Automatic Control 1

Dynamical models of physical systems

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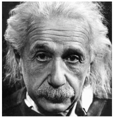
Introduction

- Objective: Develop mathematical models of physical systems often encountered in practice
- Why? Mathematical models allow us to capture the main phenomena that take place in the system, in order to analyze, simulate, and control it
- We focus on *dynamical* models of *physical* (mechanical, electrical, thermal, hydraulic) systems

- Remember: A physical model for control design purposes should be
  - *Descriptive:* able to capture the main features of the system
  - *Simple:* the simpler the model, the simpler will be the synthesized control algorithm

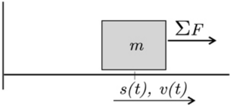
"Make everything as simple as possible, but not simpler."
– Albert Einstein



Albert Einstein
(1879-1955)

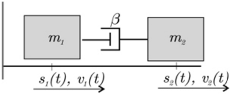
Today you will learn some basics of the art of modeling dynamical systems ...

Mechanical systems – Linear motion



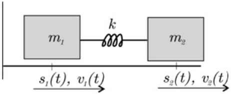
Newton's Law:

$$\sum F(t) = m\frac{dv(t)}{dt} = m\frac{d^2s(t)}{dt^2}$$



Viscous friction:

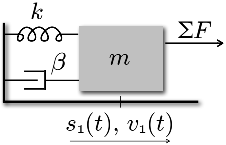
$$F_1(t) = \beta(v_2(t) - v_1(t)) = -F_2(t)$$



Elastic coupling:

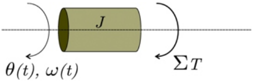
$$F_1(t) = k(s_2(t) - s_1(t)) = -F_2(t)$$

- $s_i(t), v_i(t) =$ *position* and *velocity* of body i, with respect to a fixed (inertial) reference frame
- $F_i(t) =$ *force* acting on body i
- $m, \beta, k =$ *mass*, *viscous friction coefficient*, *spring constant*

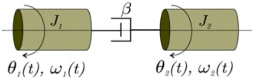


- Special case: $s_2(t) \equiv 0$, $v_2(t) \equiv 0$
  $F_1(t) = -ks_1(t)$, $F_1(t) = -\beta v_1(t)$

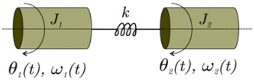
Mechanical systems – Rotational motion



Newton's Law:
$$\sum \tau(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$



Viscous friction:
$$\tau_1(t) = \beta(\omega_2(t) - \omega_1(t)) = -\tau_2(t)$$

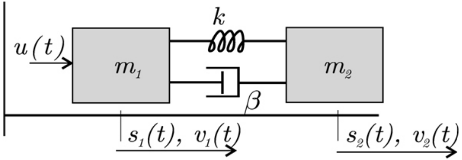


Elastic coupling:
$$\tau_1(t) = k(\theta_2(t) - \theta_1(t)) = -\tau_2(t)$$

- $\theta_i(t)$, $\omega_i(t)$ = *angular position* and *angular velocity* of body i, with respect to a fixed (inertial) reference frame
- $\tau_i(t)$: *torque* acting on body i
- J, β, k: *inertia*, *viscous friction coefficient*, *spring constant*

Example of mechanical system

Two masses connected by spring-damper (no dry friction with the surface)



Dynamics of mass m_1:

$$m_1 \frac{dv_1(t)}{dt} = u(t) + k(s_2(t) - s_1(t)) + \beta(v_2(t) - v_1(t))$$

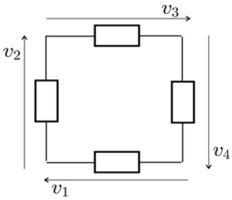
Dynamics of mass m_2:

$$m_2 \frac{dv_2(t)}{dt} = -k(s_2(t) - s_1(t)) - \beta(v_2(t) - v_1(t))$$

Note: viscous and elastic forces always oppose motion

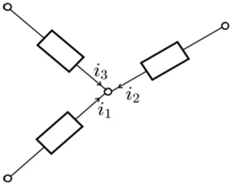
Electrical systems

Kirchhoff's voltage law: balance of voltages on a closed circuit



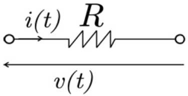
$$v_1 + v_2 + v_3 + v_4 = 0$$

Kirchhoff's current law: balance of the currents at a node

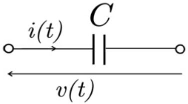


$$i_1 + i_2 + i_3 = 0$$

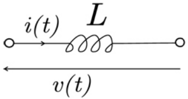
Electrical systems



Resistor: $v(t) = Ri(t)$



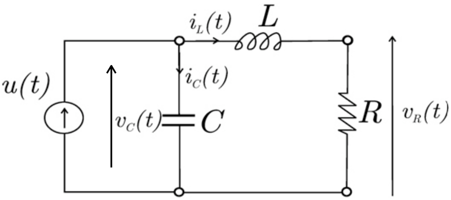
Capacitor: $i(t) = C\frac{dv(t)}{dt}$



Inductor: $v(t) = L\frac{di(t)}{dt}$

- $v(t)$: *voltage* across the component
- $i(t)$: *current* through the component
- R, C, L: *resistance*, *capacitance*, *inductance*

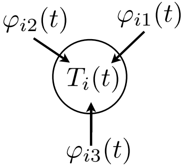
Example of electrical system



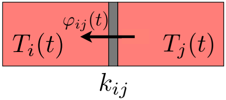
Kirchhoff's current law: $i_C(t) = C\frac{dv_C(t)}{dt} = u(t) - i_L(t)$

Kirchhoff's voltage law: $v_c(t) - L\frac{di_L(t)}{dt} - Ri_L(t) = 0$

Thermal systems



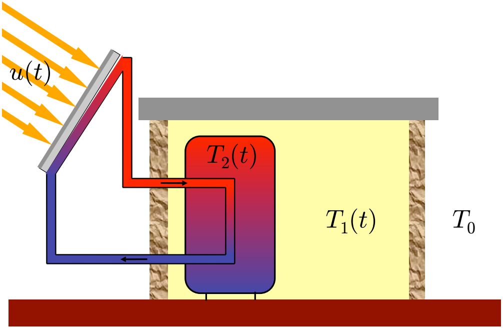
Heat transfer: energy balance
$\sum_j \varphi_{ij}(t) = C_i \frac{dT_i(t)}{dt}$



Conduction and/or convection
$\varphi_{ij}(t) = k_{ij}(T_j(t) - T_i(t))$

- $T_i(t)$, C_i = *temperature* and *heat capacity* of body i
- k_{ij} = *heat exchange coefficient* ($R_{ij} = 1/k_{ij}$ = *thermal resistance*)
- $\varphi_{ij}(t)$ = *thermal power* (=heat flow) from body j to body i

Example of thermal system



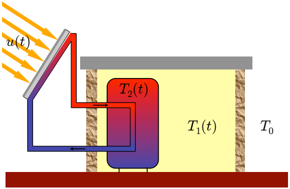
Heat transfer: energy balance

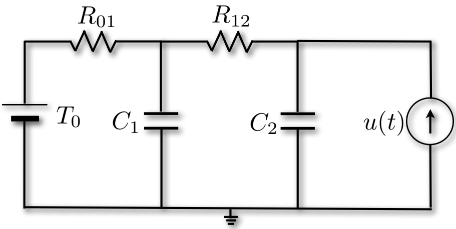
$$C_1\dot{T}_1(t) = -k_{01}(T_1(t) - T_0) + k_{12}(T_2(t) - T_1(t))$$
$$C_2\dot{T}_2(t) = -k_{12}(T_2(t) - T_1(t)) + u(t)$$

Electrical equivalent of thermal systems

thermal model	electrical model
reference temperature	ground
body	node
thermal capacitance	electrical capacitance connected to ground
thermal resistance	electrical resistance between nodes
thermal flow	current
temperature	voltage
thermal power input	current generator
constant temperature body	voltage generator

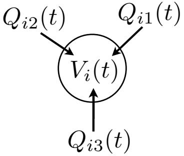
For the previous example:





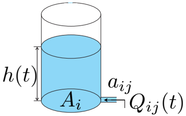
Hydraulic systems

Assumptions: the fluid is perfect (no shear stresses, no viscosity, no heat conduction), and subject only to gravity. Only one fluid is considered with constant density ρ (incompressible fluid). The orifices in the tanks are always at the bottom. The external pressure is constant (atmospheric pressure)



Mass (volume) balance

$$\sum_j Q_{ij}(t) = \frac{dV_i(t)}{dt} = A_i \frac{dh_i(t)}{dt}$$

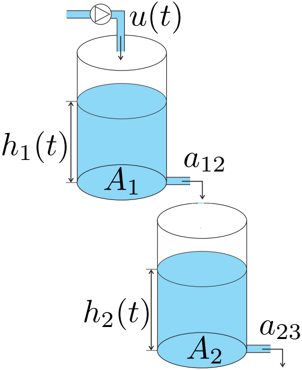


Torricelli' law

$$Q_{ij}(t) = -a_{ij}\sqrt{2gh_i(t)}$$

- A_i, $h_i(t)$ = *base area* and *fluid level* in tank i
- $Q_{ij}(t)$, a_{ij} = *volume flow* from tank j to tank i, area of orifice
- g: gravitational acceleration

Example of hydraulic system



Mass (volume) balance

$$A_1\dot{h}_1(t) = -a_{12}\sqrt{2gh_1(t)} + u(t)$$
$$A_2\dot{h}_2(t) = a_{12}\sqrt{2gh_1(t)} - a_{23}\sqrt{2gh_2(t)}$$

Choice of state variables

To obtain a state-space model one must choose state variables. How?

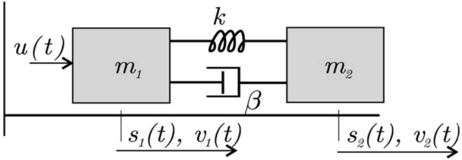
Rule of thumb: # state variables = # of energy storage elements

type	element	energy	state
mechanical	mass	kinetic energy: $\frac{1}{2}mv^2$	velocity
	spring	potential elastic energy: $\frac{1}{2}ks^2$	position
electrical	inductor	potential magnetic energy: $\frac{1}{2}Li^2$	current
	capacitor	potential electric energy: $\frac{1}{2}Cv^2$	voltage
thermal	body	internal energy: CT	temperature
hydraulic	tank	potential gravitational energy: ρgh	height

Choice of state variables also depends on selected output variables of interest ...

Example of mechanical system

Two masses connected by spring-damper (no dry friction on surface)



Dynamics of mass m_1:

$$m_1 \frac{dv_1(t)}{dt} = u(t) + k(s_2(t) - s_1(t)) + \beta(v_2(t) - v_1(t))$$

Dynamics of mass m_2:

$$m_2 \frac{dv_2(t)}{dt} = -k(s_2(t) - s_1(t)) - \beta(v_2(t) - v_1(t))$$

Example of mechanical system

Case 1. Output: $y = v_2$. Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_2 - s_1 \\ v_1 \\ v_2 \end{bmatrix}$$

$$y = x_3$$

$$\dot{x}_1(t) = x_3(t) - x_2(t)$$

$$\dot{x}_2(t) = \frac{k}{m_1}x_1(t) + \frac{\beta}{m_1}(x_3(t) - x_2(t)) + \frac{1}{m_1}u(t)$$

$$\dot{x}_3(t) = -\frac{k}{m_2}x_1(t) - \frac{\beta}{m_2}(x_3(t) - x_2(t))$$

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 & 1 \\ \frac{k}{m_1} & -\frac{\beta}{m_1} & \frac{\beta}{m_1} \\ -\frac{k}{m_2} & \frac{\beta}{m_2} & -\frac{\beta}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t)$$

Example of mechanical system

Case 2. Output: $y = s_2$. Choice of state variables

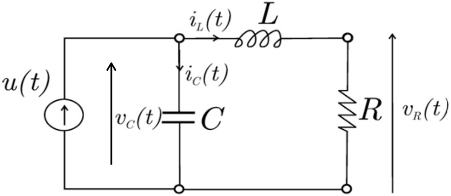
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s_1 \\ v_1 \\ s_2 \\ v_2 \end{bmatrix}$$

$$y = x_3$$

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{k}{m_1}(x_3(t) - x_1(t)) + \frac{\beta}{m_1}(x_4(t) - x_2(t)) + \frac{1}{m_1}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{m_2}(x_1(t) - x_3(t)) + \frac{\beta}{m_2}(x_2(t) - x_4(t))
\end{aligned}$$

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{\beta}{m_1} & \frac{k}{m_1} & \frac{\beta}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{\beta}{m_2} & -\frac{k}{m_2} & -\frac{\beta}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t)
\end{aligned}$$

Example of electrical system



Kirchhoff's current law: $i_C = C\frac{dv_C}{dt} = u(t) - i_L$

Kirchhoff's voltage law: $L\frac{di_L}{dt} + Ri_L - v_c = 0$

Example of electrical system

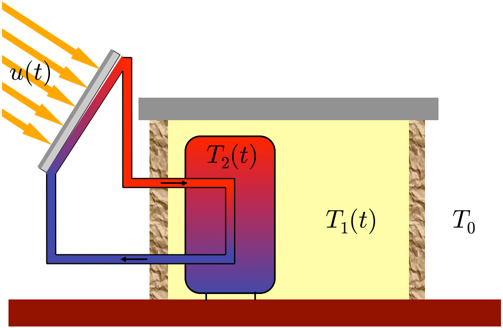
System output: $y = v_R = Ri_L$. Choice of state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$
$$y = Rx_2$$

$$\dot{x}_1(t) = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t)$$
$$\dot{x}_2(t) = \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & R \end{bmatrix} x(t)$$

Example of thermal system



Heat transfer: energy balance

$$C_1\dot{T}_1(t) = -k_{01}(T_1(t) - T_0) + k_{12}(T_2(t) - T_1(t))$$
$$C_2\dot{T}_2(t) = -k_{12}(T_2(t) - T_1(t)) + u(t)$$

Example of thermal system

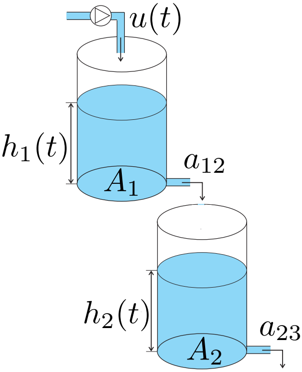
System output: $y = T_1 - T_0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1 - T_0 \\ T_2 - T_0 \end{bmatrix}$$
$$y = T_1 - T_0$$

$$\dot{x}_1(t) = \frac{1}{C_1}(-k_{12} - k_{01})x_1(t) + \frac{k_{12}}{C_1}x_2(t)$$
$$\dot{x}_2(t) = \frac{k_{12}}{C_2}x_1(t) - \frac{k_{12}}{C_2}x_2(t) + \frac{1}{C_2}u(t)$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{k_{12}+k_{01}}{C_1} & \frac{k_{12}}{C_1} \\ \frac{k_{12}}{C_2} & -\frac{k_{12}}{C_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{C_2} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Example of hydraulic system



Mass (volume) balance

$$A_1\dot{h}_1(t) = -a_{12}\sqrt{2gh_1(t)} + u(t)$$
$$A_2\dot{h}_2(t) = a_{12}\sqrt{2gh_1(t)} - a_{23}\sqrt{2gh_2(t)}$$

Example of hydraulic system

System output: h_2. Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$
$$y = h_2$$

$$\dot{x}_1(t) = -\frac{a_{12}}{A_1}\sqrt{2gx_1(t)} + \frac{1}{A_1}u(t)$$
$$\dot{x}_2(t) = \frac{a_{12}}{A_2}\sqrt{2gx_1(t)} - \frac{a_{23}}{A_2}\sqrt{2gx_2(t)}$$

The model is nonlinear !

- We want to linearize the model around the equilibrium point (x_{1r}, x_{2r}), corresponding to the constant input u_r
- The linearized model will be useful to control the system near the equilibrium point

Example of hydraulic system

- Zero state derivatives

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), u(t)) = -\frac{a_{12}}{A_1}\sqrt{2gx_1(t)} + \frac{1}{A_1}u(t) = 0 \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), u(t)) = \frac{a_{12}}{A_2}\sqrt{2gx_1(t)} - \frac{a_{23}}{A_2}\sqrt{2gx_2(t)} = 0 \\
y(t) &= \gamma(x_1(t), x_2(t), u(t)) = x_2(t)
\end{aligned}
$$

- Substitute $u(t) = u_r$ and get $x_{1r} = \frac{u_r^2}{2ga_{12}^2}$, $x_{2r} = \frac{u_r^2}{2ga_{23}^2}$, $y_r = x_{2r}$
- Linearize

$$
A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \rightarrow \begin{array}{l} \text{substitute } u = u_r, \\ x_1 = x_{1r},\ x_2 = x_{2r} \end{array} \rightarrow A = \begin{bmatrix} -\frac{a_{12}^2 g}{A_1 u_r} & 0 \\ \frac{a_{12}^2 g}{A_2 u_r} & -\frac{a_{23}^2 g}{A_2 u_r} \end{bmatrix}
$$

Example of hydraulic system

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \rightarrow \begin{array}{c} \text{substitute } u = u_r, \\ x_1 = x_{1r},\ x_2 = x_{2r} \end{array} \rightarrow B = \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix}$$

Note that here the input enters the state-update equation linearly, so there is no need to compute $\frac{\partial f_i}{\partial u}$ to get B

$$C = \begin{bmatrix} \frac{\partial \gamma}{\partial x_1} & \frac{\partial \gamma}{\partial x_2} \end{bmatrix} \rightarrow \begin{array}{c} \text{substitute } u = u_r, \\ x_1 = x_{1r},\ x_2 = x_{2r} \end{array} \rightarrow C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

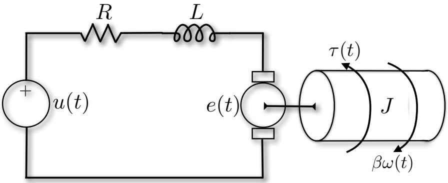
The output equation is also linear, and one can directly obtain C

The overall linearized system (with $\Delta x(t) = x(t) - x_r$, $\Delta u(t) = u(t) - u_r$, and $\Delta y(t) = y(t) - y_r$) is

$$\begin{aligned} \Delta\dot{x}(t) &= \begin{bmatrix} -\frac{a_{12}^2 g}{A_1 u_r} & 0 \\ \frac{a_{12}^2 g}{A_2 u_r} & -\frac{a_{23}^2 g}{A_2 u_r} \end{bmatrix} \Delta x(t) + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} \Delta u(t) \\ \Delta y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \Delta x(t) \end{aligned}$$

Electrical DC motor

Example of a (very common) system involving mechanical and electrical models



- Electrical part: $L\frac{di(t)}{dt} + Ri(t) + e(t) = u(t)$
  The back emf $e(t)$ is proportional to the motor speed: $e(t) = K\omega(t)$
- Mechanical part: $J\frac{d\omega(t)}{dt} + \beta\omega(t) = \tau(t)$
  The torque $\tau(t)$ is proportional to the armature current: $\tau(t) = Ki(t)$
- Overall model

$$L\frac{di(t)}{dt} = u(t) - Ri(t) - K\omega(t)$$
$$J\frac{d\omega(t)}{dt} = Ki(t) - \beta\omega(t)$$

Electrical DC motor

Case 1. System output: $y = \omega$

Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ i \end{bmatrix}$$

$$y = \omega$$

$$\dot{x}_1(t) = \frac{K}{J}x_2(t) - \frac{\beta}{J}x_1(t)$$

$$\dot{x}_2(t) = \frac{1}{L}u(t) - \frac{R}{L}x_2(t) - \frac{K}{L}x_1(t)$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{\beta}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Electrical DC motor

Case 2. System output: $y = \theta$, angular position

Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix}$$

$$y = \theta$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{K}{J}x_3(t) - \frac{\beta}{J}x_2(t) \\ \dot{x}_3(t) &= \frac{1}{L}u(t) - \frac{R}{L}x_3(t) - \frac{K}{L}x_2(t) \end{aligned}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{\beta}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)$$