K we can decide its eigenvalues arbitrarily

Zeros of closed-loop system

Fact

Linear state feedback does not change the zeros of the system: $N_K(z) = N(z)$

Example for $x \in \mathbb{R}^3$:

- Change the coordinates to canonical reachability form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} k_3 & k_2 & k_1 \end{bmatrix}$$

- Compute $N(z)$

$$\mathrm{Adj}(zI - A)B = \begin{bmatrix} z^2 + a_1 z + a_2 & z + a_1 & 1 \\ -a_3 & z(z + a_1) & z \\ -a_3 z & -a_2 z - a_3 & z^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}$$

- $\mathrm{Adj}(zI - A)B$ does not depend on the coefficients a_1, a_2, a_3 !
- Then $\mathrm{Adj}(zI - A - BK)B$ also does not depends on $a_1 - k_1$, $a_2 - k_2$, $a_3 - k_3$!
- Hence $N(z) = C\,\mathrm{Adj}(zI - A)B = C\,\mathrm{Adj}(zI - A - BK)B = N_K(z)$, $\forall K' \in \mathbb{R}^n$

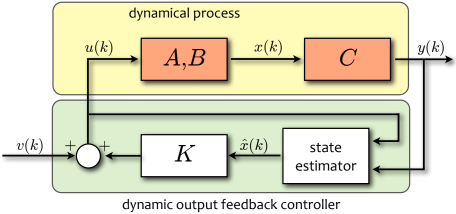
Potential issues in state feedback control

Measuring the entire state vector may be

- too expensive (many sensors)
- even impossible (high temperature, high pressure, inaccessible environment)

Can we use the estimate $\hat{x}(k)$ instead of $x(k)$ to close the loop ?

Dynamic compensator



- Assume the open-loop system is completely observable (besides being reachable)
- Construct the linear state observer

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$$

- Set $u(k) = K\hat{x}(k) + v(k)$
- The dynamics of the error estimate $\tilde{x}(k) = x(k) - \hat{x}(k)$ is

$$\tilde{x}(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) + L(Cx(k) - C\hat{x}(k)) = (A - LC)\tilde{x}(k)$$

The error estimate does not depend on the feedback gain K !

Closed-loop dynamics

- Let's combine the dynamics of the system, observer, and feedback gain

$$\begin{cases} x(k+1) &= Ax(k)+Bu(k) \\ \hat{x}(k+1) &= A\hat{x}(k)+Bu(k)+L(y(k)-C\hat{x}(k)) \\ u(k) &= K\hat{x}(k)+v(k) \\ y(k) &= Cx(k) \end{cases}$$

- Take $x(k)$, $\tilde{x}(k)$ as state components of the closed-loop system

$$\begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

(it is indeed a change of coordinates)

- The closed-loop dynamics is

$$\begin{cases} \begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(k) \\ y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} \end{cases}$$

Closed-loop dynamics

- The transfer function from $v(k)$ to $y(k)$ is

$$\begin{aligned} G(z) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} zI-A-BK & BK \\ 0 & zI-A+LC \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} (zI-A-BK)^{-1} & \star \\ 0 & (zI-A+LC)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= C(zI-A-BK)^{-1}B = \frac{N(z)}{D_K(z)} \end{aligned}$$

- Even if we substituted $x(k)$ with $\hat{x}(k)$, the input-output behavior of the closed-loop system didn't change !

The closed-loop poles can be assigned arbitrarily using **dynamic** output feedback, as in the state feedback case

The closed-loop transfer function does not depend on the observer gain L

Separation principle

Separation principle

The design of the control gain K and of the observer gain L can be done independently

- Watch out ! $G(z) = C(zI - A - BK)^{-1}B$ only represents the I/O (=input/output) behavior of the closed-loop system
- The complete set of poles of the closed-loop system are given by

$$\det\left(zI - \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}\right) = \det(zI - A - BK)\det(zI - A + LC) = D_K(z)D_L(z)$$

- A zero/pole cancellation of the observer poles has occurred:

$$G(z) = \begin{bmatrix} C & 0 \end{bmatrix}\left(zI - \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}\right)^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix} = \frac{N(z)D_L(z)}{D_K(z)D_L(z)}$$

Transient effects of the estimator gain

- The estimator gain L seems irrelevant ...
- However, consider the effect of a nonzero initial condition $\begin{bmatrix} x(0) \\ \tilde{x}(0) \end{bmatrix}$ for $v(k) \equiv 0$

$$
\begin{aligned}
y(0) &= Cx(0) \\
y(1) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(0) \\ \tilde{x}(0) \end{bmatrix} \\
&= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} (A+BK)x(0)-BK\tilde{x}(0) \\ (A-LC)\tilde{x}(0) \end{bmatrix} = C(A+BK)x(0) - CBK\tilde{x}(0) \\
y(2) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(1) \\ \tilde{x}(1) \end{bmatrix} \\
&= C(A+BK)x(1) - CBK\tilde{x}(1) \\
&= C(A+BK)^2x(0) - C(A+BK)BK\tilde{x}(0) - CBK(A-\mathbf{L}C)\tilde{x}(0)
\end{aligned}
$$

- The observer gain L has an effect on the transient !

Choosing the estimator gain

- Intuitively, if $\hat{x}(k)$ is a poor estimate of $x(k)$ then the control action will also be poor

Rule of thumb: place the observer poles $\approx$ 10 times faster than the controller poles

- Optimal methods exist to choose the observer poles (Kalman filter)
- Fact: The choice of L is very important for determining the sensitivity of the closed-loop system with respect to input and output noise

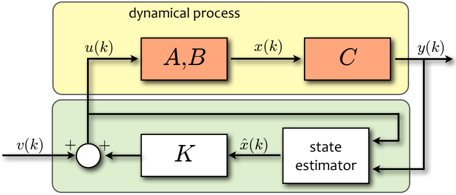
Zero/pole cancellations

- We have zero/pole cancellations, the system has uncontrollable and/or unobservable modes
- Intuitively:
  - $\tilde{x}$ does not depend on $v \Rightarrow \tilde{x}$ is not controllable
  - y depends on $\tilde{x}$ during transient $\Rightarrow \tilde{x}$ observable
- The reachability matrix R is

$$\begin{aligned} R &= \left[\begin{bmatrix} B \\ 0 \end{bmatrix} \middle| \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \middle| \cdots \middle| \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}^{2n-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \right] \\ &= \begin{bmatrix} B & (A+BK)B & \cdots & (A+BK)^{2n-1}B \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

- Since (A,B) is reachable, $\text{rank}(R) = n < 2n \Rightarrow$ uncontrollable modes
- The observability matrix Θ doesn't have a similar structure

Dynamic compensator



- The state-space equations of the *dynamic compensator* are

$$\begin{cases} \hat{x}(k+1) = (A+BK-LC)\hat{x}(k)+Bv(k)+Ly(k) \\ u(k) = K\hat{x}(k)+v(k) \end{cases}$$

- Equivalently, its transfer function is given by (superposition of effects)

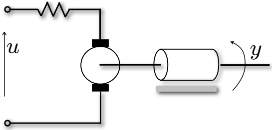
$$U(z) = (K(zI-A-BK+LC)^{-1}B+I)V(z) + \underbrace{K(zI-A-BK+LC)^{-1}L}_{\text{dynamic output feedback}}Y(z)$$

MATLAB

```
» con=-reg(sys,K,L)
```

Example: Control of a DC Motor

$$G(s) = \frac{K}{s^3 + \beta s^2 + \alpha s}$$



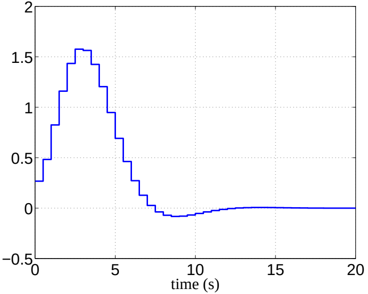
MATLAB

```
K=1; beta=.3; alpha=1;
G=tf(K,[1 beta alpha 0]);

ts=0.5; % sampling time
Gd=c2d(G,ts);
sysd=ss(Gd);
[A,B,C,D]=ssdata(sysd);

% Controller
polesK=[-1,-0.5+0.6*j,-0.5-0.6*j];
polesKd=exp(ts*polesK);
K=-place(A,B,polesKd);

% Observer
polesL=[-10, -9, -8];
polesLd=exp(ts*polesL);
L=place(A',C',polesLd)';
```

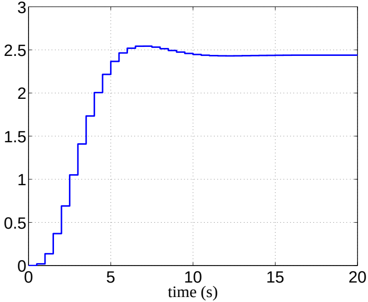
MATLAB

```
% Closed-loop system, state=[x;xhat]

bigA=[A,B*K;L*C,A+B*K-L*C];
bigB=[B;B];
bigC=[C,zeros(1,3)];
bigD=0;
clsys=ss(bigA,bigB,bigC,bigD,ts);

x0=[1 1 1]';     % Initial state
xhat0=[0 0 0]'; % Initial estimate
T=20;
initial(clsys, [x0;xhat0],T);
pause

t=(0:ts:T)';
v=ones(size(t));
lsim(clsys,v);
```

Example: Control of a DC Motor



$$x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v(k) \equiv 0$$



$$x(0) = \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v(k) \equiv 1$$

English-Italian Vocabulary

	
root locus	*luogo delle radici*
separation principle	*principio di separazione*
dynamic compensator	*compensatore dinamico*

Translation is obvious otherwise.