Simple and Certifiable QP Algorithms for Embedded Linear MPC

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Outline

- Motivating applications (MPC for aerospace industry)
- QP solvers for embedded MPC
- A new "accelerated dual gradient projection" algorithm
- Numerical examples
- Conclusions

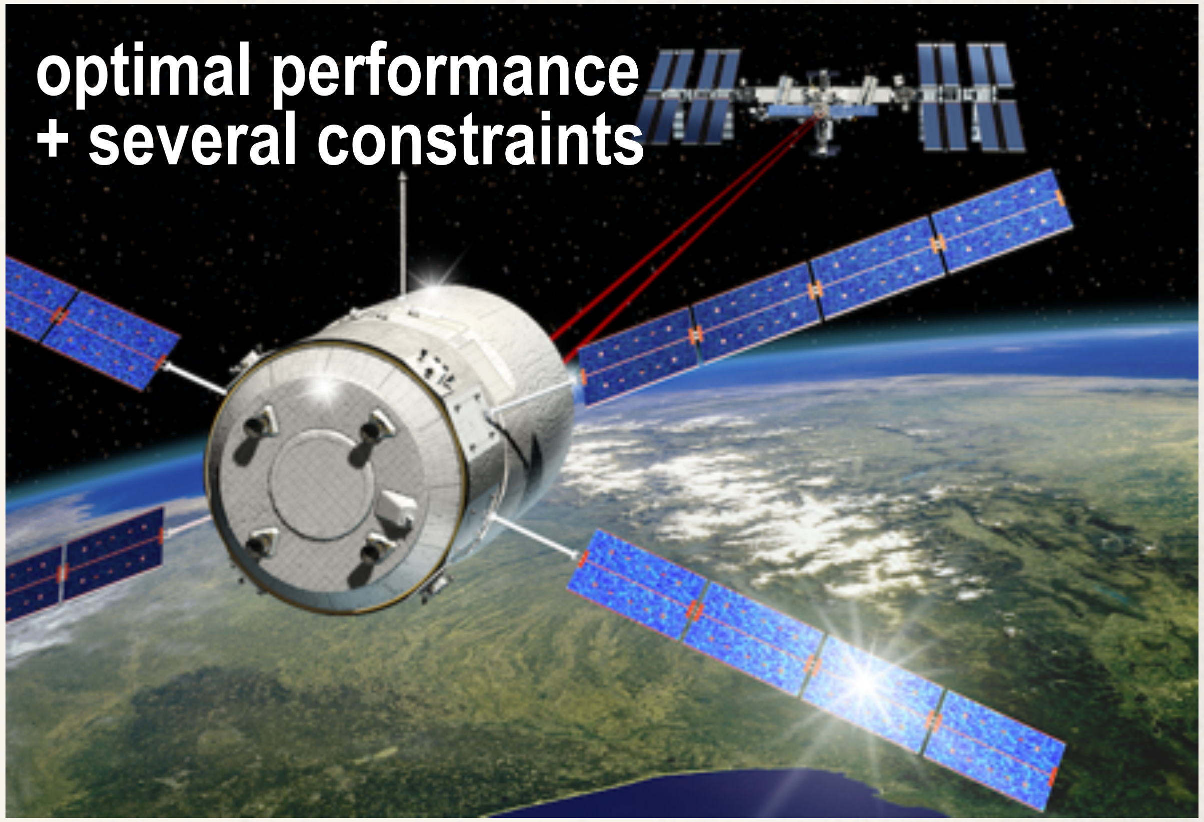
ESA-ESTEC, Noordwijk, The Netherlands

MPC for aero applications

- Plenty of potential MPC applications in aerospace industry !

  - Rendezvous and docking
  - Satellite formation flying
  - Unmanned aerial vehicles
  - Planetary rover locomotion (traction, guidance)
  - EDL (Entry Descent and Landing)
  - Ascent closed-loop guidance
  - In orbit servicing
  - ... and many others !

  http://www.esa.int/SPECIALS/Technology/



ATV Johannes Kepler docks with International Space Station

- Only a few research groups active

  - Richards, Trodden (Bristol, UK)
  - Maciejowski, Hartley (Cambridge, UK)
  - Balas, Borrelli, Keviczky (Minnesota)
  - ...
  - Murray, Dunbar (Caltech)
  - How, Breger, Tillerson (MIT)
  - Bemporad, Pascucci, Rocchi (IMT Lucca)

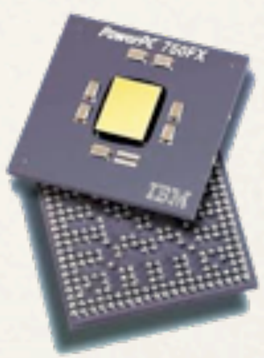
- Aerospace industry has very demanding **certification standards**

MPC for aero applications

- ESA funded project “**ORCSAT**” (2009-2011)
  - Main emphasis on **real-time implementation** capabilities of MPC (architecture, automatic generation of C and HDL code, timing)
  - Applications: orbit synchronization, impulsive hopping
  - New MATLAB Toolbox **MPCTOOL**, adding many new features on top of MPC Toolbox (such as LTV-MPC based on LP/QP)

AT697F processor not good, co-processor is needed

FPGA implementation from EML code not good

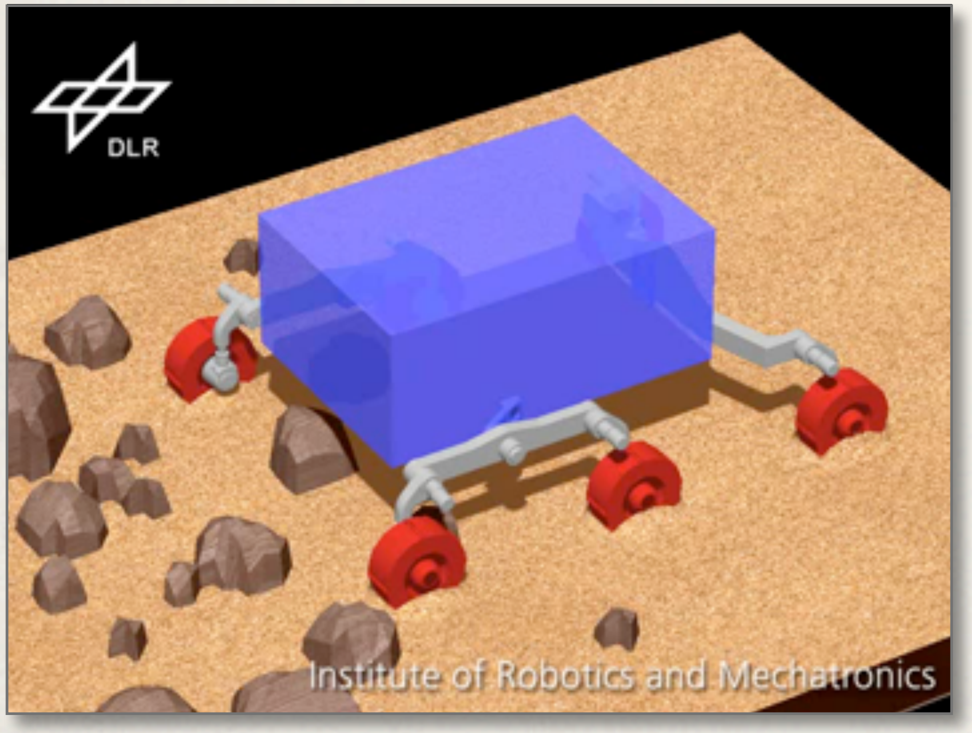


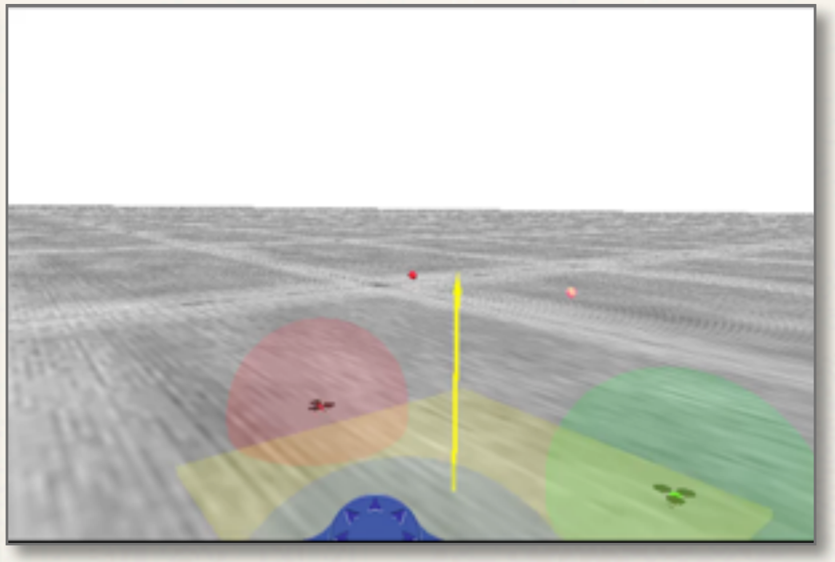
LTV-MPC implementation required PowerPC750FX

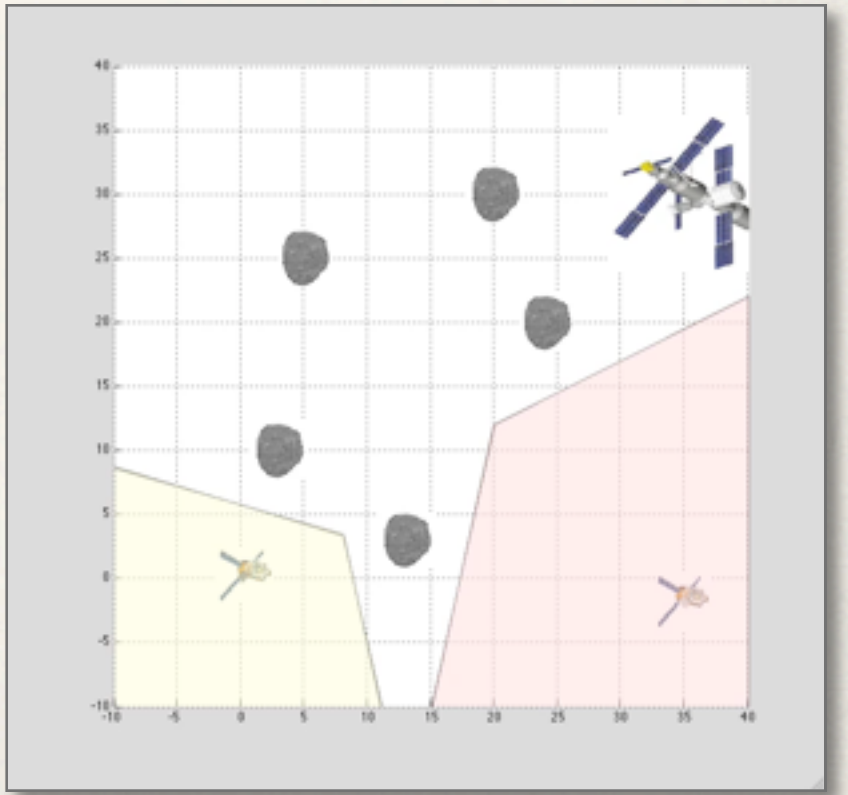
ORCSAT - On-line Reconfiguration Control System and Avionics Technologies

MPC for aero applications

- ESA funded project “**ROBMPC**” (2010-2011)
  - Main goal: explore MPC capabilities in new space apps
  - Applications: Cooperating UAVs, planetary rover locomotion (traction & guidance)
  - New MATLAB Toolbox **MPCSofT** for **LTV-MPC design**







ROBMPC - Robust MPC for Space Constrained Systems

(Bemporad, Rocchi, CDC 2011)

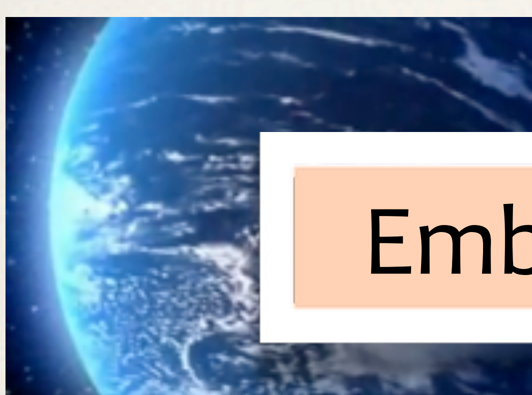
Embedded optimization in aerospace

- **PRISMA** project for autonomous formation flying (S. Persson, S. Veldman, P. Bodin, 2009)

http://www.lsespace.com/about-prisma.aspx

- Objective: **minimize fuel consumption** subject to keeping motion within a **box constraint** (solved by **linear programming**)

$$\begin{array}{ll} \min\limits_{dV(t)} & \sum_{t=0}^{\infty} \|dV(t)\|_1 \\ \text{s.t.} & \|r - y_r\|_1 \leq y_{\text{box}} \end{array}$$



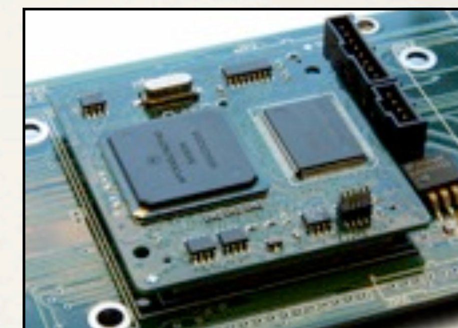
Embedded LP is **operational in space** !

How to make MPC technology really “fly” (in industry) ?

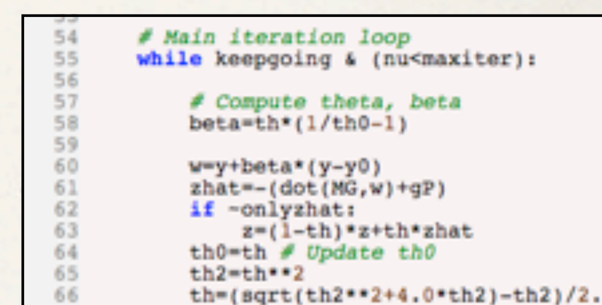
Requirements for Embedded MPC algorithms

- **Speed**: fast enough to provide a solution within short sampling intervals (such as 10-100 ms)

- Require **simple hardware** (microcontroller/FPGA) and little memory to store problem **data *and* code**



- **Code simple** enough to be **software-certifiable**



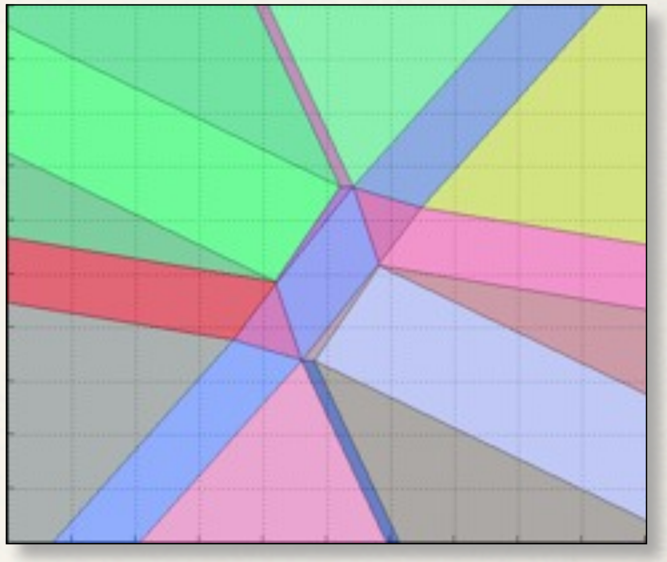
- Tightly estimate **worst-case execution time** to certify **hard real-time system** properties

Explicit model predictive control

- Presolve QP problem **off-line** for all states $x(t)$ in a given range via **multi-parametric programming**

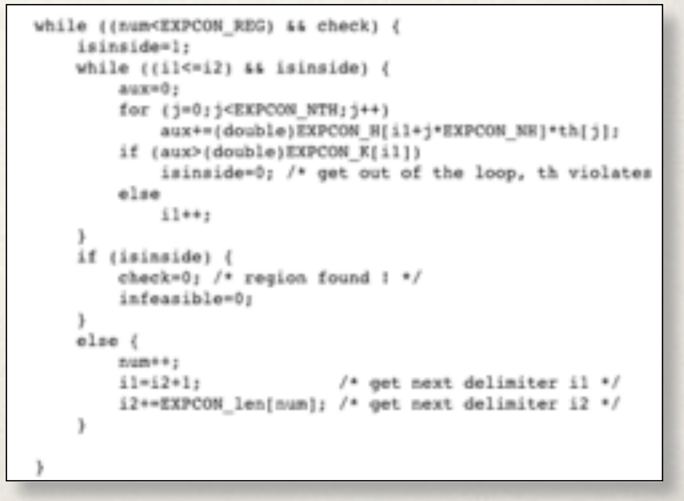
Contributors: Alessio, Baotic, Bemporad, Borrelli, Christophersen, De Doná, Dua, Filippi, Goodwin, Grancharova, Grieder, Johansen, Jones, Kerrigan, Kvasnica, Mayne, Maciejowski, Morari, Munoz de la Pena, Patrinos, Petersen, Pistikopulos, Rakovic, Sarimveis, Seron, Slupphaug, Spjøtvold, Tøndel, Zeilinger, ...

✓ Control law is **piecewise affine**
(and also continuous for linear MPC)



✓ Code **very simple**, **fast**, and **certifiable**

✖ Limited to **small problems**
(number of regions can grow exponentially with number of constraints)

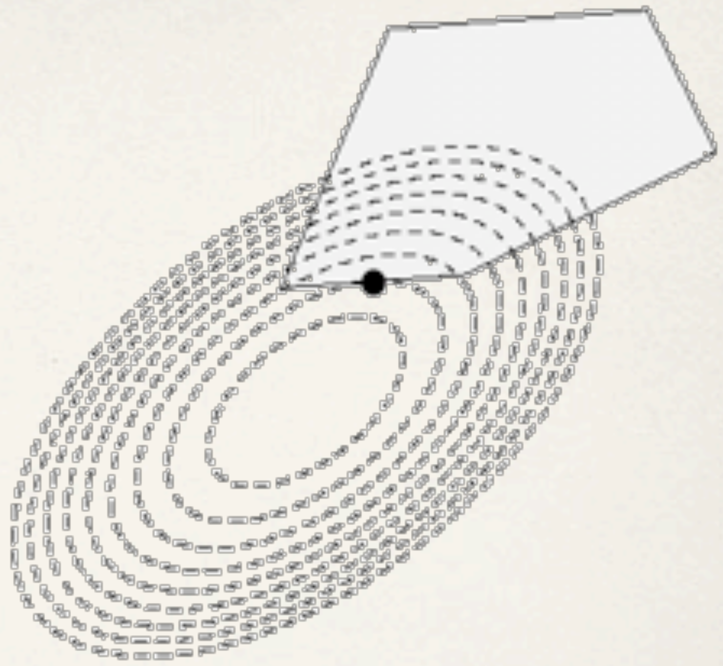


```
while ((num<EXPCON_REG) && check) {
     isinside=1;
     while ((i1<=i2) && isinside) {
          aux=0;
          for (j=0;j<EXPCON_NTH;j++)
               aux+=(double)EXPCON_H[i1+j*EXPCON_NH]*th[j];
          if (aux>(double)EXPCON_K[i1])
               isinside=0; /* get out of the loop, th violates
          else
               i1++;
     }
     if (isinside) {
          check=0; /* region found ! */
          infeasible=0;
     }
     else {
          num++;
          i1=i2+1;                /* get next delimiter i1 */
          i2+=EXPCON_len[num]; /* get next delimiter i2 */
     }

}
```

✖ Not applicable to **time-varying** models

On-line QP solution methods



- **active set** methods
  (best for **small/medium size** problems)

- **interior point** methods
  (best for **large-scale** problems)

- **gradient projection** methods (Goldstein, 1964) (Levitin & Poljak, 1965) (Nesterov, 1983)

- **conjugate gradient** methods

- **augmented Lagrangian** methods
  (and *alternating direction method of multipliers*)

Accelerated Dual Gradient Projection (GPAD)

Results based on (Patrinos, Bemporad, CDC 2012)
(Patrinos, Bemporad, journal version, submitted 2012)

Linear MPC - QP problem setup

- MPC problem (possibly LTV, Linear Time Varying):

$$
\begin{aligned}
V^*(p) = \min_{z} \quad & \ell_N(x_N) + \sum_{k=0}^{N-1} \ell_k(x_k, u_k) \\
\text{s.t.} \quad & x_0 = p \\
& x_{k+1} = A_k x_k + B_k u_k + f_k \\
& F_k x_k + G_k u_k \leq c_k \\
& \qquad k = 0, \ldots, N-1 \\
& F_N x_N \leq c_N
\end{aligned}
$$

quadratic costs

linear (affine) prediction model

linear inequality constraints

- QP problem:

$$
\begin{aligned}
V^\star(p) \triangleq \min_{z} \quad & \frac{1}{2} z' M z + (Cp + g)' z + \frac{1}{2} p' Y p \\
\text{s.t.} \quad & Gz \leq Ep + b
\end{aligned}
$$

$z =$ optimization vector

$p = x(t)$ parameter vector

Dual QP problem

$$\begin{aligned} V^\star(p) \triangleq \min_z \quad & \frac{1}{2}z'Mz + (Cp+g)'z + \frac{1}{2}p'Yp \\ \text{s.t.} \quad & Gz \leq Ep + b \end{aligned}$$

original (primal) QP

$$\begin{aligned} H &= GM^{-1}G' \\ D &= GM^{-1}C + E \\ d &= GM^{-1}g + b \end{aligned}$$

$$\begin{aligned} \Phi^\star(p) \triangleq \min_y \quad & \frac{1}{2}y'Hy + (Dp+d)'y + d_p \\ \text{s.t.} \quad & y \geq 0 \end{aligned}$$

dual QP

- Dual QP has simpler constraint set (=orthant) to project onto !

$$\text{proj}_{\{y \geq 0\}}\ y = \max\{y, 0\}$$

- **Key idea:** apply standard **fast gradient projection** algorithm to solve dual QP problem

(Nesterov, 1983)

Fast gradient projection algorithm

- **Off-line** operations: = linear algebra computations on QP matrices

$$M_G \triangleq M^{-1}G' \qquad G_L \triangleq \frac{1}{L}G$$
$$M_C \triangleq M^{-1}C \qquad E_L \triangleq \frac{1}{L}E$$
$$m_g \triangleq M^{-1}g \qquad b_L \triangleq \frac{1}{L}b$$

L = max eigenvalue of H
or
$$L = \sqrt{\sum_{i,j=1}^{m} |H_{i,j}|^2}$$ Frobenius norm

- **On-line** operations: initialization given current p=x(t)

$$g_P \triangleq M_C p + m_g, \quad p_D \triangleq -E_L p - b_L$$
$$y_0 = y_{-1} = 0, \quad z_{-1} = 0, \quad \theta_0 = \theta_{-1} = 1$$

- **On-line** operations: iterations

$$\theta_{\nu+1} = \frac{\sqrt{\theta_\nu^4+4\theta_\nu^2}-\theta_\nu^2}{2}$$
$$\beta_\nu = \theta_\nu(\theta_{\nu-1}^{-1} - 1)$$

$$\begin{cases} w_\nu = y_\nu + \beta_\nu(y_\nu - y_{\nu-1}) \\ \hat{z}_\nu = -M_G w_\nu - g_P \\ z_\nu = (1-\theta_\nu)z_{\nu-1} + \theta_\nu \hat{z}_\nu \\ y_{\nu+1} = \max\{w_\nu + G_L\hat{z}_\nu + p_D, 0\} \end{cases}$$

$\theta, \beta \in \mathbb{R}$, ν=iteration counter

Complexity (code, operations)

- Main on-line operations involve only **simple linear algebra**

$$\begin{cases} w_\nu &= y_\nu + \beta_\nu (y_\nu - y_{\nu-1}) \\ \hat{z}_\nu &= -M_G w_\nu - g_P \\ z_\nu &= (1-\theta_\nu) z_{\nu-1} + \theta_\nu \hat{z}_\nu \\ y_{\nu+1} &= \max\{w_\nu + G_L \hat{z}_\nu + p_D, 0\} \end{cases}$$

matrix-vector products

projection operation = simple comparison !

- Few lines of **`(Embedded) MATLAB`** or **`Python/Numpy`** code !

- Operations can be easily **parallelized** (for example on **`GPU`**)

Complexity (code, operations)

$$\begin{cases} w_\nu &= y_\nu + \beta_\nu(y_\nu - y_{\nu-1}) \\ \hat{z}_\nu &= -M_G w_\nu - g_P \\ z_\nu &= (1-\theta_\nu) z_{\nu-1} + \theta_\nu \hat{z}_\nu \\ y_{\nu+1} &= \max\{w_\nu + G_L \hat{z}_\nu + p_D, 0\} \end{cases}$$

- Each iteration has complexity $\mathrm{O}(N^2)$ $[N =$ prediction horizon$]$ (matrix-vector product)

- Using Riccati-like iterations, complexity gets down to $O(N)$

(Patrinos, Bemporad, CDC 2012)

- For **LTV problems,** all matrices must be computed on-line

Termination criteria

- We are interested in quality of **primal solution**

$$\begin{aligned} V(z,p) - V^\star(p) &\le \varepsilon_V \\ \max_{i \in \mathbb{N}_{[1,m]}} G_i z - E_i p - b_i &\le \varepsilon_g \end{aligned}$$

$$\begin{aligned} \varepsilon_V &\ge 0 \quad \text{optimality tolerance} \\ \varepsilon_g &\ge 0 \quad \text{feasibility tolerance} \end{aligned}$$

$$V(z,p) = \frac{1}{2} z' M z + (Cp + g)' z + \frac{1}{2} p' Y p$$

primal cost function

- V^* is unknown, so we use the duality gap as a **stopping criterion**:

$$\begin{aligned} V(z_\nu, p) - \Phi(y_{\nu+1}, p) &\le \varepsilon_V \\ \max_{i \in \mathbb{N}_{[1,m]}} G_i z_\nu - E_i p - b_i &\le \varepsilon_g \end{aligned}$$

$$\Phi(y,p) = \frac{1}{2} y' H y + (Dp + d)' y + d_p$$

dual cost function

Bounds on the number of iterations

(Patrinos, Bemporad, 2012)

- Feasibility:

$$G z_\nu - Ep - b \leq \varepsilon_g \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \forall \nu \geq \left\lceil \sqrt{\frac{8L\|y^*(p)\|}{\varepsilon_g}} \right\rceil - 2$$

- Optimality:

$\forall y^*(p)$ solving dual problem

$$\begin{gathered} V(z_\nu, p) - \Phi(y_{\nu+1}) \leq \varepsilon_V \\ \Downarrow \\ V(z_\nu, p) - V^\star(p) \leq \varepsilon_V \end{gathered} \quad \Longrightarrow \quad \forall \nu \geq \left\lceil \sqrt{\frac{2L}{\varepsilon_V}} \|y^*(p)\| \right\rceil - 2$$

(and also $V(z_\nu, p) - V^\star(p) \geq -\varepsilon_V$, see proof)

- A bound on $\|z_\nu - z^*(p)\|$ exists too

The bounds depend on optimal dual solution $y^*(p)$ ...

Bounds on the number of iterations

- A global bound on $\|y^*(p)\|$ for a given set $\mathcal{P}$ of parameters can be computed **off-line** by solving a **MILP** or a **LPCC** (Linear Program with linear Complementarity Constraints)

(Patrinos, Bemporad, 2012)

- When no constraints are active, solution is found in **one step:** $y_0 = y^* = 0, \;\; z^* = -M^{-1}(Cp+g)$ is the optimal solution

- More generally: QP optimality conditions can be stated as

$$y^* = \max\{y^* - \frac{1}{L}(Hy^* + Dp + d), 0\}$$

$$z^* = -M^{-1}(G'y^* + Cp + g)$$

Solving QP = finding a solution of the system of PWA equations

GPAD vs other fast gradient projection methods

- Nesterov's method applied to primal problem, only input-constrained MPC with simple input constraints (Richter, Jones, Morari, CDC 2009)

- Relax state equations, solve the dual. Only simple constraints and terminal sets. (Richter, Morari, Jones, CDC 2011, IEEE TAC 2012)

- Richter's algorithms must be run for a constant number of iterations (= theoretical bound), no termination criteria based on primal solution

- GPAD relaxes linear inequality constraints, not state equations. Can handle **arbitrary polyhedral constraints on inputs/states** (and polyhedral terminal sets). Practical **termination criteria based on primal solution** are provided. Different **theoretical bounds** are proved.

GPAD vs Interior Point (IP) methods

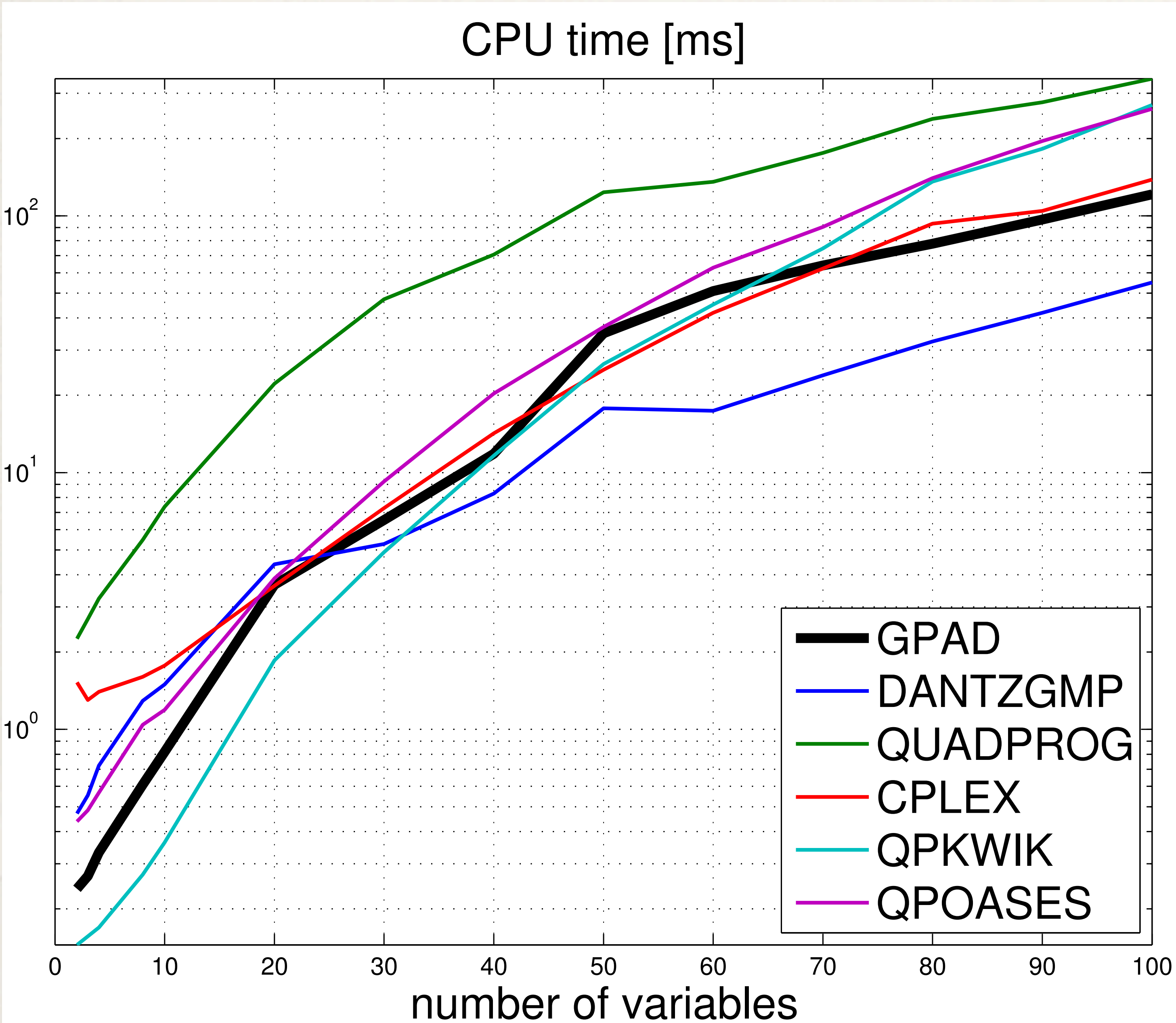
- GPAD iterations only require a **matrix-vector product,** IP requires solving a linear system

- GPAD can reach a solution of **moderate accuracy quite fast**, faster than a simple gradient method (both in practice and in theory). This is often enough for MPC purposes

- GPAD has much **tighter theoretical complexity** bounds than IP

- GPAD is **much simpler to code** !

Numerical results: speed



Embedded MATLAB code

```matlab
while keepgoing && (nu<maxiter),
    beta=th*(inv(th0)-1);
    w=y+beta*(y-y0);
    zhat=-(MG*w+gP);
    z=(1-th)*z+th*zhat;

    th0=th; % Update th0
    th2=th^2;
    th=(sqrt(th2^2+4*th2)-th2)/2;
    y0=y;
    y=max(w+GL*zhat+pD,0);

    gaphat=.5*(y'*H*y+zhat'*M*zhat)+...
        dp+c'*zhat+Dpd'*y;
    violhat=max(G*zhat-b);
    if gaphat<=epsV && violhat<=epsG,
        keepgoing=0;
    end
    nu=nu+1;
end
```

Average CPU time on random QP problems with n variables and $2n$ constraints

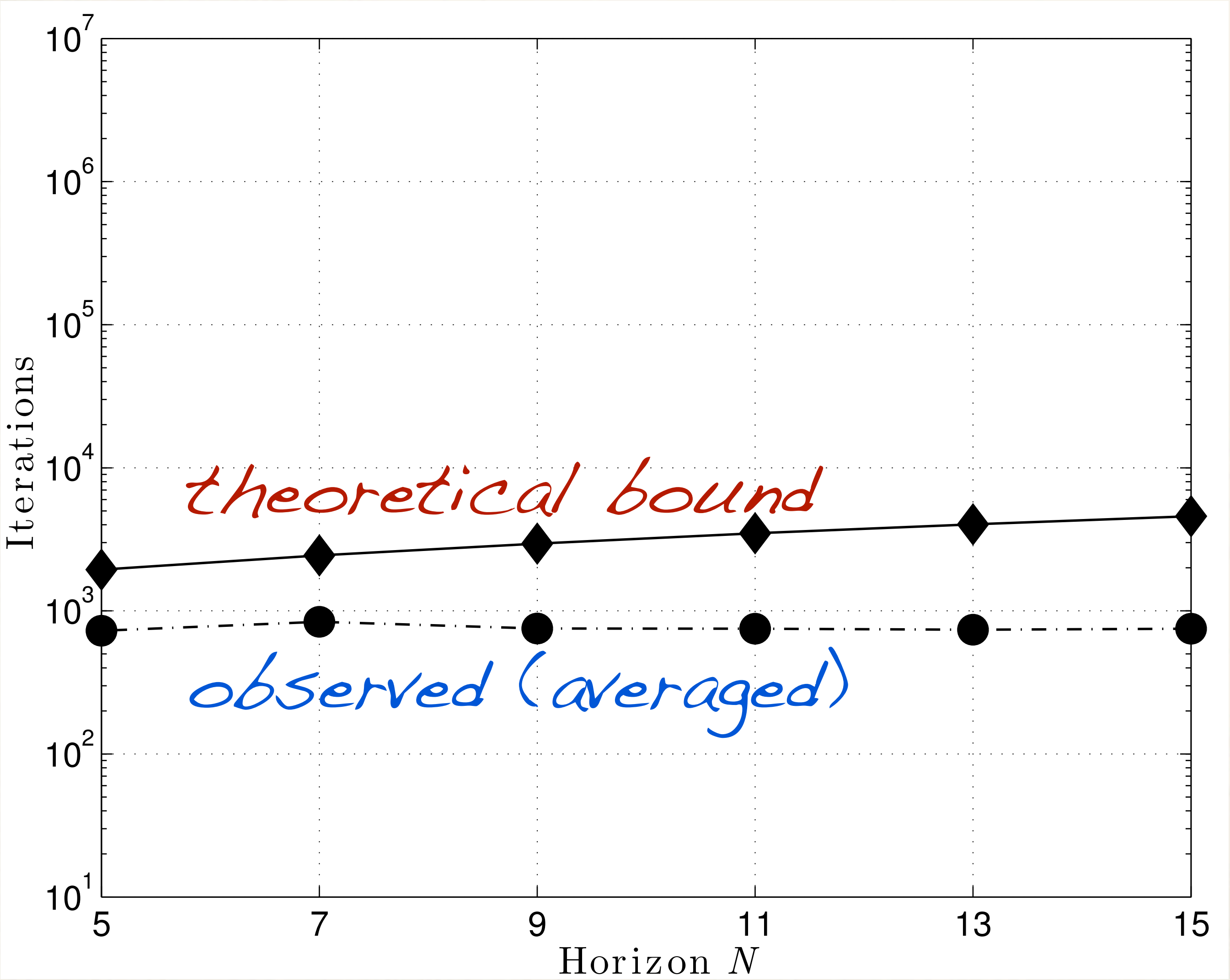
[standard algorithm with complexity $\mathrm{O}(n^2)$]

Numerical results: bounds

Ball & plate example, 2 states, 1 input, bound constraints on states and inputs

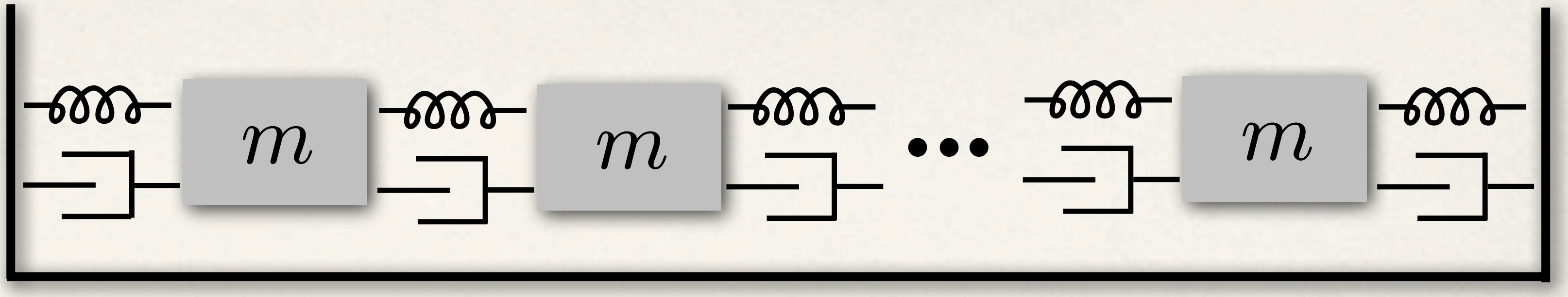
(Richter, Jones, Morari, 2011)

(Waldvogel,2010)



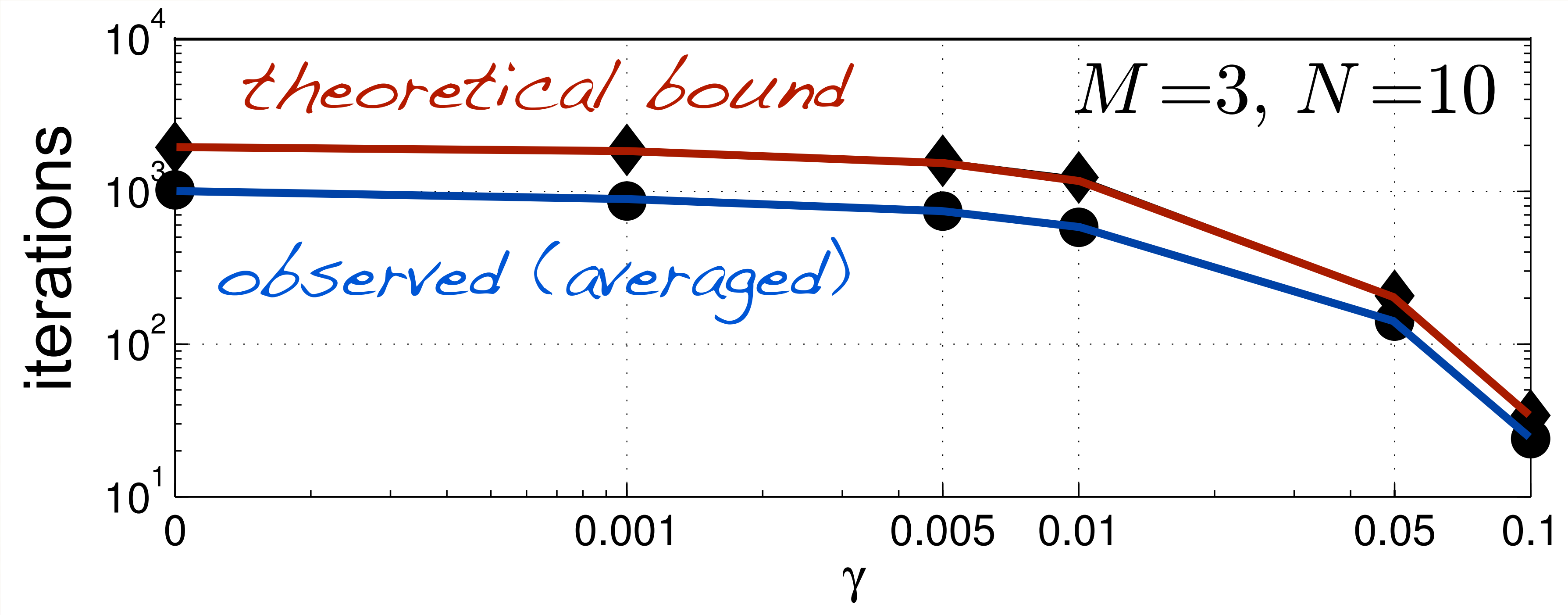
Numerical results: more complex example

Linear system:



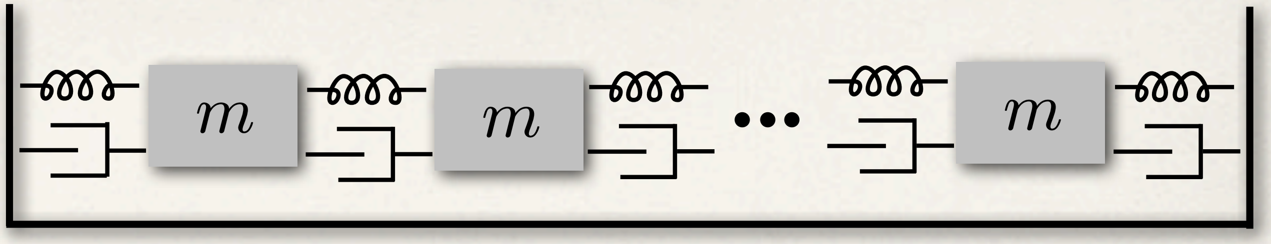
(Wang, Boyd, 2008)

- $2M$ states, M-1 inputs
- state and input constraints
- terminal set



γ = shrinking of parameter set, $\mathcal{P}_\gamma \triangleq \mathrm{dom}\{z \in \mathcal{Z}(p) | g(z) + \gamma \leq 0\}$

Numerical results: more complex example



CPU time [ms]

M	N	GPAD		Gurobi 5.0	
		avg	max	avg	max
2	10	0.40	1.09	4.48	
2	30	0.76	3.22	5.64	9.77
4	10	1.41	2.63	5.61	7.49
6	30	8.35	15.52	15.22	23.57
8	20	8.65	14.95	16.42	18.36
15	10	11.00	16.62	21.52	22.95
20	20	39.01	82.32	82.71	134.05
30	30	128.1	218.13	202.35	465.80

number of iterations

M	N	GPAD		Gurobi 5.0	
		avg	max	avg	max
2	10	19.90	60	5.42	7
2	30	20.40	100	5.50	7
4	10	41.80	80	6.05	8
6	30	52.20	100	6.57	8
8	20	52.20	90	6.51	8
15	10	54.00	80	6.56	7
20	20	61.60	140	6.85	8
30	30	62.00	100	6.94	8

QP **interior point** solver of `Gurobi 5.0`

tolerances: $\epsilon_V = \epsilon_g = 10^{-3}$

results averaged on 100 random states p

Conclusions

- GPAD solver for **QP tailored to embedded MPC** applications
  - Very **simple to code**
  - **Tight bounds** on estimated real-time execution
  - **Quality control of solution** (primal optimality & feasibility criteria)
  - Although not conceived for speed, still **reasonably fast**

- Current ongoing work
  - Robustness to computation errors (=fixed-point implementation)
  - Closed-loop stability of MPC under lower-quality solutions
  - Combined GPAD and Newton's method (for faster convergence)
  - Applications to aero industry (with ESA / Astrium / A3R)

Conclusions

Research on QP started ~60 years ago …

ON MINIMIZING A CONVEX FUNCTION SUBJECT TO LINEAR INEQUALITIES

By E. M. L. BEALE

Admiralty Research Laboratory, Teddington, Middlesex

SUMMARY

THE minimization of a convex function of variables subject to linear inequalities is discussed briefly in general terms. Dantzig's Simplex Method is extended to yield finite algorithms for minimizing either a convex quadratic function or the sum of the t largest of a set of linear functions, and the solution of a generalization of the latter problem is indicated. In the last two sections a form of linear programming with random variables as coefficients is described, and shown to involve the minimization of a convex function.

(Beale, 1955)

… still research on new **QP solvers for MPC** can have most impact !