Automatic Control 1

Continuous-time linear systems

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Dynamical models

- A *dynamical system* is an object (or a set of objects) that evolves over time, possibly under external excitations.

  Examples: a car, a robotic arm, a population of animals, an electrical circuit, a portfolio of investments, etc.

- The way the system evolves is called the *dynamics* of the system.
- A *dynamical model* of a system is a set of mathematical laws explaining in a compact form and in *quantitative* way how the system evolves over time, usually under the effect of external excitations.
- Main questions about a dynamical system:
  1. Understanding the system ("How X and Y influence each other ?")
  2. Simulation ("What happens if I apply action Z on the system ?")
  3. Design ("How to make the system behave the way I want ?")

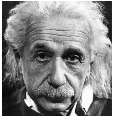
Dynamical models

- *Qualitative* models only useful in non-technical domains (examples: politics, advertisement, psychology,...)
- *Experiments* provide an answer, but have limitations:
  1. maybe too expensive (example: launch a space shuttle)
  2. maybe too dangerous (example: a nuclear plant)
  3. maybe impossible (the system doesn't exist yet!)
- In contrast, mathematical models allows us to:
  1. capture the main phenomena that take place in the system (example: Newton's law – a force on a mass produces an acceleration)
  2. analyze the system (relations among dynamical variables)
  3. simulate the system (=make predictions) about how the system behaves under certain conditions and excitations (in analytical form, or on a computer)

Dynamical models

- Working on a model has almost zero cost compared to real experiments (just mathematical thinking, paper writing, computer coding)
- However, a simulation (or any other inference obtained from the model) is as better as the dynamical model is closer to the real system
- Conflicting objectives:
  1. *Descriptive* enough to capture the main behavior of the system
  2. *Simple* enough for analyzing the system

"Make everything as simple as possible, but not simpler."
– Albert Einstein



Albert Einstein
(1879-1955)

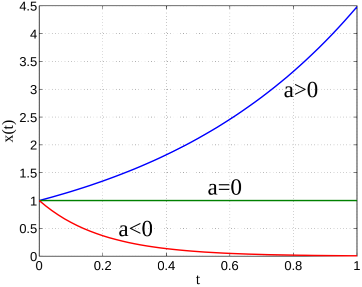
Making a good model is an **art** ! (that you are learning ...)

Ordinary differential equations (ODEs)

- First order differential equation (=the simplest dynamical model):

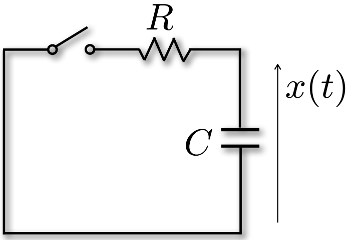
$$\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \qquad \begin{array}{l} a \in \mathbb{R}, \quad \dot{x} \triangleq \frac{dx}{dt} \\ x_0 \in \mathbb{R} \end{array}$$

- Its unique *solution* is $x(t) = e^{at}x_0$



Examples

$x(t) = \text{voltage}$

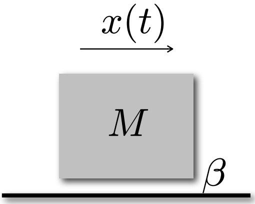


Kirchhoff's voltage law:

$$-RC\dot{x}(t) - x(t) = 0$$

$$x(t) = x(0)e^{-\frac{t}{RC}}$$

$x(t) = \text{velocity}$



Newton's law:

$$-\beta x(t) = M\dot{x}(t)$$

$$x(t) = x(0)e^{-\frac{\beta}{M}t}$$

First order differential equations with inputs

- Introduce the forcing signal $u(t)$

$$\begin{cases} \dot{x}(t) = ax(t) + bu(t) \\ x(0) = x_0 \end{cases} \qquad \begin{array}{l} a, b \in \mathbb{R},\ u(t) \in \mathbb{R} \\ x_0 \in \mathbb{R} \end{array}$$

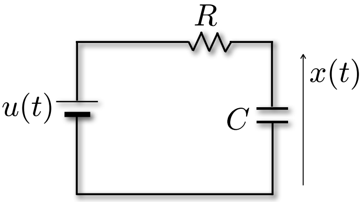
- The unique solution $x(t)$ is

$$x(t) = \underbrace{e^{at}x_0}_{\textit{natural response}} + \underbrace{\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}_{\textit{forced response}}$$

$$\begin{array}{rcll} x_\ell(t) & = & e^{at}x_0 & \text{effect of the initial condition} \\ x_f(t) & = & \int_0^t e^{a(t-\tau)}bu(\tau)d\tau & \text{effect of the input signal} \end{array}$$

Examples

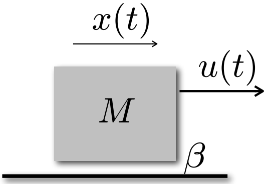
$x(t) = \text{voltage}$



$$u(t) - RC\dot{x}(t) - x(t) = 0$$

$$\dot{x}(t) = -\frac{1}{RC}x(t) - \frac{1}{RC}u(t)$$

$x(t) = \text{velocity}$



$$-\beta x(t) + u(t) = M\dot{x}(t)$$

$$\dot{x}(t) = -\frac{\beta}{M}x(t) + \frac{1}{M}$$

Continuous-time linear systems

- System of n first-order differential equations with inputs

$$\begin{cases} \dot{x}_1(t) & = & a_{11}x_1(t) + \ \ldots \ + a_{1n}x_n(t) & +b_1u(t) \\ \dot{x}_2(t) & = & a_{21}x_1(t) + \ \ldots \ + a_{2n}x_n(t) & +b_2u(t) \\ \vdots & & \vdots & \vdots \\ \dot{x}_n(t) & = & a_{n1}x_1(t) + \ \ldots \ + a_{nn}x_n(t) & +b_nu(t) \\ x_1(0) = x_{10}, & \ldots & x_n(0) = x_{n0} & \end{cases}$$

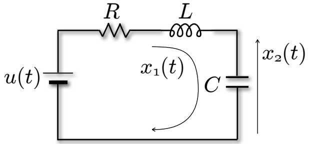
- Setting $x = [x_1 \ \ldots \ x_n]' \in \mathbb{R}^n$, the equivalent matrix form is the so-called *linear system*

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)}$$

with initial condition

$$x(0) = x_0 = [x_{10} \ \ldots \ x_{n0}]' \in \mathbb{R}^n$$

Example: RLC circuit



$$\begin{cases} u(t) - Rx_1(t) - L\frac{dx_1(t)}{dt} - x_2(t) = 0 & \text{Kirchhoff's voltage law} \\ x_1(t) = C\frac{dx_2(t)}{dt} & \text{Kirchhoff's current law} \end{cases}$$

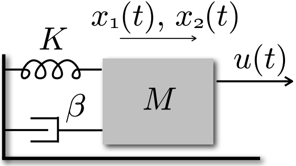
Rewrite as the 2nd order linear system

$$\begin{cases} \frac{dx_1(t)}{dt} = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) + \frac{1}{L}u(t) \\ \frac{dx_2(t)}{dt} = \frac{1}{C}x_1(t) \end{cases}$$

or in matrix form

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} u(t)$$

Example: Mass-spring-damper system



$$\begin{cases} \dot{x}_1(t) = x_2(t) & \text{velocity = derivative of traveled distance} \\ M\dot{x}_2(t) = u - \beta x_2(t) - Kx_1(t) & \text{Newton's law} \end{cases}$$

Rewrite as the 2[nd] order linear system

$$\begin{cases} \frac{dx_1(t)}{dt} = x_2(t) \\ \frac{dx_2(t)}{dt} = -\frac{\beta}{M}x_2(t) - \frac{K}{M}x_1(t) + \frac{1}{M}u(t) \end{cases}$$

or in matrix form

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{\beta}{M} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u(t)$$

Linear algebra recalls

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$ square matrix of order n, $A \in \mathbb{R}^{n\times n}$

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$ identity matrix of order n

- Characteristic equation of A:

$$\det(\lambda I - A) = 0$$

- Characteristic polynomial of A:

$$P(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

Linear algebra recall

- The *eigenvalues* of $A \in \mathbb{R}^{n\times n}$ are the roots $\lambda_1, \ldots, \lambda_n$ of its characteristic polynomial

$$\det(\lambda_i I - A) = 0, \quad i = 1, 2, \ldots, n$$

- An *eigenvector* of A is any vector $v_i \in \mathbb{R}^n$ such that

$$Av_i = \lambda_i v_i$$

  for some $i = 1, 2, \ldots, n$.

- Diagonalization of A:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = T^{-1}AT, \ T = \left[v_1 | v_2 | \ldots | v_n\right]$$

  (not all matrices A are diagonalizable, see Jordan normal form)

Linear algebra recall

- Example:

$$A = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}, \quad \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -3 \\ 5 & \lambda - 2 \end{vmatrix} = \lambda^2 - 3\lambda + 17$$

Eigenvalues: $\lambda_1 = \frac{3}{2} + j\frac{\sqrt{59}}{2}$, $\lambda_2 = \frac{3}{2} - j\frac{\sqrt{59}}{2}$

- Complex numbers recall:
  - *Imaginary unit*: $j \triangleq \sqrt{-1}$
  - *Cartesian form*: $c = a + jb, \quad c \in \mathbb{C}, a, b \in \mathbb{R}$
  - *Real part* of c: $\Re c = a$
  - *Imaginary part* of c: $\operatorname{Im} c = b$
  - *Conjugate* of c: $\bar{c} = a - jb$
  - *Polar form*: $c = \rho e^{j\theta}, \quad \rho \geq 0, \theta \in \mathbb{R}$
  - *Modulus* or *magnitude*: $|c| = \sqrt{a^2 + b^2} = \rho$
  - *Angle* or *phase*: $\angle c = \theta$
  - *Complex exponential*: $e^c = e^{a+jb} = e^a e^{jb} = e^a(\cos b + j \sin b)$

Lagrange's formula

- For the continuous-time linear system $\dot{x} = Ax + Bu$ with initial condition $x(0) = x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$

$$x(t) = \underbrace{e^{At}x_0}_{\text{natural response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$$

- The *exponential matrix* is defined as

$$e^{At} \triangleq I + At + \frac{A^2t^2}{2} + \ldots + \frac{A^nt^n}{n!} + \ldots$$

MATLAB

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» E=expm(A*t)
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- If $A \in \mathbb{R}^{n\times n}$ is diagonalizable, $A = T\Lambda T^{-1}$, then

$$\Lambda = T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \Rightarrow e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

Eigenvalues and modes

- Let $u(t) \equiv 0$ and assume A diagonalizable
- The state trajectory is the natural response

$$\begin{aligned} x(t) &= e^{At}x(0) = Te^{\Lambda t}\underbrace{T^{-1}x_0}_{\alpha} = [v_1 \dots v_n] \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ & \ddots & \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix} \alpha \\ &= \begin{bmatrix} v_1 e^{\lambda_1 t} & \dots & v_n e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i \end{aligned}$$

where v_i=eigenvector of A, λ_i=eigenvalue of A, $\alpha = T^{-1}x(0) \in \mathbb{R}^n$

- The evolution of the system depends on the eigenvalues λ_i of A, called *modes* of the system (sometimes we also refer to $e^{\lambda_i t}$ as the i-th mode)
- A mode λ_i is called *excited* if $\alpha_i \neq 0$

Differential equations of order n

$$\frac{dy^{(n)}(t)}{dt^n} + a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}} + \cdots + a_1\dot{y}(t) + a_0 y(t) = 0$$

By setting $x_1(t) \triangleq y(t)$, $x_2(t) \triangleq \dot{y}(t)$, $\ldots$, $x_n(t) \triangleq y^{n-1}(t)$, this is equivalent to the system of n first-order equations

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & x_3(t) \\ \vdots & & \vdots \\ \dot{x}_n(t) & = & -a_0x_1(t) + \ \ldots \ - a_{n-1}x_n(t) \\ x(0) & = & [y(0)\,\dot{y}(0)\ldots\, y^{n-1}(0)]' \end{array}\right.$$

Example:

$$\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 0$$

$$\begin{array}{rcl} x_1(t) & = & y(t) \\ x_2(t) & = & \dot{y}(t) \end{array} \Rightarrow \left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & -5x_1(t) - 2x_2(t) \\ x(0) & = & [y(0)\,\dot{y}(0)]' \end{array}\right.$$

n^{th}-order linear ODE with input

$$\frac{dy^{(n)}(t)}{dt^n} + a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}} + \cdots + a_1\dot{y}(t) + a_0 y(t)$$
$$= b_{n-1}\frac{du^{(n-1)}(t)}{dt} + b_{n-2}\frac{du^{(n-2)}(t)}{dt} + \cdots + b_1\dot{u}(t) + b_0 u(t)$$

One can verify by inspection that the given n^{th}-order ODE is equivalent to the following 1^{st}-order linear system of ODEs:

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & x_3(t) \\ \vdots & & \vdots \\ \dot{x}_n(t) & = & -a_0x_1(t) + \ \ldots \ - a_{n-1}x_n(t) + u(t) \\ y(t) & = & b_0x_1(t) + \ \ldots \ + b_{n-1}x_n(t) \end{array}\right. \quad \Rightarrow$$

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \end{bmatrix}, \; D = 0$$

The operation of transforming a n^{th}-order ODE into a linear system of 1^{st}-order ODEs is called *state-space realization*. There are infinitely many realizations.

Examples of state-space realizations

- Example 1

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t) + 2\dot{u}(t)$$

$$\Rightarrow \left\{ \begin{array}{rcl} \frac{d}{dt}x(t) & = & \left[\begin{array}{cc} 0 & 1 \\ -1 & 2 \end{array}\right] x(t) + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u(t) \\ y(t) & = & \left[\begin{array}{cc} 1 & 2 \end{array}\right] x(t) \end{array} \right.$$

Double check:

$$\begin{array}{rcl} \dot{y} & = & \left[\begin{array}{cc} 1 & 2 \end{array}\right]\dot{x} = \left[\begin{array}{cc} 1 & 2 \end{array}\right]\left(\left[\begin{array}{cc} 0 & 1 \\ -1 & 2 \end{array}\right] x(t) + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u(t)\right) = \left[\begin{array}{cc} -2 & 5 \end{array}\right] x(t) + 2u(t) \\ \ddot{y} & = & \left[\begin{array}{cc} -2 & 5 \end{array}\right]\dot{x} + 2\dot{u} = \left[\begin{array}{cc} -5 & 8 \end{array}\right] x(t) + 5u(t) + 2\dot{u}(t) \end{array}$$

$$\begin{array}{rcl} \ddot{y}(t) - 2\dot{y}(t) + y(t) & = & \left[\begin{array}{cc} -5 & 8 \end{array}\right] x(t) + 5u(t) + 2\dot{u}(t) - 2\left(\left[\begin{array}{cc} -2 & 5 \end{array}\right] x(t) + 2u(t)\right) \\ & & + \left[\begin{array}{cc} 1 & 2 \end{array}\right] x(t) \\ & = & \left[\begin{array}{cc} -5+4+1 & 8-10+2 \end{array}\right] x(t) + (5-4)u(t) + 2\dot{u}(t) \\ & = & u(t) + 2\dot{u}(t) \quad \text{ok!} \end{array}$$

Alternative state-space realization method

- In the following special case (=no input derivatives)

$$\frac{dy^{(n)}(t)}{dt^n}+a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}}+\cdots+a_1\dot{y}(t)+a_0y(t)=b_0u(t)$$

we can define the following states

$$\begin{array}{rclcl} x_1 & = & y & \rightarrow & \dot{x}_1=x_2 \\ x_2 & = & \dot{y} & \rightarrow & \dot{x}_2=x_3 \\ \vdots & = & \vdots & & \\ x_n & = & \frac{d^{n-1}y}{dt^{n-1}} & \rightarrow & \dot{x}_n=\frac{d^ny}{dt^n}=-a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}}-\cdots-a_1\dot{y}(t)-a_0y(t)+b_0u(t) \end{array}$$

and therefore set

$$A=\left[\begin{array}{ccccc} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{array}\right],\ B=\left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{array}\right]$$

$$C=\left[\begin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \end{array}\right],\ D=0$$

Other state-space realization methods

- The following state-space realization is called *controllable canonical form* [1]

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}, \; D = 0$$

- The following state-space realization is called *observable canonical form* [2]

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \; D = 0$$

[1] We will see later in the course that the pair (A, B) is completely *reachable*.

[2] We will see later in the course that the pair (A, B) is completely *observable*.

State vector

- Generally speaking, the *state* of a dynamical system is a set of variables that completely summarizes the past history of the system, and allows us to predict its future motion
- For the linear system

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+Bu(t)\\ y(t) & = & Cx(t)+Du(t)\end{array}\right.$$

  given the initial state $x(0)$ and the input signal $u(t)$, $\forall t \in [0,T]$, we already know how to compute the state $x(t)$ and the output $y(t)$ of the system, $\forall t \in [0,T]$.
- If we know the initial state $x(0)$, we can neglect the past history $u(-t)$, $x(-t)$, $\forall t \geq 0$
- The dimension n of the state $x(t) \in \mathbb{R}^n$ is called the *order* of the system

Some classes of dynamical systems

- *Causality*: a dynamical system is *causal* if $y(t)$ does not depend on future inputs $u(\tau)$ $\forall \tau > t$ (*strictly causal* if $\forall \tau \geq t$)
- A linear system is always causal, and strictly causal iff $D = 0$
- *Linear time-varying (LTV) systems*:

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{cases}$$

- When A, B, C, D are constant, the system is said *linear time-invariant* (LTI)
- *Multivariable systems*: more generally, a system can have m inputs ($u(t) \in \mathbb{R}^m$) and p outputs ($y(t) \in \mathbb{R}^p$). For linear systems, we still have

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

with

$$A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m},\ C \in \mathbb{R}^{p \times n},\ D \in \mathbb{R}^{p \times m}$$

Some classes of dynamical systems

- *Nonlinear systems*

$$\begin{cases} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & g(x(t), u(t)) \end{cases}$$

  where $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $g : \mathbb{R}^{n+m} \to \mathbb{R}^p$ are (rather arbitrary) nonlinear functions

- *Time-varying nonlinear systems* are very general classes of dynamical systems

$$\begin{cases} \dot{x}(t) & = & f(t, x(t), u(t)) \\ y(t) & = & g(t, x(t), u(t)) \end{cases}$$

Equilibrium

- Consider the continuous-time nonlinear system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$$

Definition

A state $x_r \in \mathbb{R}^n$ and an input $u_r \in \mathbb{R}^m$ are an *equilibrium pair* if for initial condition $x(0) = x_r$ and constant input $u(t) \equiv u_r$ the state remains constant: $x(t) \equiv x_r$, $\forall t \geq 0$

- Equivalent definition: (x_r, u_r) is an equilibrium pair if $f(x_r, u_r) = 0$
- x_r is called *equilibrium state*, u_r *equilibrium input*
- The definition generalizes to time-varying nonlinear systems

Stability

- Consider the nonlinear system

$$\begin{cases} \dot{x}(t) &= f(x(t), u_r) \\ y(t) &= g(x(t), u_r) \end{cases}$$

and let x_r an equilibrium state, $f(x_r, u_r) = 0$

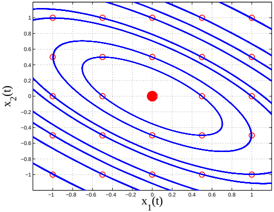
Definition

The equilibrium state x_r is *stable* if for each initial conditions $x(0)$ "close enough" to x_r, the corresponding trajectory $x(t)$ remains near x_r for all $t \geq 0$ [a]

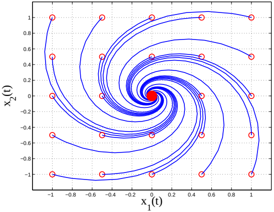
[a] Analytic definition: $\forall \epsilon > 0\ \exists \delta > 0 : \|x(0) - x_r\| < \delta \Rightarrow \|x(t) - x_r\| < \epsilon,\ \forall t \geq 0$

- The equilibrium point x_r is called *asymptotically stable* if it is stable and $x(t) \to x_r$ for $t \to \infty$
- Otherwise, the equilibrium point x_r is called *unstable*

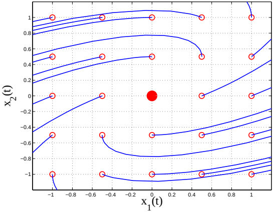
Stability of equilibria - Examples



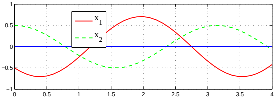
stable equilibrium

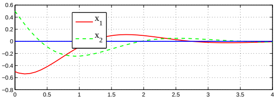


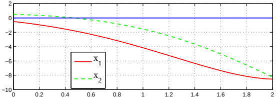
asymptotically stable equilibrium



unstable equilibrium







$$\frac{dx}{dt} = \begin{bmatrix} -2x_1(t) - 4x_2(t) \\ 2x_1(t) + 2x_2(t) \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} -x_1(t) - 2x_2(t) \\ 2x_1(t) - x_2(t) \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} 2x_1(t) - 2x_2(t) \\ x_1(t) \end{bmatrix}$$

Stability of first-order linear systems

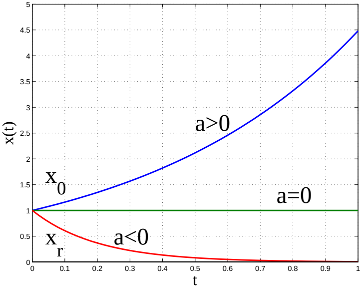
- Consider the first-order linear system

$$\dot{x}(t) = ax(t) + bu(t)$$

- $x_r = 0$, $u_r = 0$ is an equilibrium pair
- For $u(t) \equiv 0$, $\forall t \geq 0$, the solution is

$$x(t) = e^{at}x_0$$

- The origin $x_r = 0$ is
  - unstable if $a > 0$
  - stable if $a \leq 0$
  - asymptotically stable if $a < 0$



Stability of continuous-time linear systems

Since the natural response of $\dot{x} = Ax + Bu$ is $x(t) = e^{At}x_0$, the stability properties depend only on A. We can therefore talk about *system stability* of a linear system (A, B, C, D)

Theorem:

Let $\lambda_1, \ldots, \lambda_m$, $m \leq n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$. The system $\dot{x} = Ax + Bu$ is

- asymptotically stable iff $\Re\lambda_i < 0$, $\forall i = 1, \ldots, m$
- (marginally) stable if $\Re\lambda_i \leq 0$, $\forall i = 1, \ldots, m$, and the eigenvalues with null real part have equal algebraic and geometric multiplicity [a]
- unstable if $\exists\ i$ such that $\Re\lambda_i > 0$

[a] Algebraic multiplicity of λ_i = number of coincident roots λ_i of $\det(\lambda I - A)$. Geometric multiplicity of λ_i = number of linearly independent eigenvectors v_i, $Av_i = \lambda_i v_i$

The stability properties of a linear system only depend on the **real part** of the eigenvalues of matrix A

Stability of continuous-time linear systems

Proof:

- The natural response is $x(t) = e^{At}x_0$ $\left(e^{At} \triangleq I + At + \frac{A^2t^2}{2} + \ldots + \frac{A^nt^n}{n!} + \ldots\right)$
- If matrix A is diagonalizable[3] , $A = T\Lambda T^{-1}$,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \Rightarrow e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

- Take any eigenvalue $\lambda = a + jb$:

$$|e^{\lambda t}| = e^{at}|e^{jbt}| = e^{at}$$

- A is always diagonalizable if algebraic multiplicity = geometric multiplicity

$\square$

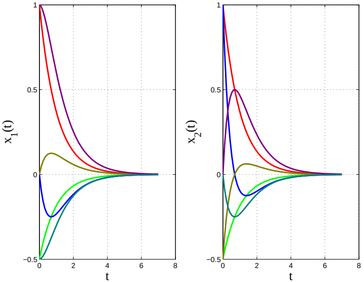
[3] If A is not diagonalizable, it can be transformed to Jordan form. In this case the natural response $x(t)$ contains modes $t^j e^{\lambda t}$, $j = 0, 1, \ldots,$ alg. multiplicity - geom. multiplicity

Example 1

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} x(t) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases} \Rightarrow \text{ eigenvalues of } A\text{: } \{-1, -2\}$$

solution:

$$\begin{cases} x_1(t) &= x_{10}(2e^{-t} - e^{-2t}) + x_{20}(-e^{-t} + e^{-2t}) \\ x_2(t) &= x_{10}(2e^{-t} - 2e^{-2t}) + x_{20}(-e^{-t} + 2e^{-2t}) \end{cases}$$



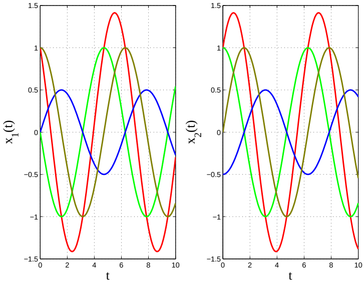
asymptotically stable

Example 2

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases} \Rightarrow \text{ eigenvalues of } A\text{: } \{+j, -j\}$$

solution:

$$\begin{cases} x_1(t) &= x_{10}\cos t - x_{20}\sin t \\ x_2(t) &= x_{10}\sin t + x_{20}\cos t \end{cases}$$



marginally stable

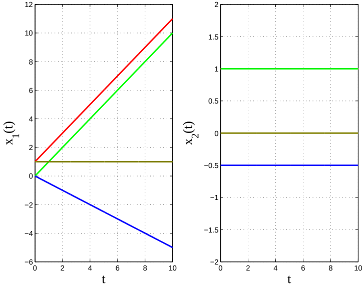
Example 3

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases} \Rightarrow \text{ eigenvalues of } A\text{: } \{0,0\}$$

solution:

$$\begin{cases} x_1(t) &= x_{10} + x_{20}t \\ x_2(t) &= x_{20} \end{cases}$$

Note: A is not diagonalizable !



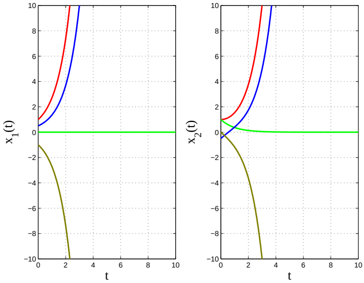
unstable

Example 4

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x(t) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases} \Rightarrow \text{ eigenvalues of } A\text{: } \{-1, 1\}$$

solution:

$$\begin{cases} x_1(t) &= x_{10}e^t \\ x_2(t) &= \frac{1}{2}x_{10}e^t + (x_{20} - \frac{1}{2}x_{10})e^{-t} \end{cases}$$



unstable

Linearization of nonlinear systems

- Consider the nonlinear system

$$\begin{cases} \dot{x}(t) & = & f(x(t),u(t)) \\ y(t) & = & g(x(t),u(t)) \end{cases}$$

- Let (x_r, u_r) be an equilibrium, $f(x_r, u_r) = 0$
- Objective: investigate the dynamic behaviour of the system for small perturbations $\Delta u(t) \triangleq u(t) - u_r$ and $\Delta x(0) \triangleq x(0) - x_r$.
- The evolution of $\Delta x(t) \triangleq x(t) - x_r$ is given by

$$\begin{aligned} \dot{\Delta} x(t) &= \dot{x}(t) - \dot{x}_r = f(x(t), u(t)) \\ &= f(\Delta x(t) + x_r, \Delta u(t) + u_r) \\ &\approx \underbrace{\frac{\partial f}{\partial x}(x_r, u_r)}_{A} \Delta x(t) + \underbrace{\frac{\partial f}{\partial u}(x_r, u_r)}_{B} \Delta u(t) \end{aligned}$$

Linearization of nonlinear systems

- Similarly

$$\Delta y(t) \approx \underbrace{\frac{\partial g}{\partial x}(x_r, u_r)}_{C} \Delta x(t) + \underbrace{\frac{\partial g}{\partial u}(x_r, u_r)}_{D} \Delta u(t)$$

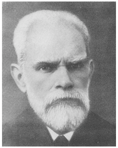
  where $\Delta y(t) \triangleq y(t) - g(x_r, u_r)$ is the perturbation of the output from its equilibrium

- The perturbations $\Delta x(t)$, $\Delta y(t)$, and $\Delta u(t)$ are (approximately) ruled by the *linearized system*

$$\begin{cases} \Delta\dot{x}(t) & = & A\Delta x(t) + B\Delta u(t) \\ \Delta y(t) & = & C\Delta x(t) + D\Delta u(t) \end{cases}$$

Lyapunov's indirect method

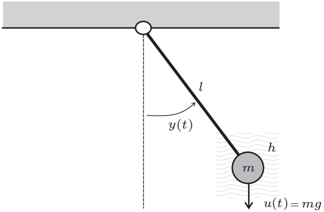
- Consider the nonlinear system $\dot{x} = f(x)$, with f differentiable, and assume $x = 0$ is equilibrium point ($f(0) = 0$)
- Consider the linearized system $\dot{x} = Ax$, with $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$
- If $\dot{x} = Ax$ is asymptotically stable, then the origin $x = 0$ is also an asymptotically stable equilibrium for the nonlinear system (locally)
- If $\dot{x} = Ax$ is unstable, then the origin $x = 0$ is an unstable equilibrium for the nonlinear system
- If A is marginally stable, nothing can be said about the stability of the origin $x = 0$ for the nonlinear system



Aleksandr Mikhailovich Lyapunov

(1857-1918)

Example: Pendulum



$y(t)$ = angular displacement
$\dot{y}(t)$ = angular velocity
$\ddot{y}(t)$ = angular acceleration
$u(t) = mg$ gravity force
$h\dot{y}(t)$ = viscous friction torque
l = pendulum length
ml^2 = pendulum rotational inertia

- mathematical model

$$ml^2\ddot{y}(t) = -lmg\sin y(t) - h\dot{y}(t)$$

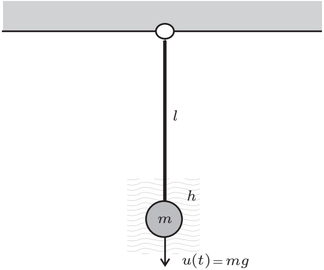
- in state-space form ($x_1 = y$, $x_2 = \dot{y}$)

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\frac{g}{l}\sin x_1 - Hx_2, \end{cases} \qquad H \triangleq \frac{h}{ml^2}$$

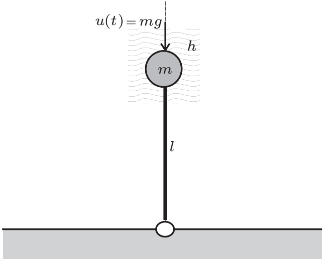
Example: Pendulum

Look for equilibrium states:

$$\begin{bmatrix} x_{2r} \\ -\frac{g}{l}\sin x_{1r} - Hx_{2r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_{2r} = 0 \\ x_{1r} = \pm k\pi,\ k = 0, 1, \ldots \end{cases}$$



$x_{2r} = 0, x_{1r} = 0, \pm 2\pi, \ldots$



$x_{2r} = 0, x_{1r} = 0, \pm\pi, \pm 3\pi, \ldots$

Example: Pendulum

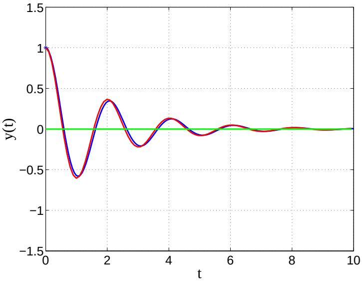
- Linearize the system around $x_{1r} = 0$, $x_{2r} = 0$

$$\Delta\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -H \end{bmatrix}}_{A} \Delta x(t)$$

- find the eigenvalues of A

$$\det(\lambda I - A) = \lambda^2 + H\lambda + \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{1}{2}\left(-H \pm \sqrt{H^2 - 4\frac{g}{l}}\right)$$

- $\Re\lambda_{1,2} < 0 \Rightarrow \dot{x} = Ax$ asymptotically stable
- by Lyapunov's indirect method $x_r = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is also an asymptotically stable equilibrium for the pendulum



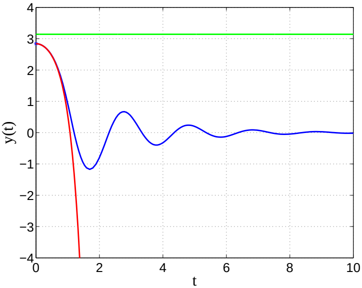
Example: Pendulum

- Linearize the system around $x_{1r} = \pi$, $x_{2r} = 0$

$$\Delta\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -H \end{bmatrix}}_{A} \Delta x(t)$$

- find the eigenvalues of A

$$\det(\lambda I - A) = \lambda^2 + H\lambda - \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{1}{2}\left(-H \pm \sqrt{H^2 + 4\frac{g}{l}}\right)$$



- $\lambda_1 < 0$, $\lambda_2 > 0 \Rightarrow \dot{x} = Ax$ unstable
- by Lyapunov's indirect method $x_r = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is also an unstable equilibrium for the pendulum

English-Italian Vocabulary

	
dynamics	*dinamica*
natural response	*risposta libera*
eigenvalue	*autovalore*
eigenvector	*autovettore*
modulus or magnitude	*modulo*
angle or phase	*fase*
nonlinear systems	*sistemi non lineari*
controllable canonical form	*forma canonica di raggiungibilità*
observable canonical form	*forma canonica di osservabilità*

Translation is obvious otherwise.