Automatic Control 1

State estimation and linear observers

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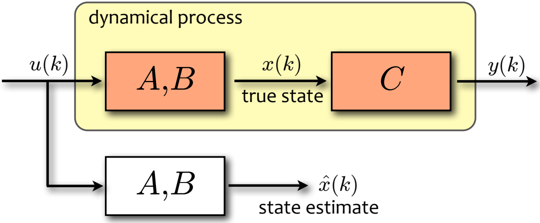
State estimation

State estimation problem

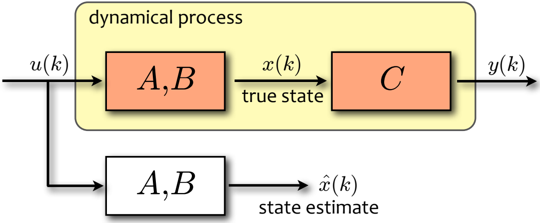
At each time k construct an estimate $\hat{x}(k)$ of the state $x(k)$, by only measuring the output $y(k)$ and input $u(k)$

- **Open-loop observer**: Build an artificial copy of the system, fed in parallel by with the same input signal $u(k)$



- The "copy" is a numerical simulator $\hat{x}(k+1) = A\hat{x}(k) + Bu(k)$ reproducing the behavior of the real system

Open-loop observer



- The dynamics of the real system and of the numerical copy are

$$
\begin{array}{rcl}
x(k+1) & = & Ax(k)+Bu(k) \\
\hat{x}(k+1) & = & A\hat{x}(k)+Bu(k)
\end{array}
$$

True process

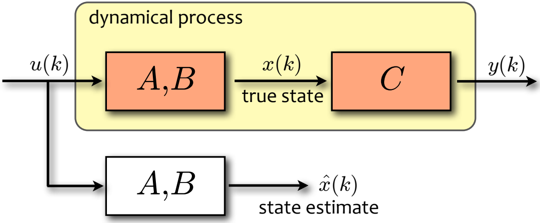
Numerical copy

- The dynamics of the *estimation error* $\tilde{x}(k) = x(k) - \hat{x}(k)$ are

$$\tilde{x}(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) = A\tilde{x}(k)$$

and then $\tilde{x}(k) = A^k(x(0) - \hat{x}(0))$

Open-loop observer

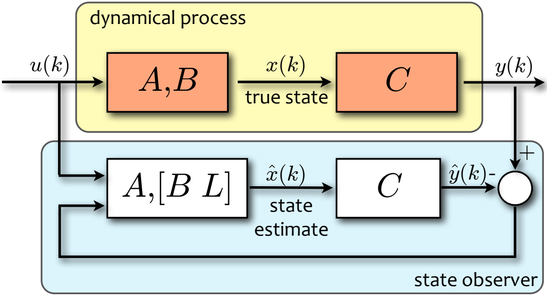


The estimation error is $\tilde{x}(k) = A^k(x(0) - \hat{x}(0))$. This is not ideal, because

- The dynamics of the estimation error are fixed by the eigenvalues of A and cannot be modified
- The estimation error vanishes asymptotically if and only if A is asymptotically stable

Note that we are not exploiting $y(k)$ to compute the state estimate $\hat{x}(k)$!

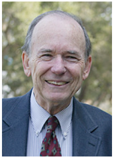
Luenberger observer



- **Luenberger observer**: Correct the estimation equation with a feedback from the estimation error $y(k) - \hat{y}(k)$

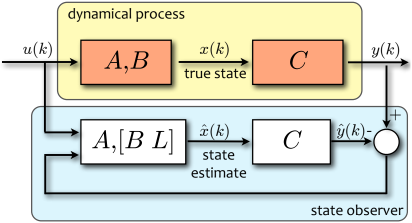
$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + \underbrace{L(y(k) - C\hat{x}(k))}_{\text{feedback on estimation error}}$$

where $L \in \mathbb{R}^{n \times p}$ is the *observer gain*



David G. Luenberger (1937-)

Luenberger observer



- The dynamics of the state estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$ is

$$\begin{aligned} \tilde{x}(k+1) &= Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) - L[y(k) - C\hat{x}(k)] \\ &= (A - LC)\tilde{x}(k) \end{aligned}$$

  and then $\tilde{x}(k) = (A - LC)^k (x(0) - \hat{x}(0))$
- Same idea for continuous-time systems $\dot{x}(t) = Ax(t) + Bu(t)$

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L[y(t) - C\hat{x}(t)]$$

  The dynamics of the state estimation error are $\frac{d\tilde{x}(t)}{dt} = (A - LC)\tilde{x}(t)$

Eigenvalue assignment of state observer

Theorem

If the pair (A,C) is observable, then the eigenvalues of $(A-LC)$ can be placed arbitrarily

Proof:

- If the pair (A,C) is completely observable, the dual system (A',C',B',D') is completely reachable
- Then we can design a compensator K for the dual system and place the eigenvalues of $(A'+C'K)$ arbitrarily
- The eigenvalues of matrix $(A'+C'K)$ = eigenvalues of its transpose $(A+K'C)$
- Define $L=-K'$. The theorem is proved. □

MATLAB

```
» L=acker(A',C',P)';
» L=place(A',C',P)';
```

where $P=[\lambda_1\lambda_2\ldots\lambda_n]$ = desired observer eigenvalues

Example of observer design

- We want to design a state observer for the continuous-time system in state-space form

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(t) \end{cases}$$

- We want to place the poles of the observer in $\{-4, -4\}$
- It is easy to verify that the system is completely observable
- Let $L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$ be the unknown observer gain
- Write the generic state estimation matrix

$$A - LC = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2}\ell_1 \\ 1 & -1 - \frac{1}{2}\ell_2 \end{bmatrix}$$

Example of observer design (cont'd)

- The characteristic polynomial of the observer is

$$\det(\lambda I - A + LC) = \lambda^2 + (2 + \frac{1}{2}\ell_2)\lambda + \frac{1}{2}\ell_2 + \frac{1}{2}\ell_1 + 1$$

- Impose the polynomial equals the desired one $(\lambda + 4)^2 = \lambda^2 + 8\lambda + 16$
- Solve the linear system of equations in ℓ_1, ℓ_2 and get

$$\ell_1 = 18,\ \ell_2 = 12$$

- The resulting Luenberger observer is

$$\frac{d\hat{x}(t)}{dt} = \begin{bmatrix} -1 & -9 \\ 1 & -7 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 18 \\ 12 \end{bmatrix} y(t)$$

Example of observer design in MATLAB

MATLAB

```
» A =[1.8097 -0.8187; 1 0];
» B = [0.5; 0];
» C =[0.1810 -0.1810];
» D = 0;

» L=place(A',C',[.5 .7])'

» eig(A-L*C)
ans = 0.7000 0.5000
```

State-space model in discrete-time

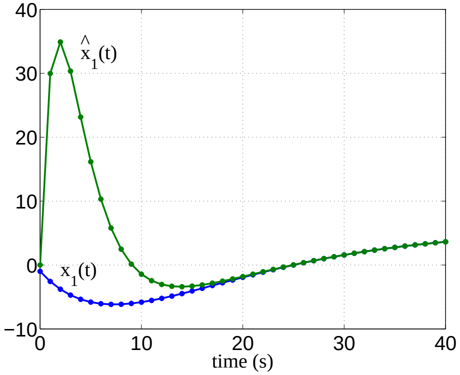
$$A = \begin{bmatrix} 1.8097 & -0.8187 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.1810 & -0.1810 \end{bmatrix}, \; D = 0$$

Resulting observer gain

$$L = \begin{bmatrix} -82.6341 \\ -86.0031 \end{bmatrix}$$

Double-check observer poles: answer is ok !

Example of observer design in MATLAB (cont'd)

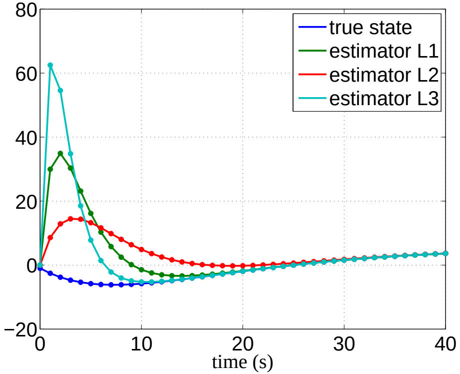


response from initial conditions
$x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $u(k) \equiv 0.1$

MATLAB

```
x=[-1;1];    % initial state
xhat=[0;0]; % initial estimate

XX=x;
XXhat=xhat;
T=40;
UU=.1*ones(1,T); % input signal

for k=0:T-1,
    u=UU(k+1);
    y=C*x+D*u;
    yhat=C*xhat+D*u;

    x=A*x+B*u;
    xhat=A*xhat+B*u+L*(y-yhat);

    XX=[XX,x];
    XXhat=[XXhat,xhat];
end

plot(0:T,[XX(1,:);XXhat(1,:)]);
```

Example of observer design in MATLAB (cont'd)



MATLAB

```
L1=place(A',C',[.5 .7])';

L2=place(A',C',[.75 .8])';

L3=place(A',C',[.4 .5])';
```

A fast observer often implies large estimation errors in the transient

Comparison of different observer gains
Response from initial conditions
$x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $u(k) \equiv 0.1$

English-Italian Vocabulary

	
estimation error	*errore di stima*
Luenberger observer	*osservatore alla Luenberger*
observer gain	*guadagno dell'osservatore*

Translation is obvious otherwise.