Automatic Control 1

Transfer functions

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Laplace transform

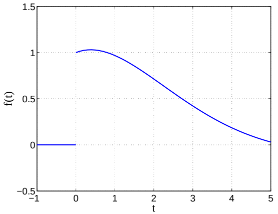
Consider a function $f(t)$, $f : \mathbb{R} \to \mathbb{R}$, $f(t) = 0$ for all $t < 0$.

Definition

The *Laplace transform* $\mathscr{L}[f]$ of f is the function $F : \mathbb{C} \to \mathbb{C}$ of complex variable $s \in \mathbb{C}$ defined by

$$F(s) = \int_0^{+\infty} e^{-st} f(t) dt$$

for all $s \in \mathbb{C}$ for which the integral exists



Pierre-Simon Laplace (1749-1827)

Once $F(s)$ is computed using the integral, it's extended to all $s \in \mathbb{C}$ for which $F(s)$ makes sense

Laplace transforms convert integral and differential equations into algebraic equations. We'll see how ...

Examples of Laplace transforms

- *Unit step*

$$f(t) = 1\!\!1(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \quad \Rightarrow \quad F(s) = \int_0^{+\infty} e^{-st} dt = -\frac{1}{s}\Big|_0^{+\infty} = \frac{1}{s}$$

- *Dirac's delta* (or *impulse function*[1])

$$f(t) = \delta(t) \triangleq \begin{cases} 0 & \text{if } t \neq 0 \\ +\infty & \text{if } t = 0 \end{cases} \quad \text{such that} \quad \int_{-\infty}^{+\infty} \delta(t) = 1$$

$$F(s) = \int_0^{+\infty} \delta(t) e^{-st} dt = e^{-s0} = 1, \ \forall s \in \mathbb{C}$$

[1]The function $\delta(t)$ is can be considered as the limit of the sequence of functions $f_\epsilon(t)$ for $\epsilon \to 0$

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{se } 0 \leq t \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

To be mathematically correct, Dirac's δ is a *distribution*, not a function

Properties of Laplace transforms

- *Linearity*

$$\mathscr{L}[k_1 f_1(t) + k_2 f_2(t)] = k_1 \mathscr{L}[f_1(t)] + k_2 \mathscr{L}[f_2(t)]$$

Example: $f(t) = \delta(t) - 2\,\mathbb{1}(t) \Rightarrow \mathscr{L}[f] = 1 - \frac{2}{s}$

- *Time delay*

$$\mathscr{L}[f(t-\tau)] = e^{-s\tau}\mathscr{L}[f(t)]$$

Example: $f(t) = 3\,\mathbb{1}(t-2) \Rightarrow \mathscr{L}[f] = \frac{3e^{-2s}}{s}$

- *Exponential scaling*

$$\mathscr{L}[e^{at} f(t)] = F(s-a), \text{ where } F(s) = \mathscr{L}[f(t)]$$

Example: $f(t) = e^{at}\,\mathbb{1}(t) \Rightarrow \mathscr{L}[f] = \frac{1}{s-a}$
Example: $f(t) = \cos(\omega t)\,\mathbb{1}(t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}\,\mathbb{1}(t) \Rightarrow \mathscr{L}[f] = \frac{s}{s^2+\omega^2}$

Properties of Laplace transforms

- *Time derivative*[2]:

$$\mathscr{L}[\frac{d}{dt}f(t)] = s\mathscr{L}[f(t)] - f(0^+)$$

Example $\Longrightarrow f(t) = \sin(\omega t)\,1\!\!1(t) \Rightarrow L[f] = \frac{\omega}{s^2+\omega^2}$

- *Multiplication by t*

$$\mathscr{L}[tf(t)] = -\frac{d}{ds}\mathscr{L}[f(t)]$$

Example $\Longrightarrow f(t) = t\,1\!\!1(t) \Rightarrow \mathscr{L}[f] = \frac{1}{s^2}$

[2] $f(0^+) = \lim_{t\to 0^+} f(t)$. If f is continuos in 0, $f(0^+) = f(0)$

Initial and final value theorems

Initial value theorem

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

Example: $f(t) = 1\!\!1(t) - t\,1\!\!1(t) \Rightarrow F(s) = \frac{1}{s} - \frac{1}{s^2}$
$f(0^+) = 1 = \lim_{s\to\infty} sF(s)$

Final value theorem

$$\lim_{t\to +\infty} f(t) = \lim_{s\to 0} sF(s)$$

Example: $f(t) = 1\!\!1(t) - e^{-t}\,1\!\!1(t) \Rightarrow F(s) = \frac{1}{s} - \frac{1}{s+1}$
$f(+\infty) = 1 = \lim_{s\to 0} sF(s)$

Convolution

- The *convolution* $h = f * g$ of two signals f and g is the signal

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

- It's easy to see that $h = f * g = g * f$
- The Laplace transform of the convolution:

$$\mathscr{L}[f(t) * g(t)] = \mathscr{L}[f(t)]\mathscr{L}[g(t)]$$

- Laplace transforms turn convolution into multiplication !

Common Laplace transforms

1	$\frac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\frac{1}{s^2}$
$\frac{t^k}{k!},\ k \geq 0$	$\frac{1}{s^{k+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} = \frac{1/2j}{s-j\omega} - \frac{1/2j}{s+j\omega}$
$\cos(\omega t + \phi)$	$\frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

courtesy of S. Boyd, `http://www.stanford.edu/~boyd/ee102/`

In MATLAB use

`F = LAPLACE(f)`

MATLAB

```
» syms t
» f=exp(2*t)+t-t^2
» F=laplace(f)

F =

1/(s-2)+1/s^2-2/s^3
```

Properties of Laplace transforms

$f(t)$	$F(s)=\int_0^\infty f(t)e^{-st}\,dt$
$f+g$	$F+G$
$\alpha f\ (\alpha\in\mathbf{R})$	αF
$\frac{df}{dt}$	$sF(s)-f(0)$
$\frac{d^kf}{dt^k}$	$s^kF(s)-s^{k-1}f(0)-s^{k-2}\frac{df}{dt}(0)-\cdots-\frac{d^{k-1}f}{dt^{k-1}}(0)$
$g(t)=\int_0^t f(\tau)\,d\tau$	$G(s)=\frac{F(s)}{s}$
$f(\alpha t),\ \alpha>0$	$\frac{1}{\alpha}F(s/\alpha)$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-\frac{dF}{ds}$
$t^kf(t)$	$(-1)^k\frac{d^kF(s)}{ds^k}$
$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
$g(t)=\begin{cases}0 & 0\le t<T\\ f(t-T) & t\ge T\end{cases},\ T\ge 0$	$G(s)=e^{-sT}F(s)$

courtesy of S. Boyd, `http://www.stanford.edu/~boyd/ee102/`

Transfer function

- Let's apply the Laplace transform to continuous-time linear systems

$$\begin{cases} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t)+Du(t) \end{cases}$$
$$x(0)=x_0$$

- Define $X(s)=\mathscr{L}[x(t)]$, $U(s)=\mathscr{L}[u(t)]$, $Y(s)=\mathscr{L}[y(t)]$
- Apply linearity and time-derivative rules

$$\begin{cases} sX(s)-x_0 & = & AX(s)+BU(s) \\ Y(s) & = & CX(s)+DU(s) \end{cases}$$

Transfer function

$$\begin{aligned} X(s) &= (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s) \\ Y(s) &= \underbrace{C(sI-A)^{-1}x_0}_{\text{Laplace transform of natural response}} + \underbrace{(C(sI-A)^{-1}B+D)U(s)}_{\text{Laplace transform of forced response}} \end{aligned}$$

Definition

The transfer function of a continuous-time linear system (A,B,C,D) is the ratio

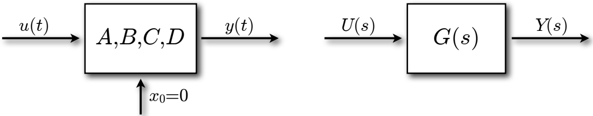
$$G(s) = C(sI-A)^{-1}B + D$$

between the Laplace transform $Y(s)$ of output and the Laplace transform $U(s)$ of the input signals *for the initial state* $x_0 = 0$

MATLAB

```
»sys=ss(A,B,C,D);
»G=tf(sys)
```

Transfer function



Example: The linear system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -10 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 2 \end{bmatrix} x(t) \end{cases}$$

has the transfer function

$$G(s) = \frac{2s + 22}{s^2 + 11s + 10}$$

Note: The transfer function does not depend on the input $u(t)$! It's only a property of the linear system.

MATLAB

```
»sys=ss([-10 1;
  0 -1],[0;1],[2 2],0);
»G=tf(sys)

Transfer function:
2 s + 22
----------
s^2 + 11 s + 10
```

Transfer functions and linear ODEs

- Consider the n^{th}-order differential equation with input

$$\frac{dy^{(n)}(t)}{dt^n} + a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}} + \cdots + a_1\dot{y}(t) + a_0 y(t) =$$
$$b_m\frac{du^{(m)}(t)}{dt^m} + b_{m-1}\frac{du^{(m-1)}(t)}{dt^{m-1}} + \cdots + b_1\dot{u}(t) + b_0 u(t)$$

- For initial conditions $y(0) = \dot{y}(0) = y^{(n-1)}(0) = 0$, we obtain immediately the transfer function from u to y

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Example

$$\ddot{y} + 11\dot{y} + 10y = 2\dot{u} + 22u$$

$$G(s) = \frac{2s + 22}{s^2 + 11s + 10}$$

MATLAB

```
»G=tf([2 22],[1 11 10])

Transfer function:
2 s + 22
----------
s^2 + 11 s + 10
```

Example

- Differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + y(t) = \dot{u}(t) + u(t)$$

- The transfer function is

$$G(s) = \frac{s+1}{s^2 + 3s + 1}$$

- The same transfer function $G(s)$ can be obtained through a state-space realization

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{array} \right.$$

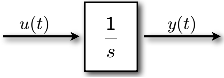
from which we compute

$$G(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+1}{s^2 + 3s + 1}$$

Some common transfer functions

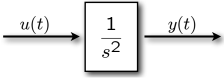
- *Integrator*

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t) \end{cases} \qquad y(t) = \int_0^t u(\tau)d\tau$$



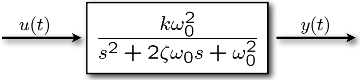
- *Double integrator*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) \\ y(t) = x_1(t) \end{cases} \qquad y(t) = \int\int_0^t u(\tau)d\tau$$



- *Damped oscillator* with frequency ω_0 rad/s and damping factor ζ

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$



Inverse Laplace transform

- The impulse response $y(t)$ is therefore the *inverse Laplace transform* of the transfer function $G(s)$, $y(t) = \mathscr{L}^{-1}[G(s)]$
- The general formula for computing the inverse Laplace transform is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

where σ is large enough that $F(s)$ is defined for $\Re s \geq \sigma$
- This formula is not used very often

In MATLAB use

```
f = ILAPLACE(f)
```

MATLAB

```
» syms s
» F=2*s/(s^2+1)
» f=ilaplace(F)

f = 2*cos(t)
```

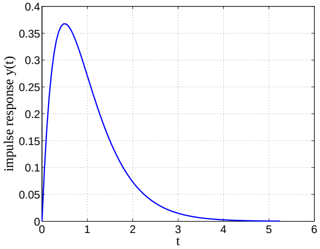
Impulse response

- Remember that an input signal $u(t)$ produces an output signal $y(t)$ whose Laplace transform $Y(s)$ is
$$Y(s) = G(s)U(s)$$
where $U(s) = \mathscr{L}[u]$, for initial state $x(0) = 0$
- Speciale case: impulsive input $u(t) = \delta(t)$, $U(s) = 1$. The corresponding output $y(t)$ is called the *impulse response*
- $G(s)$ is the Laplace transform of the impulse response $y(t)$
$$Y(s) = G(s) \cdot 1 = G(s)$$

Example:

$$G(s) = \frac{2}{s^2 + 3s + 1}$$

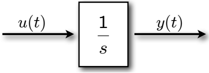
$$\mathscr{L}^{-1}[G(s)] = 2te^{-2t}$$



Examples

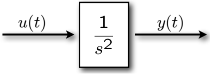
- *Integrator*

$$\begin{aligned} u(t) &= \delta(t) \\ y(t) &= \mathscr{L}^{-1}[\tfrac{1}{s}] = 1\!\!1(t) \end{aligned}$$



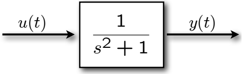
- *Double integrator*

$$\begin{aligned} u(t) &= \delta(t) \\ y(t) &= \mathscr{L}^{-1}[\tfrac{1}{s^2}] = 1\!\!1(t)t \end{aligned}$$

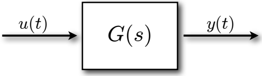


- *Undamped oscillator*

$$\begin{aligned} u(t) &= \delta(t) \\ y(t) &= \mathscr{L}^{-1}[\tfrac{1}{s^2+1}] = 1\!\!1(t)\sin t \end{aligned}$$



Poles and Zeros



- Rewrite the transfer function as the ratio of polynomials ($m < n$)

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

- The roots p_i of $D(s)$ are called the *poles* of the linear system $G(s)$
- The roots z_i of $N(s)$ are called the *zeros* of $G(s)$
- $G(s)$ is often written in zero/pole/gain form

$$G(s) = K \frac{(s - z_1) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_n)}$$

In MATLAB use `ZPK` to transform to zero/pole/gain form

Examples

- Example 1

$$G(s) = \frac{s+2}{s^3+2s^2+3s+2} = \frac{s+2}{(s+1)(s^2+s+2)}$$

poles: $\{-1, -\frac{1}{2} + j\frac{\sqrt{7}}{2}, -\frac{1}{2} - j\frac{\sqrt{7}}{2}\}$, zeros: $\{-2\}$

- Example 2

$$G(s) = \frac{2s+22}{s^2+11s+10} = \frac{2(s+11)}{(s+10)(s+1)}$$

poles: $\{-10, -1\}$, zeros: $\{-11\}$

MATLAB

```
» G=tf([2 22],[1 11 10])
» zpk(G)

Zero/pole/gain:
2 (s+11)
---------
(s+10) (s+1)
```

Partial fraction decomposition

- The *partial fraction decomposition* of a rational function $G(s)=N(s)/D(s)$ is (assuming $p_i \neq p_j$)[3]

$$G(s)=\frac{\alpha_1}{s-p_1}+\cdots+\frac{\alpha_n}{s-p_n}$$

- α_i is called the *residue*[4] of $G(s)$ in $p_i \in \mathbb{C}$

$$\alpha_i=\lim_{s\to p_i}(s-p_i)G(s)$$

- The inverse Laplace transform of $G(s)$ is easily computed by inverting each term

$$\mathscr{L}^{-1}[G(s)]=\alpha_1 e^{p_1 t}+\cdots+\alpha_n e^{p_n t}$$

[3]For multiple poles p_i with multiplicity k we have the terms

$$\frac{\alpha_{i1}}{(s-p_i)}+\cdots+\frac{\alpha_{ik}}{(s-p_i)^k},\ \alpha_{ij}=\frac{1}{(k-j)!}\lim_{s\to p_i}\frac{d^{(k-j)}}{ds^{(k-j)}}[(s-p_i)^k G(s)]$$

and the inverse Laplace transform is

$$\alpha_{i1}e^{p_i t}+\cdots+\alpha_{ik}\frac{t^{k-1}}{(k-1)!}e^{p_i t}$$

[4]Residues of conjugate poles are conjugate of each other: $p_i=\bar{p}_j \ \Rightarrow\ \alpha_i=\bar{\alpha}_j$

Linear algebra recalls

- The *inverse* of a matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

  is the matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$

- The inverse A^{-1} can be computed using the *adjugate* matrix $\mathrm{Adj}A$

$$A^{-1} = \frac{\mathrm{Adj}A}{\det A}$$

- The adjugate matrix is the transpose of the *cofactor matrix* C of A

$$\mathrm{Adj}A = C^T,\ C_{ij} = (-1)^{i+j}M_{ij}$$

  where M_{ij} is the (i,j) *cofactor* of A, that is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A

Numerical caveat

- Consider the linear system of n equalities $Ax = b$ in the unknown vector $x \in \mathbb{R}^n$ ($A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$)
- If $\det A \neq 0$, the unique solution is $x = A^{-1}B$
- However, computing A^{-1} is not a smart thing to do for finding x !
- Numerical example: `n=1000; A=rand(n,n)+10*eye(n); b=rand(n,1);`

MATLAB

```
» tic; x=inv(A)*b; toc

elapsed_time =

2.2190
```

First A is inverted, an operation that costs $O(n^3)$ arithmetic operations

MATLAB

```
» tic; x=A\ b; toc

elapsed_time =

0.8440
```

The linear system is solved using Gauss method, an operation that costs $O(n^2)$ arithmetic operations

Poles, eigenvalues, modes

- Linear system

$$\begin{cases} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t)+Du(t) \end{cases}$$
$$x(0)=0$$

$$G(s)=C(sI-A)^{-1}B+D \triangleq \frac{N_G(s)}{D_G(s)}$$

- Use the adjogate matrix to represent the inverse of $sI-A$

$$C(sI-A)^{-1}B+D=\frac{C\,\mathrm{Adj}(sI-A)B}{\det(sI-A)}+D$$

- The denominator $D_G(s)=\det(sI-A)$!

The poles of $G(s)$ coincide with the eigenvalues of A

- Well, not always ...

Poles, eigenvalues, modes

- Some eigenvalues of A may not be poles of $G(s)$ in case of *pole/zero cancellations*
- Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$\det(sI - A) = (s-1)(s+1)$

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s+1}$$

- The pole $s = 1$ has no influence on the input/output behavior of the system (but it has influence on the free response $x_1(t) = e^t x_{10}$)
- We'll better understand cancellations when investigating reachability and observability properties

Steady-state solution and DC gain

- Let A asymptotically stable. Natural response vanishes asymptotically
- Assume constant $u(t) \equiv u_r$. What is the asymptotic value $x_r = lim_{t\to\infty} x(t)$?

  Impose $0 = \dot{x}_r(t) = Ax_r + Bu_r$ and get $x_r = -A^{-1}Bu_r$

  The corresponding *steady-state* output $y_r = Cx_r + Du_r$ is

$$y_r = \underbrace{(-CA^{-1}B + D)}_{\text{DC Gain}} u_r$$

- Cf. final value theorem:

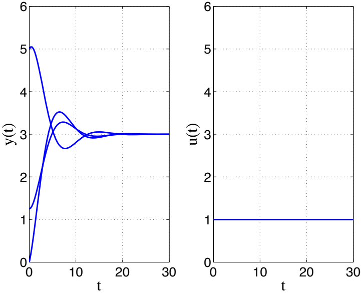
$$\begin{aligned} y_r &= \lim_{t\to+\infty} y(t) = \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} sG(s)U(s) = \lim_{s\to 0} sG(s)\frac{u_r}{s} \\ &= G(0)u_r = (-CA^{-1}B + D)u_r \end{aligned}$$

- $G(0)$ is called the *DC gain* of the system

DC gain - Example

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} x(t) \end{cases}$$

- DC gain: $-\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 3$
- Transfer function: $G(s) = \frac{2s+3}{4s^2+2s+1}$. We have G(0)=3



Output $y(t)$ for different initial conditions and input $u(t) \equiv 1$

MATLAB

```
»sys=tf([2 3],[4 2 1]);
»dcgain(sys)

ans =

    3
```

English-Italian Vocabulary

	
transfer function	*funzione di trasferimento*
Laplace transform	*trasformata di Laplace*
unit step	*gradino unitario*
delay	*ritardo*
damped oscillator	*oscillatore smorzato*
impulse response	*risposta all'impulso*
inverse Laplace transform	*antitrasformata di Laplace*
partial fraction decomposition	*decomposizione in fratti semplici*
DC gain	*guadagno in continua*
steady-state	*regime stazionario*

Translation is obvious otherwise.