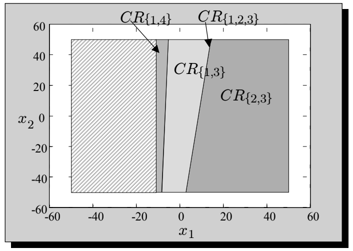
Explicit Form of Model Predictive Control

via Multiparametric Programming

Example of Multiparametric Solution

Multiparametric LP ($\xi \in \mathbb{R}^2$)

$$
\begin{array}{ll}
\min_{\xi} & -3\xi_1 - 8\xi_2 \\
\text{s.t.} & \left\{\begin{array}{lcl}
\xi_1 + \xi_2 & \leq & 13 + x_1 \\
5\xi_1 - 4\xi_2 & \leq & 20 \\
-8\xi_1 + 22\xi_2 & \leq & 121 + x_2 \\
-4\xi_1 - \xi_2 & \leq & -8 \\
-\xi_1 & \leq & 0 \\
-\xi_2 & \leq & 0
\end{array}\right.
\end{array}
$$



$$
\xi(x) = \left\{\begin{array}{llll}
\begin{bmatrix} 0.00 & 0.05 \\ 0 & 0.06 \end{bmatrix} x + \begin{bmatrix} 11.85 \\ 9.80 \end{bmatrix} & \text{if} & \begin{bmatrix} 0.02 & 0.00 \\ 0.00 & 0.02 \\ 0.00 & -0.02 \\ -0.12 & 0.01 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ -1.00 \end{bmatrix} & \text{CR}_{\{2,3\}} \\
\begin{bmatrix} 0.73 & -0.03 \\ 0.27 & 0.03 \end{bmatrix} x + \begin{bmatrix} 5.50 \\ 7.50 \end{bmatrix} & \text{if} & \begin{bmatrix} 0.00 & 0.02 \\ 0.00 & -0.02 \\ 0.12 & -0.01 \\ -0.15 & 0.00 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} & \text{CR}_{\{1,3\}} \\
\begin{bmatrix} -0.33 & 0.00 \\ 1.33 & 0 \end{bmatrix} x + \begin{bmatrix} -1.67 \\ 14.67 \end{bmatrix} & \text{if} & \begin{bmatrix} 0.00 & 0.02 \\ 0.00 & -0.02 \\ 0.15 & -0.00 \\ -0.09 & 0.00 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ -1.00 \\ 1.00 \end{bmatrix} & \text{CR}_{\{1,4\}}
\end{array}\right.
$$

Finite-Time Constrained LQR

$$
\begin{aligned}
x(t+1) &= Ax(t) + B_1 u(t) \\
y(t) &= Cx(t) + D_1 u(t)
\end{aligned}
$$

Linear Model

$$\min_U J(U, x(0)) \triangleq \sum_{k=0}^{T-1} \|y(k)\|_Q^2 + \|u(k)\|_R^2$$

Quadratic Performance Index

$$\min_U J(U, x(0)) = \tfrac{1}{2} U'HU + x'(0)FU + \cancel{\tfrac{1}{2} x'(0)Yx(0)}$$

$$\text{subject to } \; GU \leq W + Kx(0) \qquad U \triangleq \{u(0), u(1), \ldots, u(T-1)\}$$

$$
\begin{aligned}
\text{subject to } \; u_{\min} &\leq u(k) \leq u_{\max} \\
y_{\min} &\leq y(k) \leq y_{\max}
\end{aligned}
$$

Constraints

(convex) **QUADRATIC PROGRAM (QP)**

We want to show that: The optimal solution $u(k) = f_k(x(0))$ can be determined in closed form and is continuous and piecewise affine.

Multiparametric Quadratic Programming

(Bemporad et al., *Automatica*, in press)

$$
\begin{array}{ll}
\min_U & \frac{1}{2} U'HU + x'FU + \cancel{\frac{1}{2} x'Yx} \\
\text{subj.to} & GU \leq W + Kx
\end{array}
$$

$U \triangleq \{u(0), u(1), \ldots, u(N-1)\}$

$U \in \mathbf{R}^d \qquad x \in X \subseteq \mathbf{R}^n$

- Objective: solve the QP **for all** x
- Coordinate transformation: $z \triangleq U + H^{-1}F'x$

$$
\begin{array}{ll}
\min_z & \frac{1}{2} z'Hz \\
\text{subj.to} & Gz \leq W + Sx \\
& S \triangleq (K + GH^{-1}F')
\end{array}
$$

$z \in \mathbf{R}^d$

z = optimization variables, *x* = parameters

Linearity of Solution

$x_0 \in X$ ⇨ solve QP to find (z_0, λ_0)

$$\min_z \frac{1}{2} z' H z \quad \text{subj.to } Gz \le W + Sx$$

⇨ identify active constraints

⇨ form matrices $\tilde{G}, \tilde{W}, \tilde{S}$ by collecting active constraints

$\tilde{G}z - \tilde{W} - \tilde{S}x = 0, \ \tilde{\lambda} \ge 0$

KKT optimality conditions:

$$\begin{array}{l} Hz + G'\lambda = 0, \ \lambda \in \mathbf{R}^q \\ \lambda_i (G^i z - W^i - S^i x) = 0 \\ \lambda_i \ge 0, \ i = 1, \ldots, q \end{array}$$

➡

$$\begin{array}{ll} Hz + \tilde{G}'\tilde{\lambda} = 0 & (1) \\ \tilde{G}z - \tilde{W} - \tilde{S}x = 0 & (2) \end{array}$$

From (1) : $z = -H^{-1}\tilde{G}'\tilde{\lambda}$

From (2) :

$$\tilde{\lambda}(x) = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x)$$

$$z(x) = H^{-1}\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x)$$

➡ In some neighborhood of x_0, λ and z are explicit linear functions of x

Determining a Critical Region

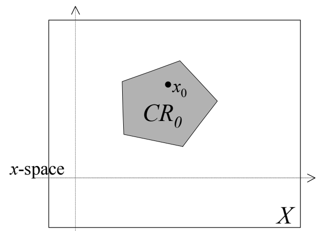
- Substitute $\tilde{\lambda}(x) = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x)$ and $z(x) = H^{-1}\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x)$

  into the constraints $\begin{array}{l} \tilde{\lambda}(x) \ge 0 \\ \tilde{G}z(x) \le \tilde{W} + \tilde{S}x \end{array}$

- Remove redundant constraints

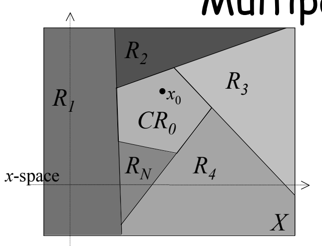
➡ critical region CR_0

$CR_0 = \{\mathcal{A}x \le \mathcal{B}\}$



- CR_0 is the set of all and only parameters x for which $\tilde{G}, \tilde{W}, \tilde{S}$ is the optimal combination of active constraints at the optimizer

Multiparametric QP



$CR_0 = \{\mathcal{A}x \le \mathcal{B}\}$

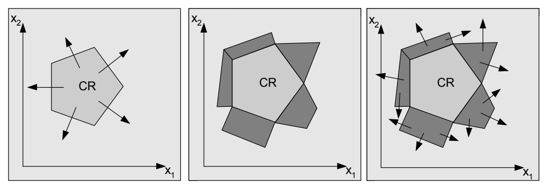
$R_i = \{x \in X : \ \mathcal{A}^i x > \mathcal{B}^i, \ \mathcal{A}^j z \le \mathcal{B}^j, \ \forall j < i\}$

Theorem: $\{CR_0, R_1, \ldots, R_N\}$ is a partition of $X \subseteq \mathbf{R}^n$

Proceed iteratively: for each region R_i repeat the whole procedure with $X \leftarrow R_i$

Alternative: New mpQP Algorithm, based on exploring neighboring regions (much more efficient !!!) (Tondel,Johansen,Bemporad, CDC2001)

Multiparametric QP



The active set of a neighboring region is found by using the active set of the current region + knowledge of the type of hyperplane we are crossing:

$Gz \le W + Sx$ ⇒ The corresponding constraint is added to the active set

$\tilde{\lambda} \ge 0$ ⇒The corresponding constraint is withdrawn from the active set

(Tøndel, Johansen, Bemporad, CDC 2001)

Convexity and Continuity

Theorem 4 *Consider the multi-parametric quadratic program (21) and let $H \succ 0$. Then the optimizer $z(x)$ is continuous and piecewise affine, and the optimal solution $V(x)$ is continuous, convex and piecewise quadratic.*

(Bemporad et al., *Automatica*, in press)

$$z(x) = \arg\min_z \frac{1}{2} z' H z \quad \text{subj.to } Gz \leq W + Sx$$

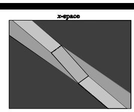
continuous, piecewise affine

$$V(x) = \min_z \frac{1}{2} z' H z \quad \text{subj.to } Gz \leq W + Sx$$

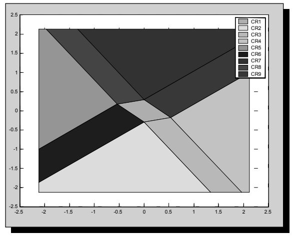
convex, continuous, piecewise quadratic

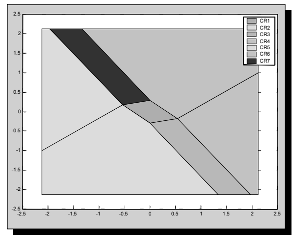
Corollary: The optimal solution $u(0) = f(x(0))$ is continuous and piecewise affine in *x(0)*

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \leq K_N \end{cases}$$



Union of Regions



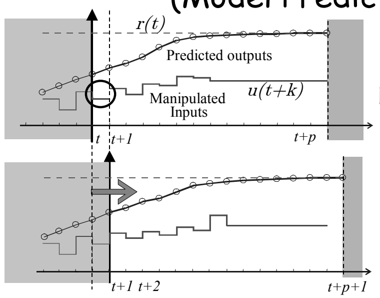


$$U(x(0)) \triangleq [u_0(x(0)), \ldots, u_{T-1}(x(0))]$$

- Regions where the first component of the solution is the same can be joined (when their union is convex). (Bemporad, Fukuda, Torrisi, *Computational Geometry*, 2001)

Corollary: By using Bellman's principle of optimality, the optimal solution $u(k) = f_k(x(k))$ is also a continuous and PWA feedback law

Receding Horizon Control
(Model Predictive Control)

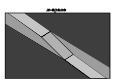


At each time *t*, only apply the first optimal move of the finite horizon optimal control problem

$$u(t) = f_0(x(t))$$

Corollary: The MPC controller $u(t) = f_0(x(t))$ is a continuous and piecewise affine state-feedback

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \leq K_N \end{cases}$$



Double Integrator Example

- System: $y(t) = \frac{1}{s^2} u(t)$ $\xrightarrow[T=1\text{ s}]{\text{sampling + ZOH}}$ $x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$, $\quad y(t) = [1 \quad 0]\, x(t)$

- Constraints: $-1 \leq u(t) \leq 1$

- Control objective: minimize $\sum_{t=0}^{\infty} y'(t) y(t) + \frac{1}{100} u^2(t)$

  $u_{t+k} = K_{LQ}\; x(t+k|t) \;\; \forall k \geq N_u$

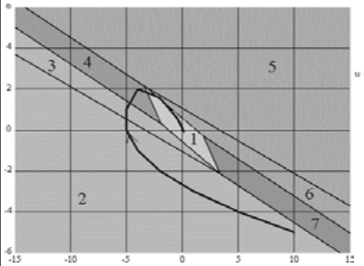
- Optimization problem: for N_u=2

$$H = \begin{bmatrix} 11.0932 & 5.4117 \\ 5.4117 & 4.3306 \end{bmatrix} \quad F = \begin{bmatrix} 4.6815 & 2.0810 \\ 10.0932 & 5.4117 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

mp-QP solution

N_u=2

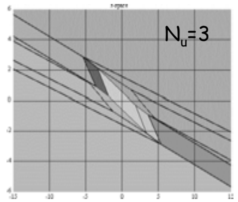


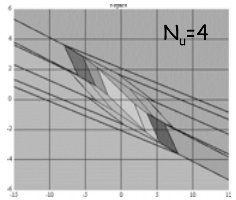
$$u = \begin{cases} [-0.4805\ -0.7691]\,x & \text{if } \begin{bmatrix} -0.4805 & -0.7691 \\ 0.4805 & 0.7691 \\ 0.1200 & -0.2886 \\ -0.1200 & 0.2886 \end{bmatrix} x \le \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix} \quad \text{(Region \#1)} \\ 1.0000 & \text{if } \begin{bmatrix} 2.0810 & 5.4117 \\ 4.6815 & 10.0932 \end{bmatrix} x \le \begin{bmatrix} -9.7423 \\ -16.5048 \end{bmatrix} \quad \text{(Region \#2)} \\ [-0.4220\ -0.9099]\,x + 0.4878 & \text{if } \begin{bmatrix} -0.4220 & -0.9099 \\ 0.2028 & -0.4878 \\ 0.4220 & 0.9099 \end{bmatrix} x \le \begin{bmatrix} 0.5122 \\ -1.6906 \\ 1.4878 \end{bmatrix} \quad \text{(Region \#3)} \\ 1.0000 & \text{if } \begin{bmatrix} -0.4805 & -1.2496 \\ 2.0810 & 3.3306 \\ 4.6815 & 10.0932 \end{bmatrix} x \le \begin{bmatrix} 2.2496 \\ -4.3306 \\ -5.6815 \end{bmatrix} \quad \text{(Region \#4)} \\ -1.0000 & \text{if } \begin{bmatrix} -4.6815 & -10.0932 \\ -2.0810 & -5.4117 \end{bmatrix} x \le \begin{bmatrix} -16.5048 \\ -9.7423 \end{bmatrix} \quad \text{(Region \#5)} \\ [-0.4220\ -0.9099]\,x - 0.4878 & \text{if } \begin{bmatrix} 0.4220 & 0.9099 \\ -0.2028 & 0.4878 \\ -0.4220 & -0.9099 \end{bmatrix} x \le \begin{bmatrix} 0.5122 \\ -1.6906 \\ 1.4878 \end{bmatrix} \quad \text{(Region \#6)} \\ -1.0000 & \text{if } \begin{bmatrix} 0.4805 & 1.2496 \\ -2.0810 & -3.3306 \\ -4.6815 & -10.0932 \end{bmatrix} x \le \begin{bmatrix} 2.2496 \\ -4.3306 \\ -5.6815 \end{bmatrix} \quad \text{(Region \#7)} \end{cases}$$

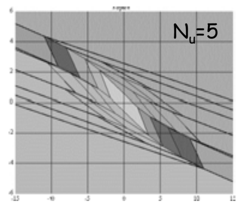
Explicit Controller ≡ Original MPC controller !

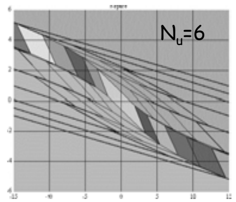
(piecewise affine law) (implicit law: need to solve QP on line)

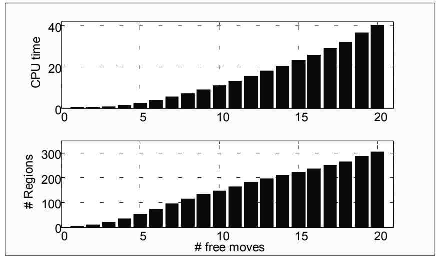
Complexity











Extensions

- Tracking of reference *r(t)* : $\delta u(t) = F(x(t), u(t-1), r(t))$
- Rejection of measured disturbance *v(t)* : $\delta u(t) = F(x(t), u(t-1), v(t))$
- Soft constraints: $u(t) = F(x(t))$

  $y_{\min} - \epsilon \le y(t+k|t) \le y_{\max} + \epsilon$
- Variable constraints: $u(t) = F(x(t), u_{\min}(t), \ldots, y_{\max}(t))$

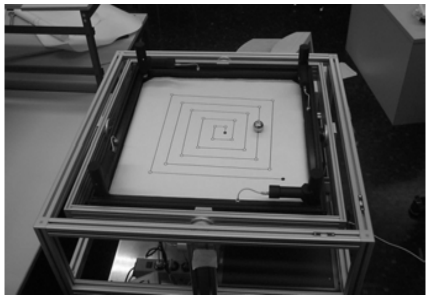
  $u_{\min}(t) \le u(t+k) \le u_{\max}(t)$
  $y_{\min}(t) \le y(t+k|t) \le y_{\max}(t)$
- Linear norms: $\min_U J(U, x(t)) \triangleq \sum_{k=0}^{p} \|Qy(t+k|t)\|_\infty + \|Ru(t+k)\|_\infty$

(Bemporad, Borrelli, Morari, 2000)

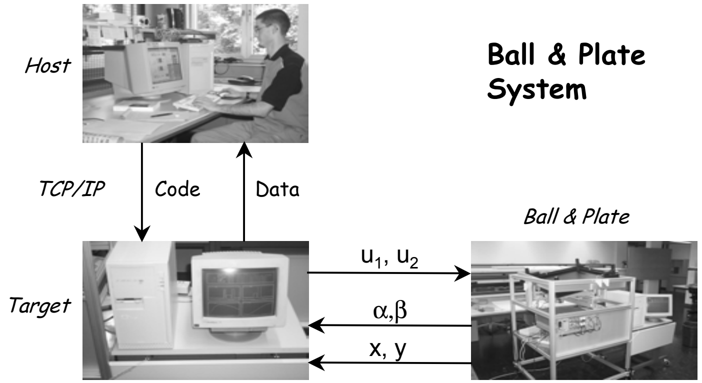
MPC Regulation of a Ball on a Plate

Task:

- Tune an MPC controller by simulation, using the ***MPC Simulink Toolbox***
- Get the ***explicit solution*** of the MPC controller.
- Validate the controller on ***experiments***.

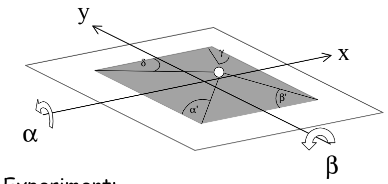


B&P Experimental Setup



Ball & Plate System

Ball&Plate Experiment

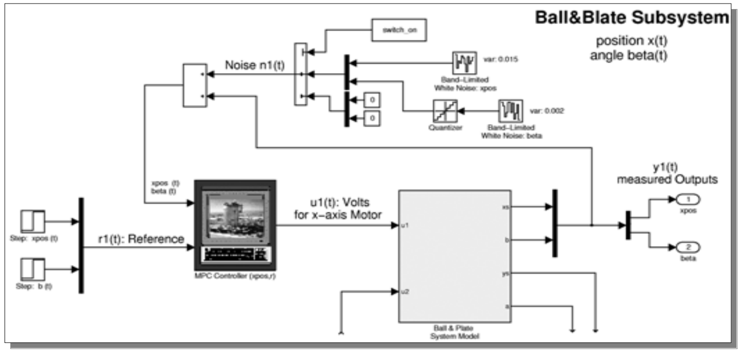


- Specification of Experiment:
  Angle: -17°… +17°   Plate: -30 cm…+30 cm
  Input Voltage: -10 V… +10 V
  Computer: PENTIUM166   Sampling Time:30 ms
- Model: LTI 14 states
  Constraints on inputs and states
- Controller   *x*: 22 Regions, *y*: 23 Regions

15th June

General Philosophy: (1) MPC Design

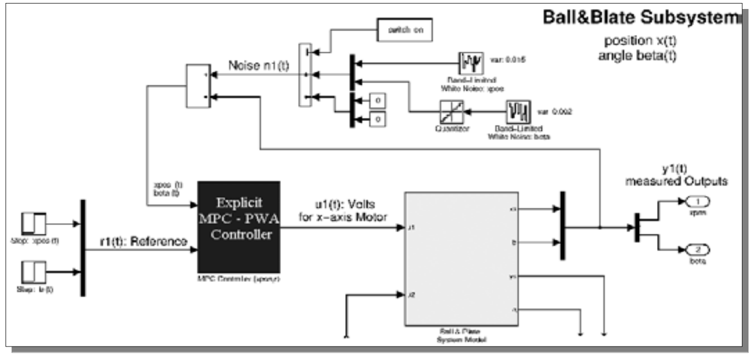
- Step 1: Tune the MPC controller (in simulation)



E.g: MPC Toolbox for Matlab   (Bemporad, Morari, Ricker, 2002)

General Philosophy: (2) Implementation

- Step 2: Solve mp-QP and implement Explicit MPC

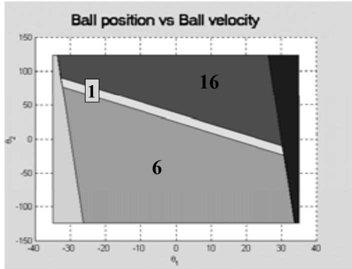


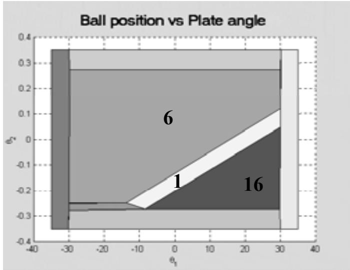
E.g: Real-Time Workshop + xPC Toolbox

Explicit MPC Solution

Controller: x: 22 Regions, y: 23 Regions

***x*-MPC:** sections at $\alpha_x=0, \dot{\alpha}_x=0, u_x=0, r_x=18, r_\alpha=0$





Region 1: LQR Controller (near Equilibrium)

Region 6: Saturation at -10

Region 16: Saturation at +10

MPC Regulation of a Ball on a Plate

Design Steps:

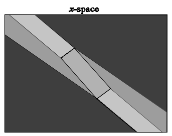
- Tune an MPC controller by simulation, using the ***MPC Simulink Toolbox*** .
- Get the ***explicit solution*** of the MPC controller.
- Validate the controller on ***experiments***.

15th June

Comments on Explicit MPC

- Multiparametric Quadratic Programs (mp-QP) can be solved efficiently
- Model Predictive Control (MPC) can be solved off-line via mp-QP
- Explicit solution of MPC controller $u = f(x)$ is Piecewise Affine

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if} \quad H_N x \leq K_N \end{cases}$$



⇒ **Eliminate heavy on-line computation for MPC**

⇒ **Make MPC suitable for fast/small/cheap processes**

MILP Formulation of MPC

$$\min \sum_{k=0}^{T-1} \|Qy(t+k+1|t)\|_\infty + \|Ru(t+k)\|_\infty$$
$$\text{s.t.MLD dynamics}$$

- Introduce slack variables: $\min |x| \Rightarrow \begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \epsilon \geq x \\ & \epsilon \geq -x \end{array}$

$$\begin{array}{lll} \epsilon_k^x & \geq & \|Qy(t+k|t)\|_\infty \\ \epsilon_k^u & \geq & \|Ru(t+k)\|_\infty \end{array} \Rightarrow \begin{array}{llll} \epsilon_k^x & \geq & [Qy(t+k|t)]_i & i=1,\ldots,p, \; k=1,\ldots,T-1 \\ \epsilon_k^x & \geq & -[Qy(t+k|t)]_i & i=1,\ldots,p, \; k=1,\ldots,T-1 \\ \epsilon_k^u & \geq & [Ru(t+k))]_i & i=1,\ldots,m, \; k=0,\ldots,T-1 \\ \epsilon_k^u & \geq & -[Ru(t+k))]_i & i=1,\ldots,m, \; k=0,\ldots,T-1 \end{array}$$

- Set $\xi \triangleq [\epsilon_1^x, \ldots, \epsilon_{N_y}^x, \epsilon_1^u, \ldots, \epsilon_{T-1}, U, \delta, z]$

Mixed Integer Linear Program (MILP)

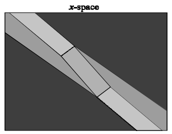
$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u$$
$$\text{s.t.} \, G\xi \leq W + Sx(t)$$

Multiparametric MILP

$$\begin{array}{ll}\min\limits_{\xi=\{\xi_c,\xi_d\}} & f'\xi_c + d'\xi_d \\ \text{s.t.} & G\xi_c + E\xi_d \le W + Fx\end{array} \qquad \begin{array}{l}\xi_c \in \mathbf{R}^n \\ \xi_d \in \{0,1\}^m\end{array}$$

- mp-MILP can be solved (by alternating MILPs and mp-LPs) (Dua, Pistikopoulos, 1999)
- **Theorem:** The multiparametric solution $\xi^*(x)$ is piecewise affine
- **Corollary: The MPC controller is piecewise affine in** x

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \le K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \le K_N \end{cases}$$



- Remarks on explicit MPC law:
    - **Automatic partitioning** of state-space (no gridding!)
    - **Stability** guarantee (value function=PWL Lyapunov function)

Hybrid Control Examples (Revisited)

Hybrid MPC - Example

Switching System:

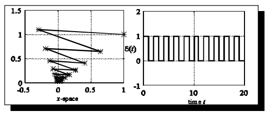
$$x(t+1) = 0.8\begin{bmatrix}\cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t)\end{bmatrix} x(t) + \begin{bmatrix}0\\1\end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1]\, x(t)$$

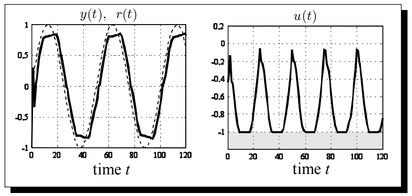
$$\alpha(t) = \begin{cases} \frac{\pi}{3} & \text{if } [1 \ \ 0]x(t) \ge 0 \\ -\frac{\pi}{3} & \text{if } [1 \ \ 0]x(t) < 0 \end{cases}$$

Constraint: $-1 \le u(t) \le 1$

Open loop:



Closed loop:



Hybrid MPC - Example

- MLD system

State $x(t)$	2 variables
Input $u(t)$	1 variables
Aux. binary vector $\delta(t)$	1 variables
Aux. continuous vector $z(t)$	4 variables

- mp-MILP optimization problem

$$\min_{\{v_0^1\}} J(v_0^1, x(t)) \triangleq \sum_{k=0}^{1} \|Q_1(v(k)-u_e)\|_\infty + \|Q_2(\delta(k|t)-\delta_e)\|_\infty + \|Q_3(z(k|t)-z_e)\|_\infty + \|Q_4(x(k|t)-x_e)\|_\infty$$

subject to constraints

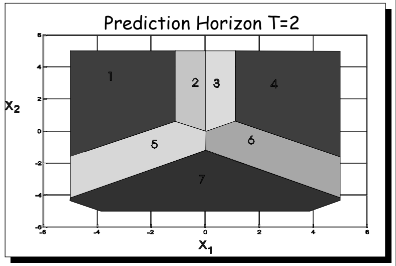
to be solved in the region $\begin{array}{rcl} -5 & \le x_1 \le & 5 \\ -5 & \le x_2 \le & 5 \end{array}$

- Computational complexity of mp-MILP

Linear constraints	84
Continuous variables	20
Binary variables	2
Parameters	2
Time to solve mp-MILP	3 min
Number of regions	7

mp-MILP Solution

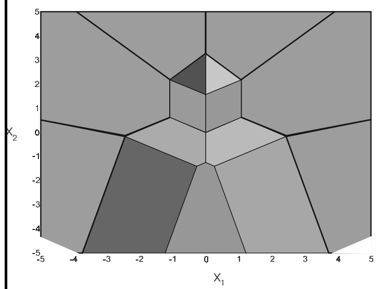
$$u = \begin{cases} -1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.7296 \\ -1.0000 & 0.0000 \\ 0.0000 & 1.0000 \\ 94.5067 & -1.0000 \end{bmatrix} x \le \begin{bmatrix} -2.1680 \\ 5.0000 \\ 5.0000 \\ -105.1385 \end{bmatrix} \\ & \text{(Region \#1)} \\ [0.9238\ 0.0000]\, x & \text{if } \begin{bmatrix} -188.1504 & 1.0000 \\ -20.3158 & -34.1997 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \le \begin{bmatrix} 204.3621 \\ 1.0000 \\ 0.0000 \\ 5.0000 \end{bmatrix} \\ & \text{(Region \#2)} \\ [-0.9238\ -0.0000]\, x & \text{if } \begin{bmatrix} -217.00 & 1.0000 \\ 38.1919 & -64.29 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \le \begin{bmatrix} 4.9708 \\ 1.0000 \\ 1.0000 \\ 5.0000 \end{bmatrix} \\ & \text{(Region \#3)} \\ -1.0000 & \text{if } \begin{bmatrix} -188.15 & -1.0000 \\ -1.0000 & -1.6858 \\ 1.0000 & 0.0000 \\ 1.0000 & 133.68 \end{bmatrix} x \le \begin{bmatrix} -204.39 \\ -2.1673 \\ 5.0000 \\ 665.60 \end{bmatrix} \\ & \text{(Region \#4)} \\ [0.4619\ -0.8000]\, x & \text{if } \begin{bmatrix} 1.0000 & -1.7193 \\ -1.0000 & 1.7295 \\ -1.0000 & 0.0000 \\ 13.4029 & 23.6383 \\ 43.4069 & 0.0000 \end{bmatrix} x \le \begin{bmatrix} 2.1387 \\ 2.1174 \\ 5.0000 \\ -1.0000 \\ -1.0000 \end{bmatrix} \\ & \text{(Region \#5)} \\ [-0.4619\ -0.8000]\, x & \text{if } \begin{bmatrix} 1.0000 & 1.7175 \\ -20.2527 & 34.1997 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.7369 \\ 106.3247 & 1.0000 \end{bmatrix} x \le \begin{bmatrix} 2.0413 \\ -1.0000 \\ 0.0000 \\ 2.2345 \\ 525.0259 \end{bmatrix} \\ & \text{(Region \#6)} \\ 1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.6446 \\ -1.0000 & 1.7449 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.6788 \\ 1.0000 & -265.137 \\ 1.0000 & 1.7369 \\ 1.0000 & 0.0000 \end{bmatrix} x \le \begin{bmatrix} 12.0940 \\ -2.2448 \\ 5.0000 \\ 12.3141 \\ 1329.55 \\ -2.2345 \\ 5.0000 \end{bmatrix} \\ & \text{(Region \#7)} \end{cases}$$



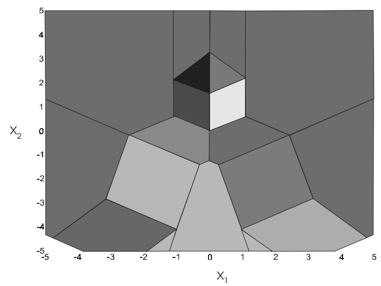
PWA law $\equiv$ MPC law

Linear constraints	84
Continuous variables	20
Binary variables	2
Parameters	2
Time to solve mp-MILP	3 min
Number of regions	7

mp-MILP Solution



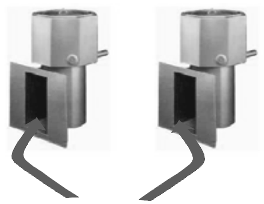
Prediction Horizon T=3



Prediction Horizon T=4

Hybrid Control - An Example

(Hedlund and Rantzer, CDC1999)



u_0

$$\begin{bmatrix} \dot T_1 \\ \dot T_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + B_i u_0$$

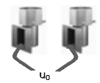
$$B_i = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if first furnace heated} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if second furnace heated} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if no heating} \end{cases}$$

- Objective:
  - Control the temperature $T_1,\ T_2$ to a given set-point
- Constraints:
  - Only three operation modes:
    1- Heat only the first furnace
    2- Heat only the second furnace
    3- Do not heat any furnaces

Amount of heating power u_0 is constant

Alternate Heating of Two Furnaces

(Bemporad, Borrelli, Morari, 2000)



- mp-MILP optimization problem

$$\min_{\{u(t),u(t+1),u(t+2)\}} \sum_{k=0}^{2} \|R(u(t+k+1)-u(t+k))\|_\infty + \|Q(x(t+k|t)-x_e)\|_\infty$$

- Parameter set

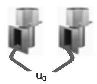
$$\begin{array}{rcl} -1 & \le x_1 \le & 1 \\ -1 & \le x_2 \le & 1 \\ 0 & \le u_0 \le & 1 \end{array}$$

parameterized !

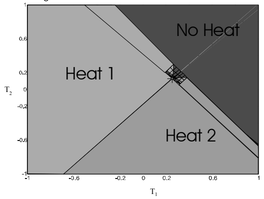
- Computational complexity of mp-MILP

Linear constraints	168
Continuous variables	33
Binary variables	6
Parameters	3
Time to solve mp-MILP	5 min
Number of regions	105

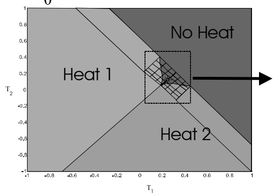
mp-MILP Solution

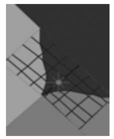


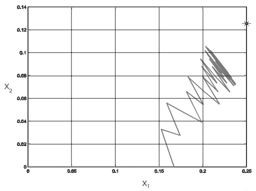
u_0=0.4



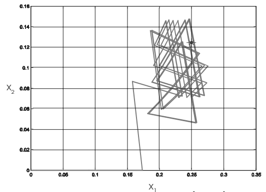
u_0=0.8







Set point cannot be reached



Set point is reached

NOTE: The control action of explicit and of implicit MPC are totally equal !

Hybrid Control Example: Traction Control System

Vehicle Traction Control

Improve driver's ability to control a vehicle
under adverse external conditions (wet or icy roads)

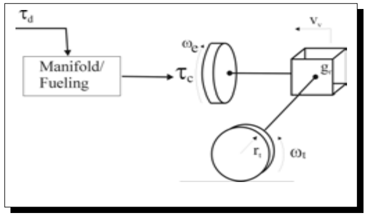
Model
nonlinear, uncertain, constraints

Controller
suitable for real-time implementation

MLD hybrid framework + optimization-based control strategy

Eidgenössische Technische Hochschule Zürich

Simple Traction Model



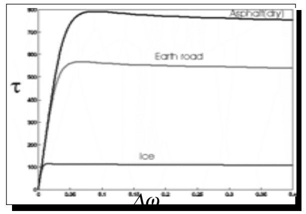
• Mechanical system

$$\dot{\omega}_e = \frac{1}{J_e}\left(\tau_c - b_e\omega_e - \frac{\tau_t}{g_r}\right)$$

$$\dot{v}_v = \frac{\tau_t}{m_v r_t}$$

• Manifold/fueling dynamics

$$\tau_c = b_i \tau_d (t - \tau_f)$$

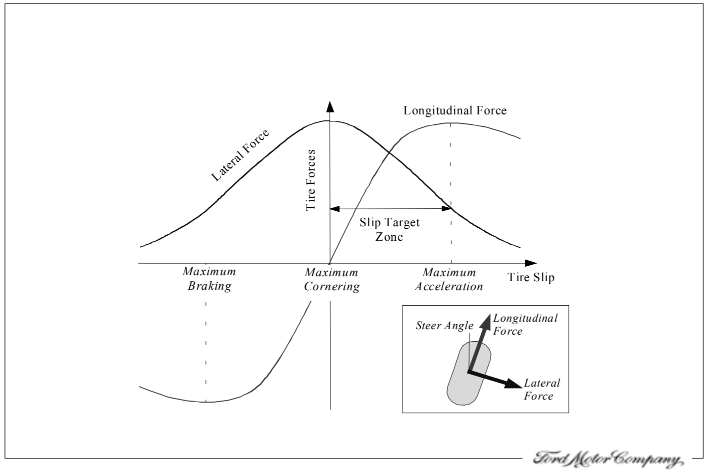


• Tire torque τ_t is a function of slip $\Delta\omega$ and road surface adhesion μ

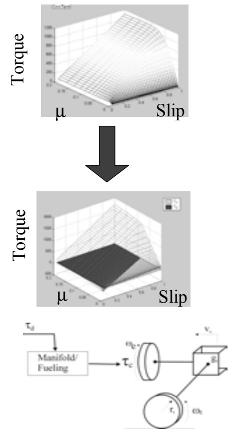
$$\Delta\omega = \frac{\omega_e}{g_r} - \frac{v_v}{r_t}$$

Wheel slip

TIRE FORCE CHARACTERISTICS



Hybrid Model



Nonlinear tire torque $\tau_t = f(\Delta\omega, \mu)$

PWA Approximation

(PWL Toolbox, Julian, 1999) (Ferrari et al., HSCC'01)

HYSDEL (Hybrid Systems Description Language) → Mixed-Logical Dynamical (MLD) Hybrid Model (discrete time)

MLD Model

$$
\begin{aligned}
x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \\
y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \\
E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5
\end{aligned}
$$

State x(t)	9	variables
Input u(t)	1	variable
Aux. Binary Var. d(t)	3	variables
Aux. Continuous variables z(t)	4	variables

→ The MLD matrices are automatically generated in Matlab format by HYSDEL

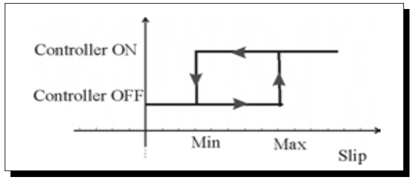
Performance and Constraints

- Control objective:

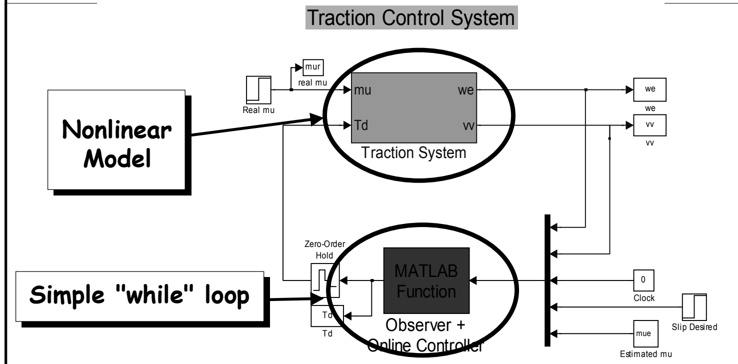
$$\min \sum_{k=0}^{N} |\Delta\omega(k|t) - \Delta\omega_{des}|$$

subj. to. MLD Dynamics

- Constraints:
  - Limits on the engine torque: $-20Nm \leq \tau_d \leq 176Nm$
- Logic Constraint:
  - Hysteresis



Simulation - Control Scheme



EXPERIMENTAL APPARATUS

EXPERIMENTAL APPARATUS

Experiment

- >500 regions
- 20ms sampling time
- Pentium 266Mhz + Labview

EXPERIMENTAL RESULTS

EXPERIMENTAL RESULTS

EXPERIMENTAL RESULTS

Comments from Ford:

- Performance of the MPC controller is quite good given the limited development time, oversimplified plant model used, and minimal development iterations.
- The MPC controller requires much less supervision by logical constructs than controllers developed with traditional techniques.
- Further testing in real-world environments is needed for a complete comparison with traditional controllers.

Ford Motor Company