Automatic Control 1

Linear State Feedback Control

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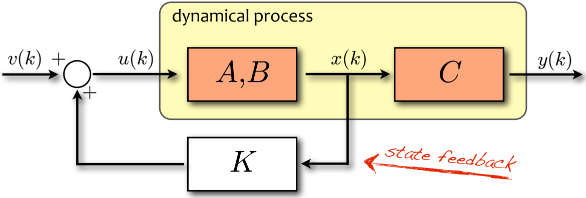
Stabilization by state feedback

- **Main idea**: design a device that makes the process (A,B,C) asymptotically stable by manipulating the input u to the process



- If measurements of the state vector are available, we can set

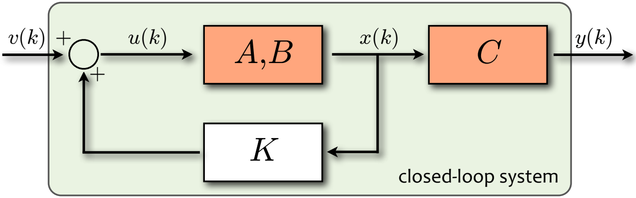
$$u(k) = k_1x_1(k) + k_2x_2(k) + \ldots + k_nx_n(k) + v(k)$$

- $v(k)$ is an exogenous signal exciting the closed-loop system

Problem

Find a *feedback gain* $K = [k_1 \; k_2 \; \ldots \; k_n]$ that makes the closed-loop system asymptotically stable

Stabilization by state feedback



- Let $u(k) = Kx(k) + v(k)$. The overall system is

$$
\begin{aligned}
x(k+1) &= (A+BK)x(k) + Bv(k) \\
y(k) &= (C+DK)x(k) + Dv(k)
\end{aligned}
$$

Theorem

(A,B) reachable $\Rightarrow$ the eigenvalues of $(A+BK)$ can be decided **arbitrarily**

Eigenvalue assignment problem

Fact

(A,B) reachable $\Leftrightarrow$ (A,B) is algebraically equivalent to a pair $(\tilde{A},\tilde{B})$ in *controllable canonical form*

$$\tilde{A} = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The transformation matrix T such that $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$ is

$$T = [B\ AB\ \dots A^{n-1}B] \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where $a_1, a_2, \dots, a_{n-1}$ are the coefficients of the characteristic polynomial

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = \det(\lambda I - A)$$

- Let (A,B) reachable and assume $m=1$ (single input)
- Characteristic polynomials:

$$\begin{array}{rcl} p_A(\lambda) & = & \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \text{ (open-loop eigenvalues)} \\ p_d(\lambda) & = & \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0 \text{ (desired closed-loop eigenvalues)} \end{array}$$

- Suppose (A,B) in controllable canonical form

$$A = \left[\begin{array}{cccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array}\right]$$

- As $K = \left[k_1 \ \ldots \ k_n\right]$, we have

$$A+BK = \left[\begin{array}{cccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -(a_0-k_1) & -(a_1-k_2) & \ldots & -(a_{n-1}-k_n) \end{array}\right]$$

- The characteristic polynomial of $A+BK$ is therefore

$$\lambda^n + (a_{n-1} - k_n)\lambda^{n-1} + \ldots + (a_1 - k_2)\lambda + (a_0 - k_1)$$

- To match $p_d(\lambda)$ we impose

$$a_0 - k_1 = d_0 \,,\; a_1 - k_2 = d_1 \,,\; \ldots \,,\; a_{n-1} - k_n = d_{n-1}$$

Procedure

If (A, B) is in controllable canonical form, the feedback gain

$$K = \left[\; a_0 - d_0 \;\; a_1 - d_1 \;\; \ldots \;\; a_{n-1} - d_{n-1} \;\right]$$

makes $p_d(\lambda)$ the characteristic polynomial of $(A + BK)$

- If (A, B) is not in controllable canonical form we must set

$$\begin{aligned} \tilde{K} &= \begin{bmatrix} a_0 - d_0 & a_1 - d_1 & \dots & a_{n-1} - d_{n-1} \end{bmatrix} \\ K &= \tilde{K} T^{-1} \end{aligned}$$

← don't invert T, solve instead $T'K' = \tilde{K}'$ w.r.t. K' !

where

$$T = R \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- Explanation: a matrix M and $T^{-1}MT$ have the same eigenvalues

$$\begin{aligned} \det(\lambda I - T^{-1}MT) &= \det(T^{-1}T\lambda - T^{-1}MT) = \det(T^{-1}) \det(\lambda I - M) \det(T) \\ &= \det(\lambda I - M) \end{aligned}$$

- Since $(\tilde{A} + \tilde{B}\tilde{K}) = T^{-1}AT + T^{-1}BKT = T^{-1}(A + BK)T$, it follows that $(\tilde{A} + \tilde{B}\tilde{K})$ and $(A + BK)$ have the same eigenvalues

Ackermann's formula

- Let (A,B) reachable and assume $m = 1$ (single input)
- Characteristic polynomials:

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \quad \text{(open-loop eigenvalues)}$$
$$p_d(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0 \quad \text{(desired closed-loop eigenvalues)}$$

- Let $p_d(A) = A^n + d_{n-1}A^{n-1} + \ldots + d_1A + d_0I$ ← (This is $n \times n$ matrix !)

Ackermann's formula

$$K = -[0\ 0\ \ldots\ 0\ 1][B\ AB\ \ldots\ A^{n-1}B]^{-1}p_d(A)$$

MATLAB

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» K=-acker(A,B,P);
» K=-place(A,B,P);
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where $P = [\lambda_1\lambda_2\ldots\lambda_n]$ are the desired closed-loop poles

Pole-placement example

- Consider the dynamical system

$$x(k+1) = \begin{bmatrix} 0 & -\frac{1}{4} \\ -3 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} u(k) = Ax(k) + Bu(k)$$

- $\text{rank}\, R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{8} \\ -\frac{1}{2} & -\frac{7}{2} \end{bmatrix} = 2$: the system is reachable
- We want to assign the closed-loop eigenvalues $\frac{1}{2}$ and $\frac{1}{4}$:

$$p_d(\lambda) = (\lambda - \frac{1}{2})(\lambda - \frac{1}{4}) = \lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} = \det(\lambda I - A - BK)$$

$$\begin{aligned}
\lambda I - A - BK &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{4} \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{4} \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ -\frac{k_1}{2} & -\frac{k_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \lambda - k_1 & \frac{1}{4} - k_2 \\ 3 + \frac{k_1}{2} & \lambda - 1 + \frac{k_2}{2} \end{bmatrix}
\end{aligned}$$

- Therefore:

$$\det(\lambda I - A - BK) = \begin{vmatrix} \lambda - k_1 & \frac{1}{4} - k_2 \\ 3 + \frac{k_1}{2} & \lambda - 1 + \frac{k_2}{2} \end{vmatrix} = (\lambda - k_1)(\lambda - 1 + \frac{k_2}{2}) - (3 + \frac{k_1}{2})(\frac{1}{4} - k_2)$$

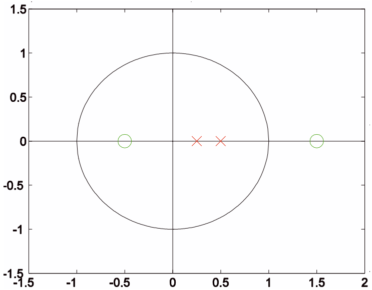
$$= \lambda^2 + (\frac{k_2}{2} - 1 - k_1)\lambda + (\frac{7k_1}{8} - \frac{3}{4} + 3k_2)$$

- Impose that the coefficients of the polynomials are the same

$$\begin{cases} \frac{k_2}{2} - 1 - k_1 = -\frac{3}{4} \\ \frac{7k_1}{8} - \frac{3}{4} + 3k_2 = \frac{1}{8} \end{cases} \Rightarrow \begin{cases} k_1 = \frac{k_2}{2} - \frac{1}{4} \\ \frac{7k_2}{16} - \frac{7}{32} - \frac{3}{4} + 3k_2 = \frac{1}{8} \end{cases}$$

- Finally, we get

$$\begin{cases} k_2 = \frac{7}{22} \\ k_1 = -\frac{1}{11} \end{cases} \Rightarrow K = \frac{1}{11}\begin{bmatrix} -1 & \frac{7}{2} \end{bmatrix}$$



o= open-loop eigenvalues
x= closed-loop eigenvalues

MATLAB

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» A=[ 0 -1/4 ; -3 1 ];
» B=[ 1 ; -1/2 ];
» K=-place(A,B,[ 1/2 1/4 ])

K = -0.0909  0.3182
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Eigenvalue assignment for unreachable systems

Theorem

If $\text{rank}(R) = n_c < n$ then $n - n_c$ eigenvalues cannot be changed by state feedback

Proof:

- Let T be the change of coordinates transforming (A, B) in canonical reachability decomposition
- Let $K \in \mathbb{R}^{m \times n}$ be a feedback gain, and let $\tilde{K} = KT$ the corresponding gain in transformed coordinates
- As observed earlier, $A + BK$ and $\tilde{A} + \tilde{B}\tilde{K}$ have the same eigenvalues
- Let $\tilde{K} = \begin{bmatrix} K_{uc} & K_c \end{bmatrix}$, $K_c \in \mathbb{R}^{m \times n_c}$. Then

$$\tilde{A} + \tilde{B}\tilde{K} = \begin{bmatrix} A_{uc} & 0 \\ A_{21} + B_c K_{uc} & A_c + B_c K_c \end{bmatrix}$$

The eigenvalues of the unreachable part A_{uc} cannot be changed !