Automatic Control 1

Z-transform

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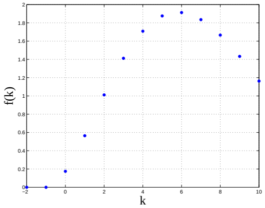
Z-transform

Consider a function $f(k)$, $f : \mathbb{Z} \to \mathbb{R}$, $f(k) = 0$ for all $k < 0$

Definition

The unilateral *Z-transform* of $f(k)$ is the function of the complex variable $z \in \mathbb{C}$ defined by

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$



Witold Hurewicz
(1904-1956)

Once $F(z)$ is computed using the series, it's extended to all $z \in \mathbb{C}$ for which $F(z)$ makes sense

Z-transforms convert difference equations into algebraic equations. It can be considered as a discrete equivalent of the Laplace transform.

Examples of Z-transforms

- *Discrete impulse*

$$f(k)=\delta(k)\triangleq\begin{cases}0 & \text{if } k\neq 0\\ 1 & \text{if } k=0\end{cases}\quad\Rightarrow\quad \mathscr{Z}[\delta]=F(z)=1$$

- *Discrete step*

$$f(k)=1\!\!1(k)\triangleq\begin{cases}0 & \text{if } k<0\\ 1 & \text{if } k\geq 0\end{cases}\quad\Rightarrow\quad \mathscr{Z}[1\!\!1]=F(z)=\frac{z}{z-1}$$

- *Geometric sequence*

$$f(k)=a^k\,1\!\!1(k)\quad\Rightarrow\quad \mathscr{Z}[f]=F(z)=\frac{z}{z-a}$$

Properties of Z-transforms

- *Linearity*

$$\mathscr{Z}[k_1 f_1(k) + k_2 f_2(k)] = k_1 \mathscr{Z}[f_1(k)] + k_2 \mathscr{Z}[f_2(k)]$$

Example: $f(k) = 3\delta(k) - \frac{5}{2^k}\,\mathbb{1}(t) \Rightarrow \mathscr{Z}[f] = 3 - \frac{5z}{z-\frac{1}{2}}$

- *Forward shift*[1]

$$\mathscr{Z}[f(k+1)\,\mathbb{1}(k)] = z\mathscr{Z}[f] - zf(0)$$

Example: $f(k) = a^{k+1}\,\mathbb{1}(k) \Rightarrow \mathscr{Z}[f] = z\frac{z}{z-a} - z = \frac{az}{z-a}$

[1] z is also called *forward shift operator*

Properties of Z-transforms

- *Backward shift* or *unit delay* [2]

$$\mathscr{Z}[f(k-1)\,1\!\!1(k)] = z^{-1}\mathscr{Z}[f]$$

Example: $f(k) = 1\!\!1(k-1) \Rightarrow \mathscr{Z}[f] = \frac{z}{z(z-1)}$

- *Multiplication by k*

$$\mathscr{Z}[kf(k)] = -z\frac{d}{dz}\mathscr{Z}[f]$$

Example: $f(k) = k\,1\!\!1(k) \Rightarrow \mathscr{Z}[f] = \frac{z}{(z-1)^2}$

[2] z^{-1} is also called *backward shift operator*

Initial and final value theorems

Initial value theorem

$$f(0) = \lim_{z\to\infty} F(z)$$

Example: $f(k) = 1\!\!1(k) - k\,1\!\!1(k) \Rightarrow F(z) = \frac{z}{z-1} - \frac{z}{(z-1)^2}$
$f(0) = \lim_{z\to\infty} F(z) = 1$

Final value theorem

$$\lim_{k\to+\infty} f(k) = \lim_{z\to1}(z-1)F(z)$$

Example: $f(k) = 1\!\!1(k) + (-0.7)^k\,1\!\!1(t) \Rightarrow F(z) = \frac{z}{z-1} + \frac{z}{z+0.7}$
$f(+\infty) = \lim_{z\to1}(z-1)F(z) = 1$

Discrete-time transfer function

- Let's apply the Z-transform to discrete-time linear systems

$$\begin{cases} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k)+Du(k) \end{cases}$$

$$x(0)=x_0$$

- Define $X(z)=\mathscr{Z}[x(k)]$, $U(z)=\mathscr{Z}[u(k)]$, $Y(z)=\mathscr{Z}[y(k)]$

- Apply linearity and forward shift rules

$$\begin{aligned} zX(z)-zx_0 &= AX(z)+BU(z) \\ Y(z) &= CX(z)+DU(z) \end{aligned}$$

Discrete-time transfer function

$$
\begin{aligned}
X(z) &= z(zI-A)^{-1}x_0 + (zI-A)^{-1}BU(z) \\
Y(z) &= \underbrace{zC(zI-A)^{-1}x_0}_{\text{Z-transform of natural response}} + \underbrace{(C(zI-A)^{-1}B+D)U(z)}_{\text{Z-transform of forced response}}
\end{aligned}
$$

Definition:

The transfer function of a discrete-time linear system (A,B,C,D) is the ratio

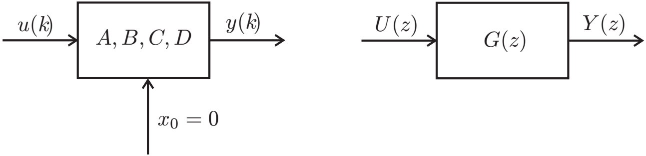
$$G(z) = C(zI-A)^{-1}B + D$$

between the Z-transform $Y(z)$ of the output and the Z-transform $U(z)$ of the input signals *for the initial state* $x_0 = 0$

MATLAB

```
»sys=ss(A,B,C,D,Ts);
»G=tf(sys)
```

Discrete-time transfer function



Example: The linear system

$$\begin{cases} x(k+1) = \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(k) \end{cases}$$

with sampling time $T_s = 0.1$ s has the transfer function

$$G(z) = \frac{-z + 1.5}{z^2 - 0.25}$$

Note: Even for discrete-time systems, the transfer function does not depend on the input $u(k)$. It's only a property of the linear system

MATLAB

```
»sys=ss([0.5 1;
  0 -0.5],[0;1],[1 -1],0,0.1);
»G=tf(sys)

Transfer function:
-z + 1.5
----------
s^2 - 0.25
```

Difference equations

- Consider the n^{th}-order difference equation forced by u

$$a_n y(k-n) + a_{n-1} y(k-n+1) + \cdots + a_1 y(k-1) + y(k)$$
$$= b_n u(k-n) + \cdots + b_1 u(k-1)$$

- For zero initial conditions we get the transfer function

$$\begin{aligned} G(z) &= \frac{b_n z^{-n} + b_{n-1} z^{-n+1} + \cdots + b_1 z^{-1}}{a_n z^{-n} + a_{n-1} z^{-n+1} + \cdots + a_1 z^{-1} + 1} \\ &= \frac{b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n} \end{aligned}$$

Difference equations

- **Example:** $3y(k-2)+2y(k-1)+y(k)=2u(k-1)$

$$G(z)=\frac{2z^{-1}}{3z^{-2}+2z^{-1}+1}=\frac{2z}{z^2+2z+3}$$

- **Note:** The same transfer function $G(z)$ is obtained from the equivalent matrix form

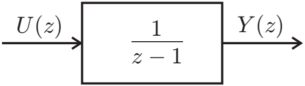
$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 2 \end{bmatrix} x(k) \end{cases}$$

$$\Rightarrow G(z)=\begin{bmatrix} 0 & 2 \end{bmatrix}\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Some common transfer functions

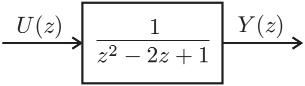
- *Integrator*

$$\begin{cases} x(k+1) = x(k) + u(k) \\ y(k) = x(k) \end{cases}$$



- *Double integrator*

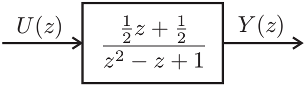
$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) \\ x_2(k+1) = x_2(k) + u(k) \\ y(k) = x_1(k) \end{cases}$$

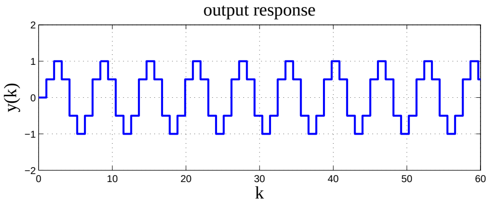


Some common transfer functions

- *Oscillator*

$$\begin{cases} x_1(k+1) &= x_1(k) - x_2(k) + u(k) \\ x_2(k+1) &= x_1(k) \\ y(k) &= \frac{1}{2}x_1(k) + \frac{1}{2}x_2(k) \end{cases}$$



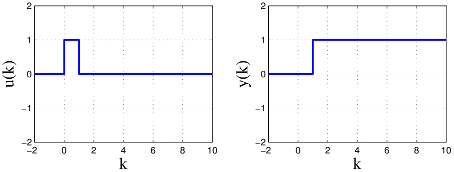


Impulse response

- Consider the impulsive input $u(k) = \delta(k)$, $U(z) = 1$. The corresponding output $y(k)$ is called *impulse response*
- The Z-transform of $y(k)$ is $Y(z) = G(z) \cdot 1 = G(z)$
- Therefore the impulse response coincides with the *inverse Z-transform* $g(k)$ of the transfer function $G(z)$

Example (integrator:)

$$\begin{aligned} u(k) &= \delta(k) \\ y(k) &= \mathscr{Z}^{-1}\left[\frac{1}{z-1}\right] = 1\!\!1(k-1) \end{aligned}$$



Poles, eigenvalues, modes

- Linear discrete-time system

$$\begin{cases} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k)+Du(k) \end{cases} \qquad G(z)=C(zI-A)^{-1}B+D \triangleq \frac{N_G(z)}{D_G(z)}$$

$$x(0)=0$$

- Use the adjugate matrix to represent the inverse of $zI-A$

$$C(zI-A)^{-1}B+D=\frac{C\,\mathrm{Adj}(zI-A)B}{\det(zI-A)}+D$$

- The denominator $D_G(z)=\det(zI-A)$!

The poles of $G(z)$ coincide with the eigenvalues of A

- Well, as in continuous-time, not always ...

Steady-state solution and DC gain

- Let A asymptotically stable ($|\lambda_i| < 1$). Natural response vanishes asymptotically
- Assume constant $u(k) \equiv u_r$, $\forall k \in \mathbb{N}$. What is the asymptotic value $x_r = lim_{k\to\infty} x(k)$?

  Impose $x_r(k+1) = x_r(k) = Ax_r + Bu_r$ and get $x_r = (I-A)^{-1}Bu_r$

  The corresponding *steady-state* output $y_r = Cx_r + Du_r$ is

$$y_r = \underbrace{(C(I-A)^{-1}B + D)}_{\text{DC gain}} u_r$$

- Cf. final value theorem:

$$\begin{aligned} y_r &= \lim_{k\to+\infty} y(k) = \lim_{z\to 1}(z-1)Y(z) \\ &= \lim_{z\to 1}(z-1)G(z)U(z) = \lim_{z\to 1}(z-1)G(z)\frac{u_r z}{z-1} \\ &= G(1)u_r = (C(I-A)^{-1}B + D)u_r \end{aligned}$$

- $G(1)$ is called the *DC gain* of the system

Example - Student dynamics

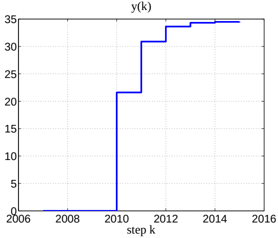
- Recall student dynamics in 3-years undergraduate course

$$\begin{cases} x(k+1) = \begin{bmatrix} .2 & 0 & 0 \\ .6 & .15 & 0 \\ 0 & .8 & .08 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 0 & .9 \end{bmatrix} x(k) \end{cases}$$

- DC gain:

$$\begin{bmatrix} 0 & 0 & .9 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & 0 & 0 \\ .6 & .15 & 0 \\ 0 & .8 & .08 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \approx 0.69$$

- Transfer function: $G(z) = \frac{0.432}{z^3 - 0.43z^2 + 0.058z - 0.0024}$, $G(1) \approx 0.69$



MATLAB

```
»A=[b1 0 0; a1 b2 0; 0 a2 b3];
»B=[1;0;0];
»C=[0 0 a3];
»D=[0];
»sys=ss(A,B,C,D,1);
»dcgain(sys)

ans =

    0.6905
```

- For $u(k) \equiv 50$ students enrolled steadily, $y(k) \to 0.69 \cdot 50 \approx 34.5$ graduate

Linear algebra recalls: Change of coordinates

- Let $\{v_1, \ldots, v_n\}$ be a *basis* of $\mathbb{R}^n$ (= n linearly independent vectors)
- The *canonical basis* of $\mathbb{R}^n$ is $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$
- A vector $w \in \mathbb{R}^n$ can be expressed as a *linear combination* of the basis vectors, whose coefficients are the *coordinates* in the corresponding basis

$$w = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} z_i v_i$$

- The relation between the coordinates $x = [x_1 \ \ldots \ x_n]'$ in the canonical basis and the coordinates $z = [z_1 \ \ldots \ z_n]'$ in the new basis is

$$x = Tz$$

where $T = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$ (= *coordinate transformation matrix*)

Algebraically equivalent systems

- Consider the linear system

$$\begin{cases} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k)+Du(k) \end{cases}$$

$$x(0)=x_0$$

- Let T be invertible and define the change of coordinates $x=Tz$, $z=T^{-1}x$

$$\begin{cases} z(k+1) &= T^{-1}x(k+1)=T^{-1}(Ax(k)+Bu(k))=T^{-1}ATz(k)+T^{-1}Bu(k) \\ y(k) &= CTz(k)+Du(k) \end{cases}$$

$$z_0=T^{-1}x_0$$

and hence

$$\begin{cases} z(k+1) &= \tilde{A}z(k)+\tilde{B}u(k) \\ y(k) &= \tilde{C}z(k)+\tilde{D}u(k) \end{cases}$$

$$z(0)=T^{-1}x_0$$

- The dynamical systems (A,B,C,D) and $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ are called *algebraically equivalent*

Transfer function of algebraically equivalent systems

- Consider two algebraically equivalent systems (A,B,C,D) and $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$

$$\begin{array}{ll} \tilde{A} = T^{-1}AT & \tilde{C} = CT \\ \tilde{B} = T^{-1}B & \tilde{D} = D \end{array}$$

- (A,B,C,D) and $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ have the same transfer functions:

$$\begin{aligned} \tilde{G}(z) &= \tilde{C}(zI-\tilde{A})^{-1}\tilde{B}+\tilde{D} \\ &= CT(zT^{-1}IT-T^{-1}AT)^{-1}T^{-1}B+D \\ &= CTT^{-1}(zI-A)TT^{-1}B+D \\ &= C(zI-A)^{-1}B+D \\ &= G(z) \end{aligned}$$

- The same result holds for continuous-time linear systems

English-Italian Vocabulary

	
Z-transform	*trasformata zeta*
forward shift operator	*operatore di anticipo*
unit delay	*ritardo unitario*

Translation is obvious otherwise.