Automatic Control 1

Reachability Analysis

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Academic year 2010-2011

Reachability

- Consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

with $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$ and initial condition $x(0) = x_0 \in \mathbb{R}^n$

- The solution is $x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^j Bu(k-1-j)$

Definition

The system $x(k+1) = Ax(k) + Bu(k)$ is *(completely) reachable* if $\forall x_1, x_2 \in \mathbb{R}^n$ there exist $k \in \mathbb{N}$ and $u(0), u(1), \ldots, u(k-1) \in \mathbb{R}^m$ such that

$$x_2 = A^k x_1 + \sum_{j=0}^{k-1} A^j Bu(k-1-j)$$

- In simple words: a system is completely reachable if from any state x_1 we can reach any state x_2 at some time k, by applying a suitable input sequence

Linear algebra recalls: Change of coordinates

- Let $\{v_1, \ldots, v_n\}$ be a *basis* of $\mathbb{R}^n$ (= n linearly independent vectors)
- The *canonical basis* of $\mathbb{R}^n$ is $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$
- A vector $w \in \mathbb{R}^n$ can be expressed as a *linear combination* of the basis vectors, whose coefficients are the *coordinates* in the corresponding basis

$$w = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} z_i v_i$$

- The relation between the coordinates $x = [x_1 \ \ldots \ x_n]'$ in the canonical basis and the coordinates $z = [z_1 \ \ldots \ z_n]'$ in the new basis is

$$x = Tz$$

where $T = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$ (= *coordinate transformation matrix*)

Algebraically equivalent systems

- Consider the linear system

$$\left\{\begin{array}{rcl} x(k+1) & = & Ax(k)+Bu(k) \\ y(k) & = & Cx(k)+Du(k) \end{array}\right.$$

$$x(0)=x_0$$

- Let T be invertible and define the change of coordinates $x=Tz$, $z=T^{-1}x$

$$\left\{\begin{array}{rcl} z(k+1) & = & T^{-1}x(k+1)=T^{-1}\left(Ax(k)+Bu(k)\right)=T^{-1}ATz(k)+T^{-1}Bu(k) \\ y(k) & = & CTz(k)+Du(k) \end{array}\right.$$

$$z_0=T^{-1}x_0$$

and hence

$$\left\{\begin{array}{rcl} z(k+1) & = & \tilde{A}z(k)+\tilde{B}u(k) \\ y(k) & = & \tilde{C}z(k)+\tilde{D}u(k) \end{array}\right.$$

$$z(0)=T^{-1}x_0$$

- The dynamical systems (A,B,C,D) and $(\bar{A},\bar{B},\bar{C},\bar{D})$ are called *algebraically equivalent*

Linear algebra recalls: Change of coordinates

- Let $A \in \mathbb{R}^{n\times n}$, $T = \left[\ v_1\ \ldots\ v_n\ \right]$ invertible. In the new coordinates $z = T^{-1}x$ the transformed matrix is

$$\tilde{A} = T^{-1}AT = T^{-1}A\left[\ v_1\ \ldots\ v_n\ \right] = T^{-1}\left[\ Av_1\ \ldots\ Av_n\ \right]$$

- Therefore, the columns of $\tilde{A}$ are the coordinates of the transformed vectors $Av_1, \ldots, Av_n$ in the new basis $\{v_1, \ldots, v_n\}$
- Special case: if $v_1, \ldots, v_n$ are the eigenvectors of A, $Av_i = \lambda_i v_i$, then $\tilde{A} = \mathrm{Diag}\left\{\lambda_1, \ldots, \lambda_n\right\}$

$$\begin{aligned}\tilde{A} &= T^{-1}AT = T^{-1}\left[\lambda_1 v_1 | \lambda_2 v_2 | \ldots \lambda_n v_n\right] \\ &= T^{-1}\left[v_1|v_2|\ldots v_n\right]\begin{bmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix} = TT^{-1}\Lambda = \Lambda\end{aligned}$$

Linear algebra recalls: Change of coordinates

Proposition

Matrices $A \in \mathbb{R}^{n\times n}$ and $\tilde{A} = T^{-1}AT$ have the same characteristic polynomials

Proof:
Recall Binet theorem

$$\det(AB) = \det A \cdot \det B$$

for any pair of square matrices A, B, and recall that

$$\det(A^{-1}) = \frac{1}{\det A}$$

for any invertible matrix A. Then

$$\begin{aligned}\det(\lambda I - \tilde{A}) &= \det(\lambda T^{-1}IT - T^{-1}AT) = \det\left(T^{-1}(\lambda I - A)T\right)\\ &= \det(T^{-1})\det(\lambda I - A)\det(T) = \det(\lambda I - A)\end{aligned}$$

$\square$

Note: we already saw that algebraically equivalent systems have the same transfer function, hence the same poles, hence the same characteristic polynomial of the state-update matrix

Linear algebra recalls

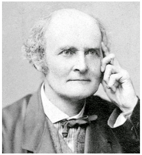
- Let $A \in \mathbb{R}^{m\times n}$
  - *image* of A: $\mathrm{Im}(A) = \{w \in \mathbb{R}^m : w = Av,\ v \in \mathbb{R}^n\}$
  - *rank* of A: rank(A)=dimension of Im(A)
  - *kernel* of A: $\ker(A) = \{v \in \mathbb{R}^n : 0 = Av\}$
- Let $A \in \mathbb{R}^{n\times n}$
  - *spectrum* of A: $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$
  - *A-invariant subspace*: $V \subseteq \mathbb{R}^n$ is such that $AV \subseteq V$, that is $Av \in V$, $\forall v \in V$

Sir William Rowan Hamilton (1805-1865)

Cayley-Hamilton Theorem

Let $\det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of $A \in \mathbb{R}^{n\times n}$. Then

$$A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I = 0$$



Arthur Cayley (1821-1895)

Reachability

- Let's focus on the problem of determining a sequence of n inputs transferring the state vector from x_1 to x_2 after n steps

$$\underbrace{x_2 - A^n x_1}_{X} = \underbrace{\left[\, B\; AB\; \dots\; A^{n-1}B \,\right]}_{R} \underbrace{\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}}_{U}$$

- This is equivalent to solve with respect to U the linear system of equations

$$RU = X$$

- The matrix $R \in \mathbb{R}^{n \times nm}$ is called the *reachability matrix* of the system
- A solution U exists if and only if $X \in \mathrm{Im}(R)$
(Rouché-Capelli theorem: a solution exists $\Leftrightarrow$ $\mathrm{rank}([R\; X]) = \mathrm{rank}(R)$)

Reachability

Theorem

The system (A,B) is completely reachable $\Leftrightarrow$ $\mathrm{rank}(R) = n$

Proof:

($\Rightarrow$) Assume (A,B) reachable, choose $x_1 = 0$ and $x_2 = x$. Then $\exists k \geq 0$ such that

$$x = \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

If $k \leq n$, then clearly $x \in \mathrm{Im}(R)$. If $k > n$, by Cayley-Hamilton theorem we have again $x \in \mathrm{Im}(R)$. Since x is arbitrary, $\mathrm{Im}(R) = \mathbb{R}^n$, so $\mathrm{rank}(R) = n$.

($\Leftarrow$) If $\mathrm{rank}(R) = n$, then $\mathrm{Im}(R) = \mathbb{R}^n$. Let $X = x_2 - A^n x_1$ and $U = \left[\ u(n-1)' \ \ldots\ u(1)'\ u(0)'\ \right]'$. The system $X = RU$ can be solved with respect to U, $\forall X$, so any state x_1 can be transferred to x_2 in $k = n$ steps. Therefore, the system (A,B) is completely reachable.

Comments on the reachability property

- The reachability property of a system only depends on A and B
- We therefore say that a pair (A, B) is *reachable* if

$$\operatorname{rank}\left(\left[\, B\; AB\; \ldots\; A^{n-1}B \,\right]\right) = n$$

MATLAB

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» R=ctrb(A,B)
» rank(R)
```

- $\operatorname{Im}(R)$ is the set of states that are reachable from the origin, that is the set of states $x \in \mathbb{R}^n$ for which there exists $k \in \mathbb{N}$ and $u(0)$, $u(1)$, …, $u(k-1) \in \mathbb{R}^m$ such that

$$x = \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

- If $\operatorname{Im}(R) = \mathbb{R}^n$, a system is completely reachable $\Leftrightarrow$ all the states are reachable from the origin in n steps (proof: set $x = x_2 - A^n x_1$)

Minimum-energy control

- Let (A,B) reachable and consider steering the state from $x(0)=x_1$ into $x(k)=x_2$, $k>n$

$$\underbrace{x_2 - A^k x_1}_{X} = \underbrace{\left[\, B \; AB \; \ldots \; A^{k-1}B \,\right]}_{R_k} \underbrace{\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}}_{U}$$

  ($R_k \in \mathbb{R}^{n\times km}$ is the reachability matrix for k steps)
- Since $\operatorname{rank}(R_k) = \operatorname{rank}(R) = n$, $\forall k>n$ (Cayley-Hamilton), we get $\operatorname{rank} R_k = \operatorname{rank}[R_k \; X] = n$
- Hence the system $X = R_k U$ admits solutions U

Problem

Determine the input sequence $\{u(j)\}_{j=0}^{k-1}$ that brings the state from $x(0)=x_1$ to $x(k)=x_2$ with minimum energy $\frac{1}{2}\sum_{j=0}^{k-1}\|u(j)\|^2 = \frac{1}{2}U'U$

Minimum-energy control

- The problem is equivalent to finding the solution U of the system of equations

$$X = R_k U$$

  with minimum norm $\|U\|$
- We must solve the optimization problem

$$U^* = \arg\min \frac{1}{2}\|U\|^2 \quad \text{subject to} \quad X = R_k U$$

- Let's apply the method of Lagrange multipliers:

$$\mathcal{L}(U,\lambda) = \frac{1}{2}\|U\|^2 + \lambda'(X - R_k U) \quad \text{Lagrangean function}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial U} &= U - R_k'\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= X - R_k U = 0 \end{aligned} \quad \Rightarrow \quad U^* = \underbrace{R_k'(R_k R_k')^{-1}}_{R_k^{\#} \,=\, \text{pseudoinverse matrix}} \cdot X$$

MATLAB
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U=pinv(Rk)*X
```

- Note that $R_k R_k'$ is invertible because $\text{rank}(R_k) = \text{rank}(R) = n$, $\forall k \geq n$

Canonical reachability decomposition

- **Goal**: Make a change of coordinates to separate reachable and unreachable states
- Let $\text{rank}(R) = n_c < n$ and consider the change of coordinates

$$T = \left[\begin{array}{cccccc} w_{n_c+1} & \dots & w_n & v_1 & \dots & v_{n_c} \end{array}\right]$$

  where $\left\{v_1, \dots, v_{n_c}\right\}$ is a basis of $\text{Im}(R)$, and $\left\{w_{n_c+1}, \dots, w_n\right\}$ is a completion to obtain a basis of $\mathbb{R}^n$
- As $\text{Im}(R)$ is A-invariant ($Ax \in \text{Im}(R)\ \forall x \in \text{Im}(R)$ follows from Cayley-Hamilton theorem), Av_i has no components along the basis vectors $w_{n_c+1}, \dots, w_n$. Since $T^{-1}Av_i$ are the new coordinates of Av_i, the first $n - n_c$ components of $T^{-1}Av_i$ are zero
- The columns of B also have zero components along $w_{n_c+1}, \dots, w_n$, because $\text{Im}(B) \subseteq \text{Im}(R)$
- In the new coordinates, the system has matrices $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$ e $\tilde{C} = CT$ in the *canonical reachability form* (a.k.a. *controllability staircase form*)

$$\tilde{A} = \left[\begin{array}{cc} A_{uc} & 0 \\ A_{21} & A_c \end{array}\right] \quad \tilde{B} = \left[\begin{array}{c} 0 \\ B_c \end{array}\right] \quad \tilde{C} = \left[\begin{array}{cc} C_{uc} & C_c \end{array}\right]$$

MATLAB
```
[At,Bt,Ct,Tinv]=
   ctrbf(A,B,C)
```

Reachability and transfer function

Proposition

The eigenvalues of A_{uc} are not poles of the transfer function $C(zI-A)^{-1}B+D$

Proof:

- Consider a matrix T changing the state coordinates to canonical reachability decomposition of (A,B)
- The transfer function is

$$
\begin{aligned}
G(z) &= C(zI-A)^{-1}B+D = \tilde{C}(zI-\tilde{A})^{-1}\tilde{B}+D \\
&= \begin{bmatrix} C_{uc} & C_c \end{bmatrix} \left(zI - \begin{bmatrix} A_{uc} & 0 \\ A_{21} & A_c \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ B_c \end{bmatrix} + D \\
&= \begin{bmatrix} C_{uc} & C_c \end{bmatrix} \begin{bmatrix} (zI-A_{uc})^{-1} & 0 \\ \star & (zI-A_c)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ B_c \end{bmatrix} + D \\
&= C_c(zI-A_c)^{-1}B_c + D
\end{aligned}
$$

- $G(z)$ does not depend on the eigenvalues of A_{uc}

Lack of reachability $\rightarrow$ zero/pole cancellations !

Reachability and transfer function

- Why are the eigenvalues of A_{uc} not appearing in the transfer function $G(z)$?
- Remember: $G(z)$ explains the forced response, i.e., the response for $x(0) = 0$
- Expressed in canonical decomposition, the system evolution is

$$\left\{\begin{array}{rcl} x_{uc}(k+1) & = & A_{uc}x_{uc}(k) \\ x_c(k+1) & = & A_c x_c(k) + B_c u(k) + A_{21}x_{uc}(k) \\ y(k) & = & C_{uc}x_{uc}(k) + C_c x_c(k) + Du(k) \end{array}\right.$$

- For $x_{uc}(0) = 0$, $x_c(0) = 0$, we get $x_{uc}(k) \equiv 0$ and

$$\left\{\begin{array}{rcl} x_c(k+1) & = & A_c x_c(k) + B_c u(k) \\ y(k) & = & C_c x_c(k) + Du(k) \\ x_c(0) & = & 0 \end{array}\right.$$

  so the forced response does not depend at all on A_{uc} !
- The input $u(k)$ only affects the output $y(k)$ through the reachable subsystem (A_c, B_c, C_c, D), not through the unreachable part A_{uc}

Canonical reachability decomposition

Proposition

$A_c \in \mathbb{R}^{n_c \times n_c}$ and $B_c \in \mathbb{R}^{n_c \times m}$ are a *completely reachable* pair

Proof:

- Consider the reachability matrix

$$\tilde{R} = \left[\ \tilde{B}\ \tilde{A}\tilde{B}\ \dots\ \tilde{A}^{n-1}\tilde{B}\ \right] = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B_c & A_cB_c & \dots & A_c^{n-1}B_c \end{bmatrix}$$

and

$$\tilde{R} = \left[\ T^{-1}B \quad T^{-1}ATT^{-1}B \quad \dots \quad T^{-1}A^{n-1}TT^{-1}B\ \right] = T^{-1}R$$

- Since T is nonsingular, $\text{rank}(\tilde{R}) = \text{rank}(R) = n_c$, so

$$\text{rank}\left[\ B_c \quad A_cB_c \quad \dots \quad A_c^{n_c-1}B_c\ \right] = n_c$$

that is (A_c, B_c) is completely reachable $\square$

Controllability

- If the system is completely reachable, we have seen that we can bring the state vector from any value $x(0) = x_1$ to any other value $x(n) = x_2$
- Let's focus on the subproblem of determining a finite sequence of inputs that brings the state to the final value $x(n) = 0$

Definition

A system $x(k+1) = Ax(k) + Bu(k)$ is *controllable* to the origin in k steps if $\forall x_0 \in \mathbb{R}^n$ there exists a sequence $u(0), u(1), \ldots, u(k-1) \in \mathbb{R}^m$ such that $0 = A^k x_0 + \sum_{j=0}^{k-1} A^j Bu(k-1-j)$

- Controllability is a weaker condition than reachability

Controllability

- The linear system of equations

$$-A^k x_0 = \underbrace{\begin{bmatrix} B\ AB\ \ldots\ A^{k-1}B \end{bmatrix}}_{R_k} \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

admits a solution if and only if $A^k x_0 \in \text{Im}(R_k)$, $\forall x_0 \in \mathbb{R}^n$

Theorem

The system is controllable to the origin in k steps if and only if

$$\text{Im}(A^k) \subseteq \text{Im}(R_k)$$

- If a system is controllable in n steps, it is also controllable in k steps for each $k > n$ (just set $u(n) = u(n+1) = \ \ldots\ = u(k-1) = 0$)
- For the same reason, if a system is controllable in k steps with $k < n$, it is also controllable in n steps (just set $u(k) = u(k+1) = \ \ldots\ = u(n-1) = 0$)

Controllability

Definition

A system is said *completely controllable* if it is controllable in n steps

Theorem

A system $x(k+1) = Ax(k) + Bu(k)$ is completely controllable if and only if all the eigenvalues of its unreachable part A_{uc} are null

Proof:

- Given an initial state $x_0 \in \mathbb{R}^n$, we must solve with respect U the linear system of equations

$$-A^n x_0 = \left[\, B\; AB\; \dots\; A^{n-1}B \,\right] U = RU, \quad U = \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Controllability

- Let $(\tilde{A},\tilde{B})$ be a canonical reachability decomposition of (A,B). Then

$$-A^n x_0 = -T\tilde{A}^n z_0 = \begin{bmatrix} T\tilde{B} & T\tilde{A}\tilde{B} & \dots & T\tilde{A}^{n-1}\tilde{B} \end{bmatrix} U = T\tilde{R}U$$

where we set $z_0 = T^{-1}x_0 = \begin{bmatrix} z_{uc} \\ z_c \end{bmatrix}$ and $T = [w_{n_r+1} \ \dots \ w_n \ v_1 \ \dots \ v_{n_r}]$

- since T is invertible, the system $T\tilde{R}U = -T\tilde{A}^n z_0$ is equivalent to $\tilde{R}U = -\tilde{A}^n z_0$, that is

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ B_c & A_cB_c & \dots & A_c^{n-1}B_c \end{bmatrix} U = -\begin{bmatrix} A_{uc}^n & 0 \\ \star & A_c^n \end{bmatrix} \begin{bmatrix} z_{uc} \\ z_c \end{bmatrix}$$

- as the pair (A_c,B_c) is completely reachable, the system above has a solution if and only is $A_{uc}^n z_{uc} = 0$ for any initial condition z_{uc}, that is $A_{uc}^n = 0$
- if A_{uc} is a *nilpotent* matrix (= all its $n-n_c$ eigenvalues are zero), by Cayley-Hamilton $A_{uc}^{n-n_c} = 0$ and therefore

$$A_{uc}^n = A_{uc}^{n_c} A_{uc}^{n-n_c} = A_{uc}^{n_c} \cdot 0 = 0$$

□

Stabilizability

Definition

A linear system $x(k+1) = Ax(k) + Bu(k)$ is called *stabilizable* if can be driven asymptotically to the origin

- Stabilizability is a weaker condition than controllability

Theorem

A linear system $x(k+1) = Ax(k) + Bu(k)$ is *stabilizable* if and only if all the eigenvalues of its unreachable part have moduli < 1

Proof:

- Take any $z_0 = \begin{bmatrix} z_{uc}(0) \\ z_c(0) \end{bmatrix} \in \mathbb{R}^n$ and an input sequence $\{u(k)\}_{k=0}^{\infty}$ that makes the reachable component $z_c(k) \to 0$ for $k \to \infty$
- If $A_{uc}^k \to 0$ for $k \to \infty$, then $z_{uc}(k) = A_{uc}^k z_{uc}(0)$ also converges to zero for $k \to \infty$

$\square$

Reachability analysis of continuous-time systems

- Similar definitions of reachability, controllability, and stabilizability can be given for continuous-time systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- No distinction between controllability and reachability in continuous-time (because no finite-time convergence of modal response exists)
- Reachability matrix and canonical reachability decomposition are identical to discrete-time
- $\operatorname{rank} R = n$ is also a necessary and sufficient condition for reachability
- A_{uc} asymptotically stable (all eigenvalues with negative real part) is also a necessary and sufficient condition for stabilizability

English-Italian Vocabulary

	
reachability	*raggiungibilità*
controllability	*controllabilità*
stabilizability	*stabilizzabilità*
controllability staircase form	*decomposizione canonica di raggiungibilità*

Translation is obvious otherwise.