Automatic Control 1

Observability analysis

Prof. Alberto Bemporad

University of Trento

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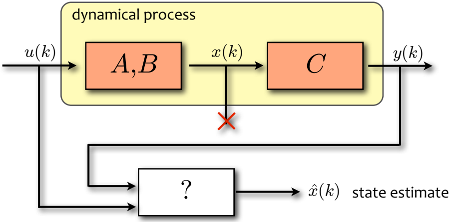
Observability



- Implementing a state feedback controller $u(k) = Kx(k)$ requires the entire state vector $x(k)$
- **Problem:** often sensors only provide the measurements of output $y(k)$
- IDEA: is it possible to estimate the state x by measuring only the output y and knowing the applied input u ?
- *Observability* analysis addresses this problem, telling us when and how the state estimation problem can be solved

Observability

- Consider

$$\begin{cases} x(k+1) & = & Ax(k)+Bu(k) \\ y(k) & = & Cx(k)+Du(k) \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and initial condition $x(0) = x_0 \in \mathbb{R}^n$ ([1])

- The solution for the output is

$$y(k, x_0, u(\cdot)) = CA^k x_0 + \sum_{j=0}^{k-1} CA^j Bu(k-1-j) + Du(k)$$

Definition

The pair of states $x_1 \neq x_2 \in \mathbb{R}^n$ is called *indistinguishable* from the output $y(\cdot)$ if for any input sequence $u(\cdot)$

$$y(k, x_1, u(\cdot)) = y(k, x_2, u(\cdot)), \forall k \geq 0$$

A linear system is called *(completely) observable* if no pair of states are indistinguishable from the output

[1] Everything here can be easily generalized to multivariable systems $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

Observability

- Consider the problem of reconstructing the initial condition x_0 from n output measurements, applying a known input sequence

$$\begin{array}{rcl} y(0) & = & Cx_0 + Du(0) \\ y(1) & = & CAx_0 + CBu(0) + Du(1) \\ & \vdots & \\ y(n-1) & = & CA^{n-1}x_0 + \sum_{j=1}^{n-2} CA^jBu(n-2-j) + Du(n-1) \end{array}$$

- Define

$$\Theta = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\text{This is a } n\times n \text{ matrix}} \qquad Y = \underbrace{\begin{bmatrix} y(0) - Du(0) \\ y(1) - CBu(0) - Du(1) \\ \vdots \\ y(n-1) - \sum_{j=1}^{n-2} CA^jBu(n-2-j) - Du(n-1) \end{bmatrix}}_{\text{This is an } n\text{-th dimensional vector}}$$

Observability

- The initial state x_0 is determined by solving the linear system

$$Y = \Theta x_0$$

  The matrix $\Theta \in \mathbb{R}^{n\times n}$ is called the *observability matrix* of the system

- If we assume perfect knowledge of the output (i.e., no noise on output measurements), we can always solve the system $Y = \Theta x_0$. In particular:
  - There is only one solution if $\text{rank}(\Theta) = n$
  - There exist infinite solutions if $\text{rank}(\Theta) < n$. In this case, all solutions are given by $x_0 + \ker(\Theta)$, where x_0 is any particular solution of the system

- Knowing x_0, one knows the state $x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^i Bu(k-1-i)$ for any k

Observability

- The system of equations $\Theta x_0 = Y$ has a solution if and only if

$$\text{rank}(\Theta) = \text{rank}([\Theta\ Y]) \qquad \text{(Rouché-Capelli Theorem)}$$

- Because we have $\Theta \in \mathbb{R}^{n\times n}$, if $\text{rank}(\Theta) = n \Rightarrow \text{rank}([\Theta\ Y]) = n$ for each Y
- The solution is unique if and only if $\text{rank}(\Theta) = n$
- Since the input $u(k)$ influences only the known vector Y, the solvability of the system $\Theta x_0 = Y$ is independent from $u(k)$
- Then, for linear systems the observability property does not depend on the input signal $u(\cdot)$, it only depends on matrix Θ (i.e., on A and C)
- We can study the observability properties of the system for $u(k) \equiv 0$

Observability

Theorem

A linear system is observable if and only if $\text{rank}(\Theta) = n$

Proof:

- ($\Rightarrow$) Assume that the system is observable, and suppose (by contradiction) that $\text{rank}(\Theta) < n$. Then $\exists\, x \neq 0$ such that $\Theta x = 0$, and then

$$Cx = 0,\ CAx = 0,\ \ldots,\ CA^{n-1}x = 0$$

By Cayley-Hamilton Theorem we have that $CA^k x = 0$, $\forall k \geq 0$, and then x is indistinguishable from the origin, which is a contradiction

- ($\Leftarrow$) Assume $\text{rank}(\Theta) = n$, and suppose (by contradiction) that $\exists x_1 \neq x_2$ that are indistinguishable from the output. Then, $CA^k x_1 = CA^k x_2$, $\forall k \geq 0$. Let $x = x_1 - x_2$. It follows that

$$Cx = 0,\ CAx = 0,\ \ldots,\ CA^{n-1}x = 0$$

i.e., $\Theta x = 0$, with $x \neq 0$, which is a contradiction

$\square$

Comments on the observability property

- As the observability property of a system depends only on matrices A and C, we call a pair (A, C) *observable* if

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$$

- It can be proved that $\ker(\Theta)$ is the set of states $x \in \mathbb{R}^n$ that are indistinguishable from the origin

$$y(k, x, u(\cdot)) = y(k, 0, u(\cdot)), \ \forall k \geq 0$$

for any input sequence $u(\cdot)$

- Since $\ker(\Theta) = \{0\}$ if and only if $\operatorname{rank}(\Theta) = n$, a system is observable if and only if there are no states that are indistinguishable from the origin $x = 0$

Canonical observability decomposition

Goal: make a change of coordinates that separate observable and unobservable states

- Let $\dim(\ker(\Theta)) = n - n_o \geq 1$ and consider the change of coordinates

$$T = \begin{bmatrix} v_{n_o+1} & \dots & v_n & w_1 & \dots & w_{n_o} \end{bmatrix}$$

  where $\left\{v_{n_o+1}, \dots, v_n\right\}$ is a basis of $\ker(\Theta)$, and $\left\{w_1, \dots, w_{n_o}\right\}$ is a completion to obtain a basis of $\mathbb{R}^n$
- By Cayley-Hamilton theorem, $\ker(\Theta)$ is A-invariant ($Ax \in \ker(\Theta)$, $\forall x \in \Theta$), and hence Av_i has no components along the basis vector $w_1, \dots, w_{n_o}$, $\forall i = n_o + 1, \dots, n$
- Note also that $Cv_i = 0$, because $\Theta v_i = 0$, $\forall i = n_o + 1, \dots, n$
- In the new coordinates the system has matrices $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$ and $\tilde{C} = CT$ in the *canonical observability form*

$$\tilde{A} = \begin{bmatrix} A_{uo} & A_{12} \\ 0 & A_o \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B_{uo} \\ B_o \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} 0 & C_o \end{bmatrix}$$

MATLAB

```
[At,Bt,Ct,Tinv]=
   obsvf(A,B,C)
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Observability and transfer function

Proposition

The eigenvalues of A_{uo} are not poles of the transfer function $C(zI-A)^{-1}B+D$

Proof:

- Consider a matrix T changing the state coordinates to canonical observability decomposition of (A,C)
- The transfer function is

$$
\begin{aligned}
G(z) &= C(zI-A)^{-1}B+D=\tilde{C}(zI-\tilde{A})^{-1}\tilde{B}+D= \\
& \begin{bmatrix} 0 & C_o \end{bmatrix}\left(zI-\begin{bmatrix} A_{uo} & A_{12} \\ 0 & A_o \end{bmatrix}\right)^{-1}\begin{bmatrix} B_{uo} \\ B_o \end{bmatrix}+D \\
&= \begin{bmatrix} 0 & C_o \end{bmatrix}\begin{bmatrix} (zI-A_{uo})^{-1} & \star \\ 0 & (zI-A_o)^{-1} \end{bmatrix}\begin{bmatrix} B_{no} \\ B_o \end{bmatrix}+D \\
&= C_o(zI-A_o)^{-1}B_o+D
\end{aligned}
$$

- $G(z)$ does not depend on the eigenvalues of A_{uo}

Lack of observability $\rightarrow$ zero/pole cancellations!

Observability and transfer function

- Why are the eigenvalues of A_{uo} not appearing in the transfer function $G(z)$?
- Expressed in canonical decomposition, the system evolution is

$$\left\{\begin{array}{rcl} x_{uo}(k+1) & = & A_{uo}x_{uo}(k)+A_{12}x_o(k)+B_{uo}u(k) \\ x_o(k+1) & = & A_o x_o(k)+B_o u(k) \\ y(k) & = & C_o x_o(k)+Du(k) \end{array}\right.$$

- The evolution of $x_o(k)$ is not affected by the unobservable states $x_{uo}(k)$

$$x_o(k)=A_o^k x_o(0)+\sum_{i=0}^{k-1}A_o^i B_o u(k-1-i)$$

so the output $y(k)=C_o x_o(k)+Du(k)$ does not depend at all on A_{uo} !

Canonical observability decomposition

Proposition

$A_o \in \mathbb{R}^{n_o \times n_o}$ and $C_o \in \mathbb{R}^{p \times n_o}$ are a completely observable pair

Proof:

- Consider the observability matrix

$$\tilde{\Theta} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & C_o \\ 0 & C_oA_o \\ \vdots & \vdots \\ 0 & C_oA_o^{n-1} \end{bmatrix} \text{ and } \tilde{\Theta} = \begin{bmatrix} CT \\ CTT^{-1}AT \\ \vdots \\ CTT^{-1}\ldots A^{n-1}T \end{bmatrix} = \Theta T$$

- Since T is nonsingular, $n - n_o = \dim\ker(\tilde{\Theta}) = \dim\ker(\Theta)$, so

$$\operatorname{rank} \begin{bmatrix} C_o \\ C_oA_o \\ \vdots \\ C_oA_o^{n_o-1} \end{bmatrix} = n_o$$

Reconstructability

- Under observability assumptions, we just saw that it is possible to determine the initial condition x_0 from n input/output measurements

$$x(0) = \Theta^{-1} Y$$

- To close the control loop at time k it is enough to know the current $x(k)$
- If the initial condition $x(0)$ is known, it is possible to calculate $x(k)$ as

$$x(k) = A^k \Theta^{-1} Y + \sum_{i=0}^{k-1} A^i B u(k-1-i)$$

- **Question:** Can we determine the current state $x(k)$ even if the system is not completely observable?

Reconstructability

Definition

A linear system $x(k+1) = Ax(k) + Bu(k)$ is called *reconstructable* in k steps if, for each initial condition x_0, $x(k)$ is uniquely determined by $\{u(j), y(j)\}_{j=0}^{k-1}$

The solutions of the system

$$Y_k \triangleq \begin{bmatrix} y(0) - Du(0) \\ y(1) - CBu(0) - Du(1) \\ \vdots \\ y(k-1) - \sum_{j=1}^{k-2} CA^j Bu(k-2-j) + Du(k-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{\Theta_k} x$$

are given by $x = x_0 + \ker(\Theta_k)$, where x_0 is the "true" (unknown) initial state

Reconstructability

- Let x_0 be the initial (unknown) "true" state, and $x = x_0 + \bar{x}$ be a generic initial state, where $\bar{x} \in \ker(\Theta_k)$. An estimation $\hat{x}(k)$ of the current state $x(k)$ is

$$\hat{x}(k) = A^k x_0 + A^k \bar{x} + \sum_{j=1}^{k-1} A^j B u(k-1-j)$$

- $\hat{x}(k)$ coincides with $x(k)$ if and only if $\bar{x} \in \ker(A^k)$. Because this must hold for any $\bar{x} \in \ker(\Theta_k)$, we can say that

A system is *reconstructable* in k steps if and only if $\ker(\Theta_k) \subseteq \ker(A^k)$

Reconstructability

Definition

A system reconstructable in n steps is called *(completely) reconstructable*

Theorem

A system is reconstructable if and only if all the eigenvalues of the nonobservable part are zero

Proof:

- Let $x = x_0 + \bar{x}$, where x_0 is the "true" (unknown) initial condition, while $\bar{x} \in \ker(\Theta)$. Then, x represents a possible generic initial condition such that $\Theta x = Y$
- Transform the system into canonical observability decomposition

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_{uo} & A_{12} \\ 0 & A_o \end{bmatrix}$$

Reconstructability

- Since $\bar{x}$ has only components along $v_{n_o+1}, \ldots, v_n$, its new coordinates

$$\bar{z} = T^{-1}\bar{x} = \begin{bmatrix} z_{uo} \\ 0 \end{bmatrix}$$

- Since $A = T\tilde{A}T^{-1}$ we get

$$A^k\bar{x} = T \begin{bmatrix} A_{uo}^k & \star \\ 0 & A_o^k \end{bmatrix} \begin{bmatrix} z_{no} \\ 0 \end{bmatrix} = T \begin{bmatrix} A_{uo}^k z_{uo} \\ 0 \end{bmatrix}$$

- Then $x(k) = A^k x_0 + A^k\bar{x} + \sum_{j=1}^{k-1} A^j Bu(k-1-j)$ is uniquely determined for all $\bar{x} \in \ker(\Theta)$ (i.e., for all $z_{uo} \in \mathbb{R}^{n-n_o}$) if and only if A_{uo} is nilpotent

$\square$

Note that although $x(0)$ is not uniquely determined, if the system is reconstructable the state $x(k)$ is uniquely determined

Detectability

Definition

A system is called *detectable* if it is reconstructable asymptotically for $t \to +\infty$

Theorem

A system is detectable if and only if A_{uo} is asymptotically stable

Proof:

- See previous slide: $A^k \bar{x} = T \begin{bmatrix} A_{uo}^k z_{uo} \\ 0 \end{bmatrix}$ converges to zero for all $\bar{x} \in \ker(\Theta)$ if and only if $\lim_{k\to\infty} A_{uo}^k = 0$. In this case

$$x(k) = A^k x_0 + A^k \bar{x} + \sum_{j=1}^{k-1} A^j Bu(k-1-j)$$

tends to be uniquely defined for any $\bar{x} \in \ker(\Theta)$

$\square$

Note: observability implies reconstructability, that implies detectability

Duality

- Given a linear system (A,B,C,D), with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, we call *dual system* the system

$$\begin{cases} \tilde{x}(k+1) & = & A'\tilde{x}(k) + C'\tilde{u}(k) \\ \tilde{y}(k) & = & B'\tilde{x}(k) + D'\tilde{u}(k) \end{cases}$$

  where $\tilde{x} \in \mathbb{R}^n$, $\tilde{u} \in \mathbb{R}^p$ and $\tilde{y} \in \mathbb{R}^m$
- The reachability [observability] matrix of the dual system is equal to the transpose of the observability [reachability] matrix of the original system

$$\begin{array}{rcl} \tilde{R} & = & \begin{bmatrix} C' & A'C' & \dots & (A')^{n-1}C' \end{bmatrix} = \Theta' \\ \tilde{\Theta} & = & \begin{bmatrix} B' \\ B'A' \\ \vdots \\ B'(A')^{n-1} \end{bmatrix} = R' \end{array}$$

- The system (A,B,C,D) is reachable [observable] if and only if its dual system (A',C',B',D') is observable [reachable]

English-Italian Vocabulary

	
observability	*osservabilità*
reconstructability	*ricostruibilità*
detectability	*rivelabilità*

Translation is obvious otherwise.