Modeling and Control of Hybrid Systems

Alberto Bemporad

Dept. of Information Engineering

University of Siena, Italy

bemporad@unisi.it

MTNS 2004
Leuven, July 8

COHES Group
Control and Optimization of Hybrid and Embedded Systems

University of Siena
(founded in 1240)

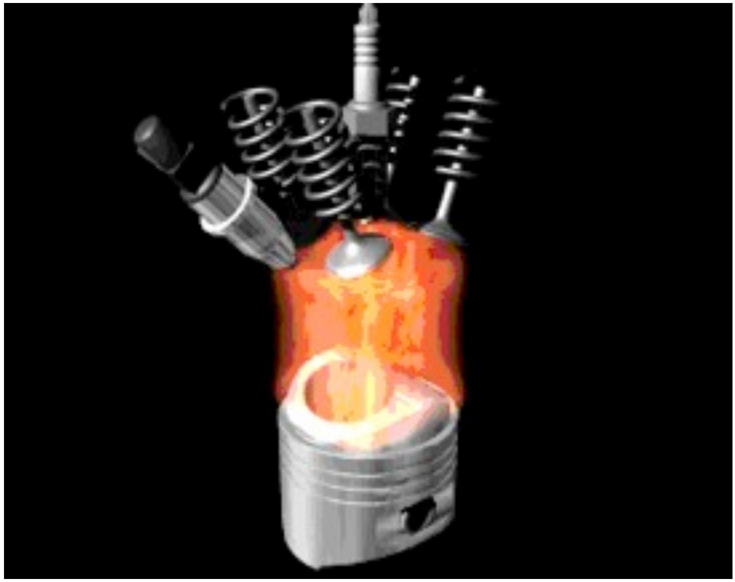
A motivating control problem ...

DISC Engine Control Problem

Objective: Develop a controller for a Direct-Injection Stratied Charge (DISC) engine.

Main control challenges:

- System has two operating **modes** (homogeneous/stratied)
- **Nonlinear** dynamics
- **Constraints** on several variables (A/F ratio, air-ow, spark)
- Control objectives and constraints depend on operating mode
- **Optimal** performance sought



(Photo: Courtesy Mitsubishi)

Main Problem Features

- **Model**: dynamical $+$ mode switching
- **Control specs**: optimal performance subject to constraints

Wish List

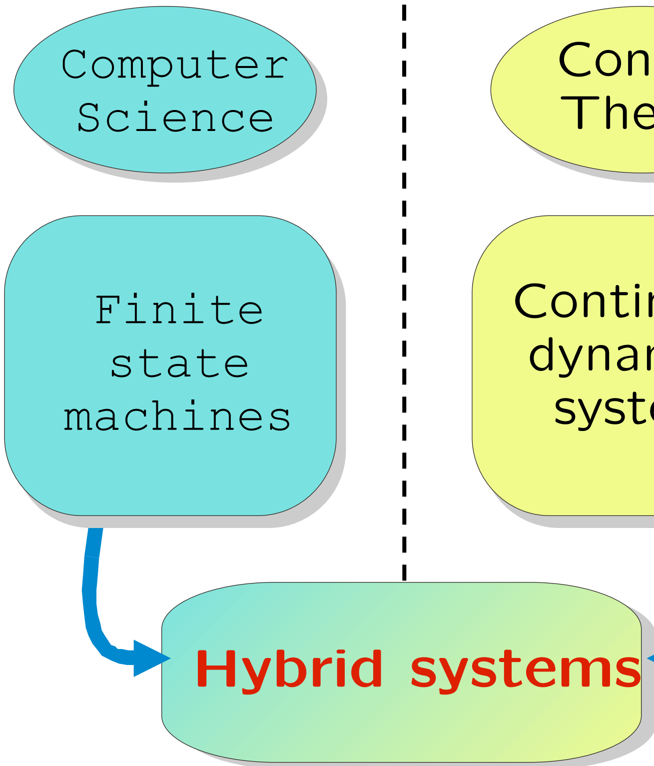
- Model paradigm for dynamical systems w/ switches
- Easy way to dene performance & constraint specs
- Solid theoretical foundation (e.g. stability guarantees)
- Come up with implementable control algorithms

Hybrid Systems

Contents

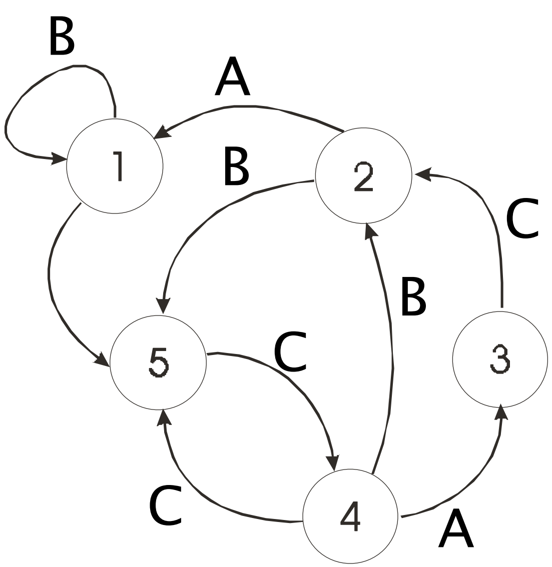
- What is a “hybrid system” ?
- Models of hybrid systems
- Controller synthesis for hybrid systems
- Applications (automotive)

Hybrid Systems



$\mathbf{x} \in \{1,2,3,4,5\}$
$\mathbf{u} \in \{A,B,C\}$

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^p$



$$\frac{dx(t)}{dt} = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Hybrid Systems

Computer Science

Control Theory

IEEE TRANSACTIONS ON AUTOMATIC CONTROL VOL. AC-11, NO. 2 APRIL, 1966

A Class of Hybrid-State Continuous-Time Dynamic Systems

H. S. WITSENHAUSEN

Abstract—A class of continuous time systems with part continuous, part discrete state is described by differential equations combined with multistable elements. Transitions of these elements between their discrete states are triggered by the continuous part of the state and not directly by inputs. The dynamic behavior of such systems, in response to piecewise continuous inputs, is defined under suitable assumptions. A general Mayer-type optimization problem is formulated. Conditions are given for a solution to be well-behaved, so that variational methods can be applied. Necessary conditions for optimality are stated and the jump conditions are interpreted geometrically.

Introduction

SOME PHYSICAL objects evolve in time according

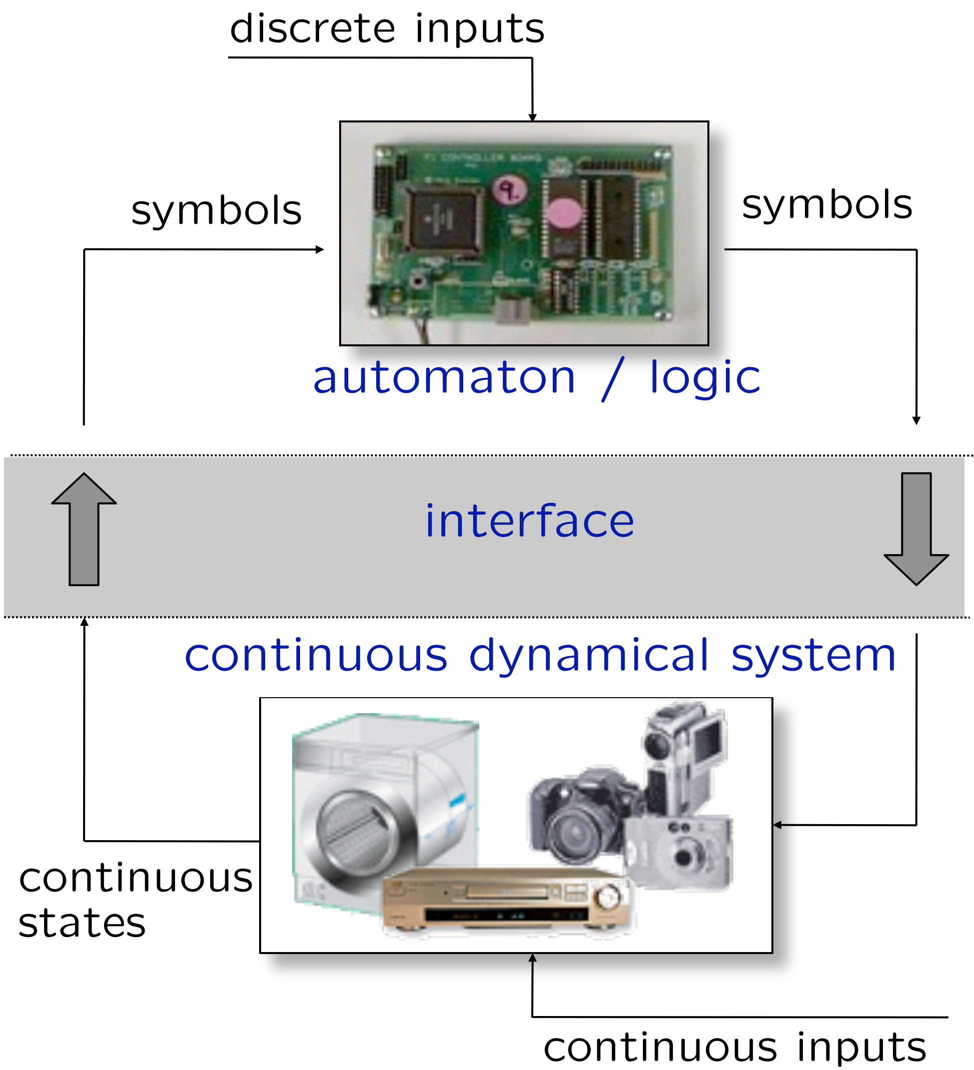
gates to process Boolean signals, 3) electronic analog switches controlled by Boolean signals.

The objective of this paper is to give a precise description of such systems, to define their dynamics, to formulate the problem of their optimum control, to introduce the notion of well-behaved solution, and to state necessary conditions for optimality (the jump conditions).

A Class of Hybrid Systems

The modifications required in otherwise continuous systems described by vector differential equations

Embedded Systems



- Consumer electronics
- Home appliances
- Oce automation
- Automobiles
- Industrial plants
- ...

“Intrinsically Hybrid” Systems

Discrete input (1,N,2,3,4) + Continuous inputs (brakes, gas, clutch) + Continuous dynamical states (velocities, torques, air-ows, fuel level)

Key Requirements for Hybrid Models

- Descriptive enough to capture the behavior of the system
  - continuous dynamics (physical laws)
  - logic components (switches, automata, logic rules)
  - interconnection between logic and dynamics
- Simple enough for solving analysis and synthesis problems

Outline

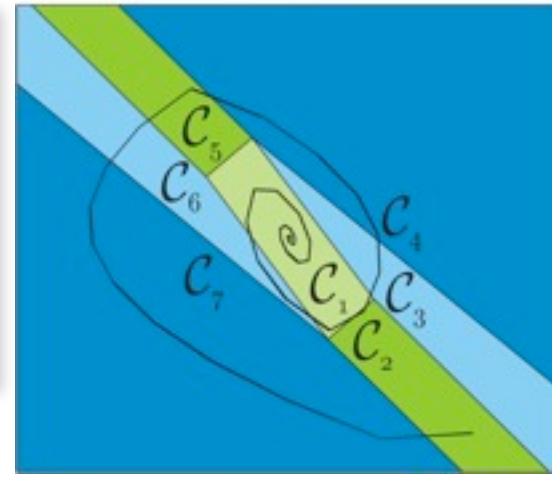
✓ What is a “hybrid system” ?

- Models of hybrid systems
- Controller synthesis for hybrid systems
- Applications (automotive)

Piecewise Ane Systems

$$
\begin{aligned}
x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\
y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\
\\
i(k) \ \text{s.t.} & \ \ H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}
\end{aligned}
$$

state+input space



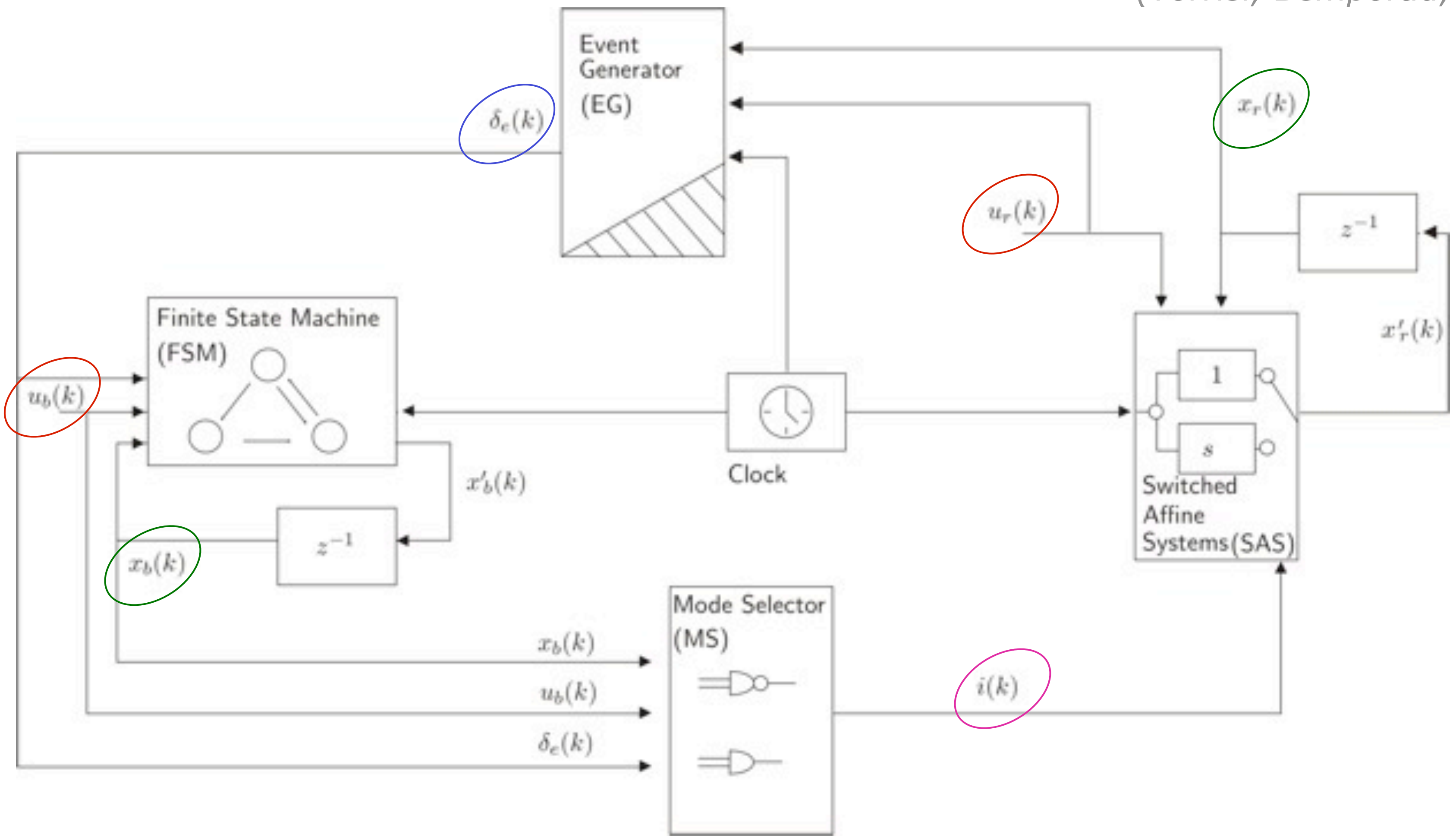
(Sontag 1981)

$x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^p$
$i(k) \in \{1, \ldots, s\}$

- Can approximate nonlinear/discontinuous dynamics arbitrarily well

Discrete Hybrid Automata

(Torrisi, Bemporad, 2004)



$x_r \in \mathbb{R}^{n_r}$ = continuous states

$x_b \in \{0,1\}^{n_b}$ = binary states

$i(k) \in \{1,\ldots,s\}$ = mode

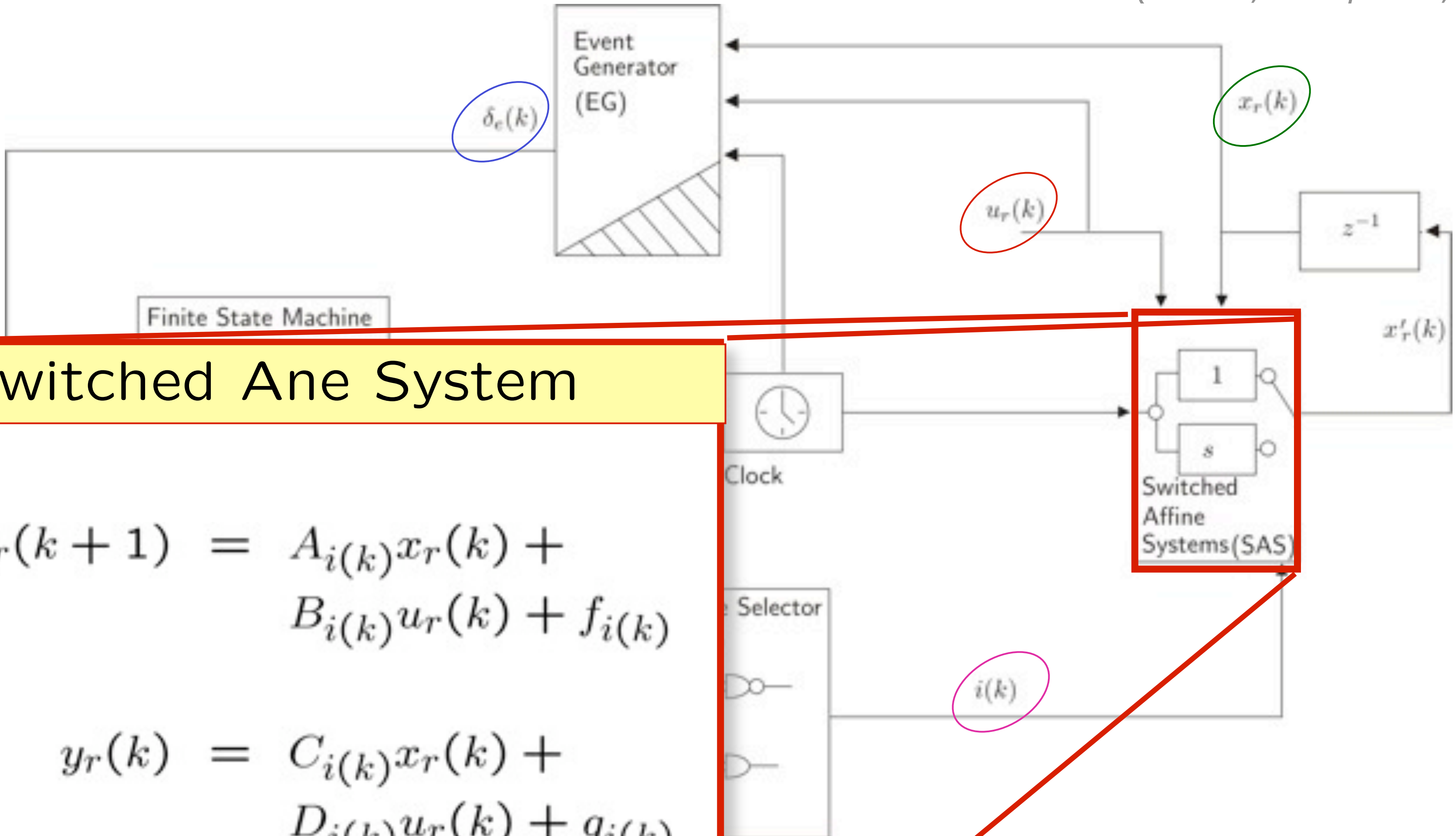
$u_r \in \mathbb{R}^{m_r}$ = continuous inputs

$u_b \in \{0,1\}^{m_b}$ = binary inputs

$\delta_e \in \{0,1\}^{n_e}$ = event conditions

Discrete Hybrid Automata

(Torrisi, Bemporad, 2004)



Switched Ane System

$$
\begin{aligned}
x_r(k+1) &= A_{i(k)}x_r(k) + \\
&\quad B_{i(k)}u_r(k) + f_{i(k)}
\end{aligned}
$$

$$
\begin{aligned}
y_r(k) &= C_{i(k)}x_r(k) + \\
&\quad D_{i(k)}u_r(k) + g_{i(k)}
\end{aligned}
$$

$$x_r \in \mathbb{R}^{n_r}, \;\; u_r \in \mathbb{R}^{m_r}, \;\; y_r \in \mathbb{R}^{p_r}$$

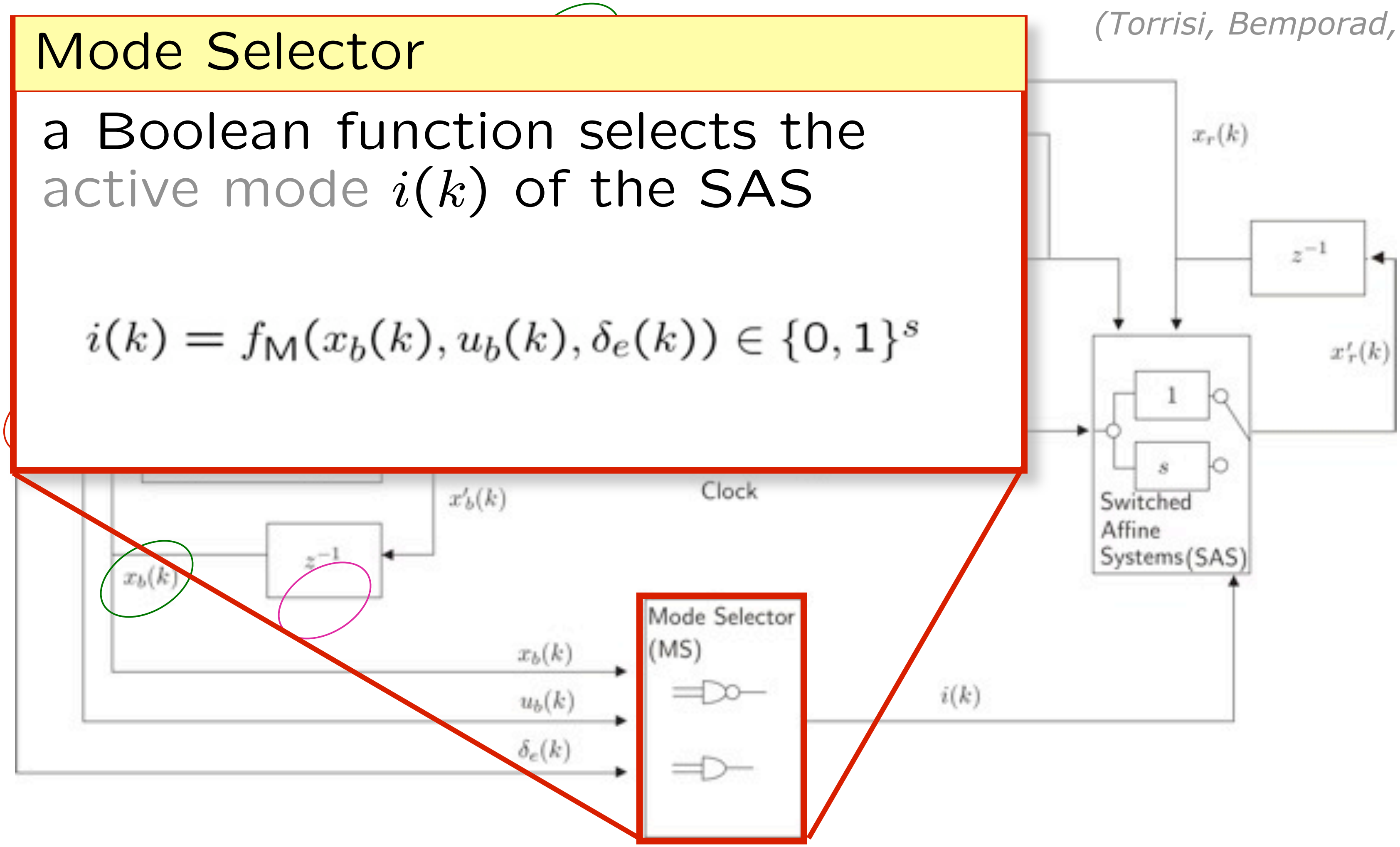
Discrete Hybrid Automata

(Torrisi, Bemporad, 2004)

Mode Selector

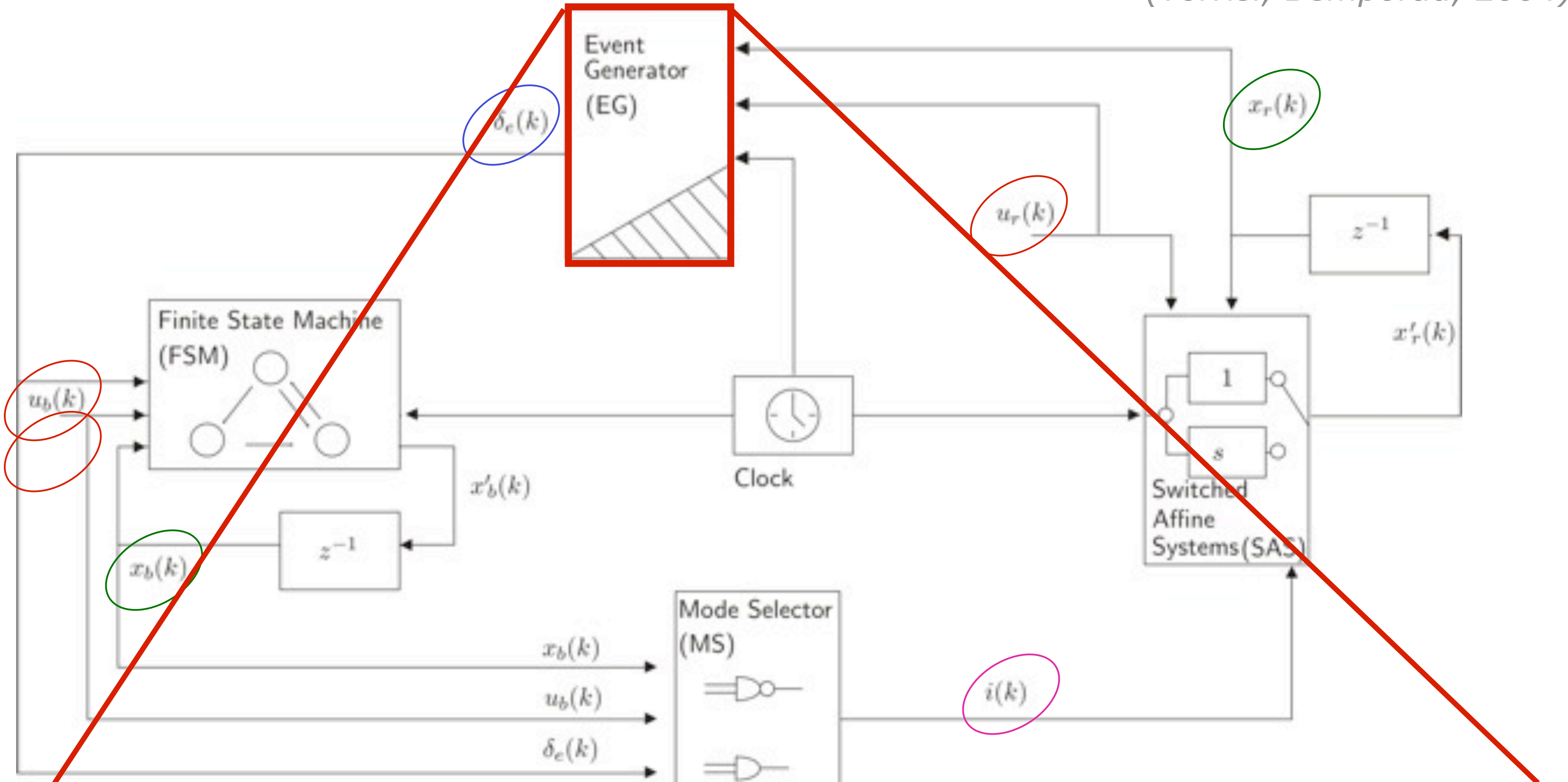
a Boolean function selects the active mode $i(k)$ of the SAS

$$i(k) = f_{\mathrm{M}}(x_b(k), u_b(k), \delta_e(k)) \in \{0, 1\}^s$$



Discrete Hybrid Automata

(Torrisi, Bemporad, 2004)



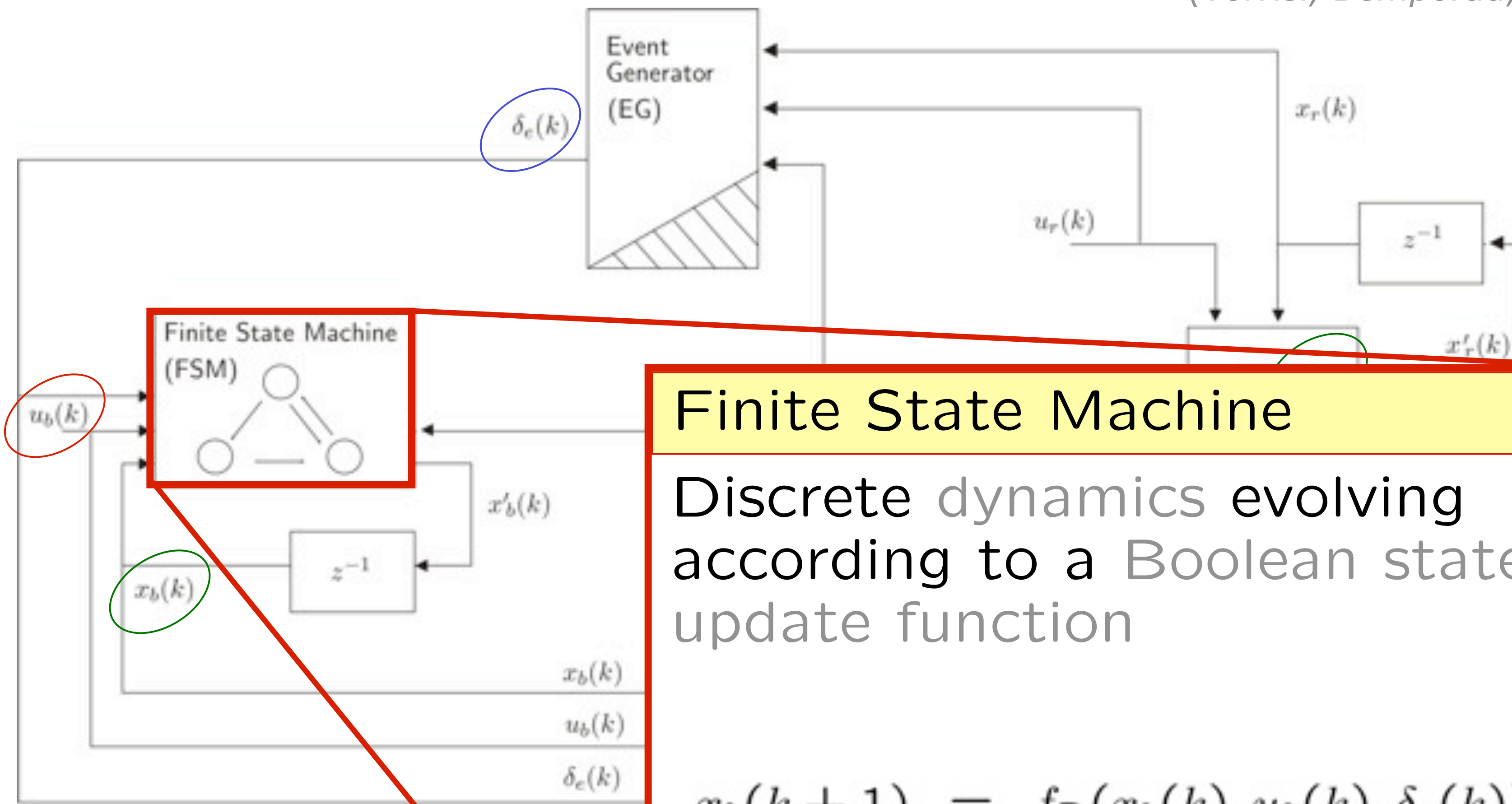
Event Generator

$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_r(k) + K^i u_r(k) + J^i k \leq W^i], \quad \delta_e^i \in \{0,1\}$$

$$x_r \in \mathbb{R}^{n_r}, \quad u_r \in \mathbb{R}^{m_r}, \quad \delta_e^i \in \{0,1\}$$

Discrete Hybrid Automata

(Torrisi, Bemporad, 2004)



Finite State Machine

Discrete dynamics evolving according to a Boolean state update function

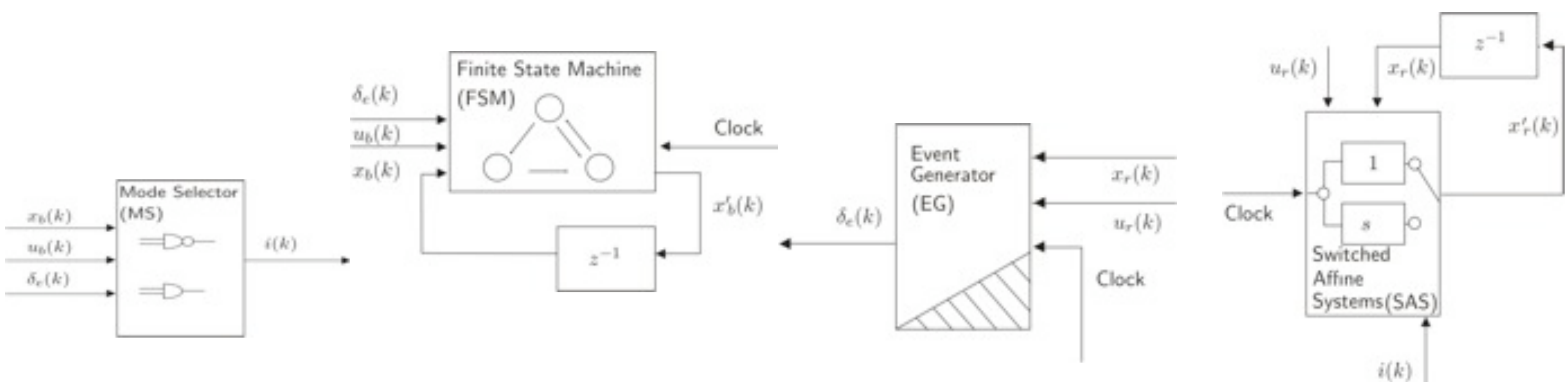
$$\begin{aligned} x_b(k+1) &= f_B(x_b(k), u_b(k), \delta_e(k)) \\ y_b(k) &= g_B(x_b(k), u_b(k), \delta_e(k)) \end{aligned}$$

$$x_b \in \{0,1\}^{n_b}, \; y_b \in \{0,1\}^{p_b}$$

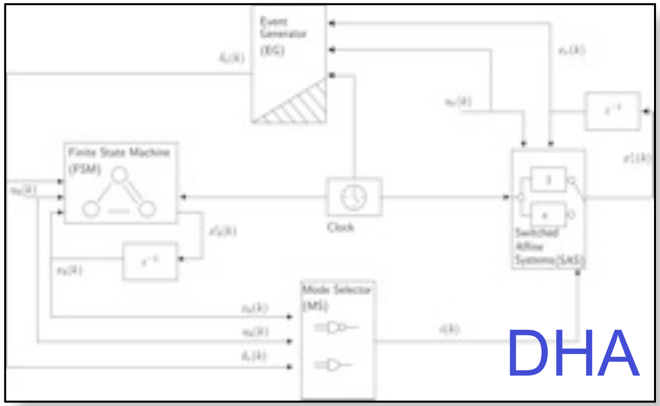
Logic and Inequalities

Glover 1975, Williams 1977

$X_1 \vee X_2$	$\delta_1 + \delta_2 \geq 1, \qquad \delta_1, \delta_2 \in \{0, 1\}$
Any logic statement $f(X) =$TRUE $\bigwedge_{j=1}^{m} \left(\vee_{i \in P_j} X_i \vee_{i \in N_j} \neg X_i \right)$ (CNF) $N_j, P_j \subseteq \{1, \ldots, n\}$	$A\delta \leq B$ $\begin{cases} 1 \leq \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) \\ \vdots \\ 1 \leq \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) \end{cases}$
$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_r(k) \leq W^i]$	$\begin{array}{rcl} H^i x_r(k) - W^i & \leq & M^i(1 - \delta_e^i) \\ H^i x_r(k) - W^i & > & m^i \delta_e^i \end{array}$
IF δ THEN $z = a_1^T x + b_1^T u + f_1$ ELSE $z = a_2^T x + b_2^T u + f_2$	$\begin{array}{rcl} (m_2 - M_1)\delta + z & \leq & a_2 x + b_2 u + f_2 \\ (m_1 - M_2)\delta - z & \leq & -a_2 x - b_2 u - f_2 \\ (m_1 - M_2)(1 - \delta) + z & \leq & a_1 x + b_1 u + f_1 \\ (m_2 - M_1)(1 - \delta) - z & \leq & -a_1 x - b_1 u - f_1 \end{array}$



Mixed Logical Dynamical Systems



Mixed Logical Dynamical (MLD) Systems

$$
\begin{aligned}
x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\
y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\
E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5
\end{aligned}
$$

(Bemporad, Morari 1999)

Continuous and binary variables

$$
\begin{aligned}
&x \in \mathbb{R}^{n_r} \times \{0,1\}^{n_b},\ u \in \mathbb{R}^{m_r} \times \{0,1\}^{m_b} \\
&y \in \mathbb{R}^{p_r} \times \{0,1\}^{p_b},\ \delta \in \{0,1\}^{r_b},\ z \in \mathbb{R}^{r_r}
\end{aligned}
$$

<u>NOTE</u>: translation from DHA to MLD has been automatized !
(HYSDEL – Hybrid Systems DEscription Language) *(Torrisi, Bemporad, 2004)*

DHA/MLD and PWA Systems

Theorem DHA/MLD systems and PWA systems are equivalent

(Bemporad, Ferrari-Trecate, Morari, 2000)
(Heemels, De Schutter, Bemporad, 2002)

- Equivalence = same initial state $x(0)$ and inputs $u(k)$ produce same states $x(k)$ and outputs $y(k)$
- PWA to DHA/MLD: Easy.
  (PWA=special case of switched ane system,
  thresholds = hyperplanes of the polyhedral partition)
- DHA/MLD to PWA: Ecient algorithms available that avoid enumeration of Boolean variables

[sequence of MILPs]

(Bemporad, IEEE-TAC, 2004)

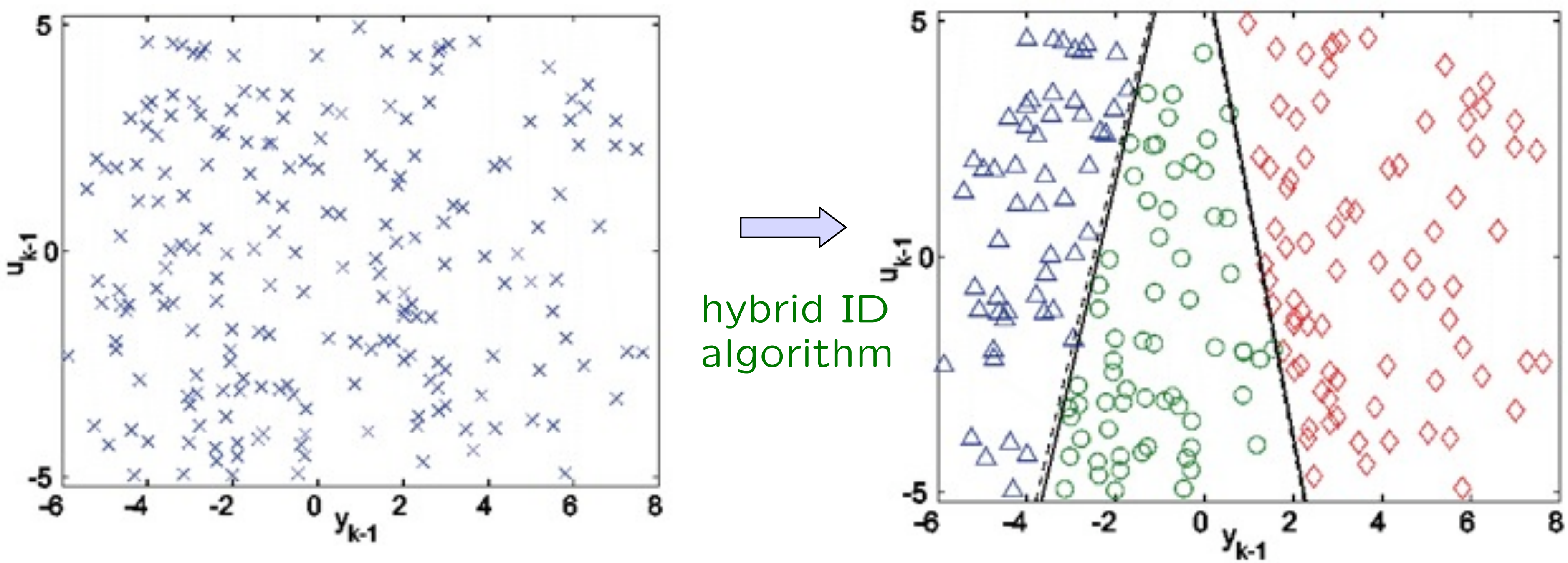
[hyperplane arrangements]

(Geyer, Torrisi, Morari, HSCC, 2003)

Getting Hybrid Models from Experimental Data

PWA Identication Problem

Given I/O data, estimate the parameters of the ane submodels **and** the partition of the PWA map



PWA Identication Problem

A. Known Guardlines (partition known, parameters unknown):
least-squares problem **EASY PROBLEM**
(Ljung's ID TBX)

B. Unknown Guardlines (partition *and* parameters unknown):
Generally non-convex, local minima **HARD PROBLEM!**

Some recent approaches to Hybrid ID:

- K-means clustering in a feature space (Ferrari-Trecate, Muselli, Liberati, Morari, 2003)
- Bayesian approach (Juloski, Heemels, Weiland, 2004)
- Mixed-integer programming (Roll, Bemporad, Ljung, 2004)
- Bounded error (partition of infeasible set of inequalities) (Bemporad, Garulli, Paoletti, Vicino, 2003)
- Algebraic geometric approach (Vidal, Soatto, Sastry, 2003)
- Hyperplane clustering in data space (Münz, Krebs, 2002)

Why are we interested in getting MLD and PWA models ?

Major Advantages of MLD/PWA Models

Many problems of analysis:

– Stability
– Safety
– Reachability
– Observability
– Well-posedness

Many problems of synthesis:

– Controller design
– Filter design / Fault detection & state estimation

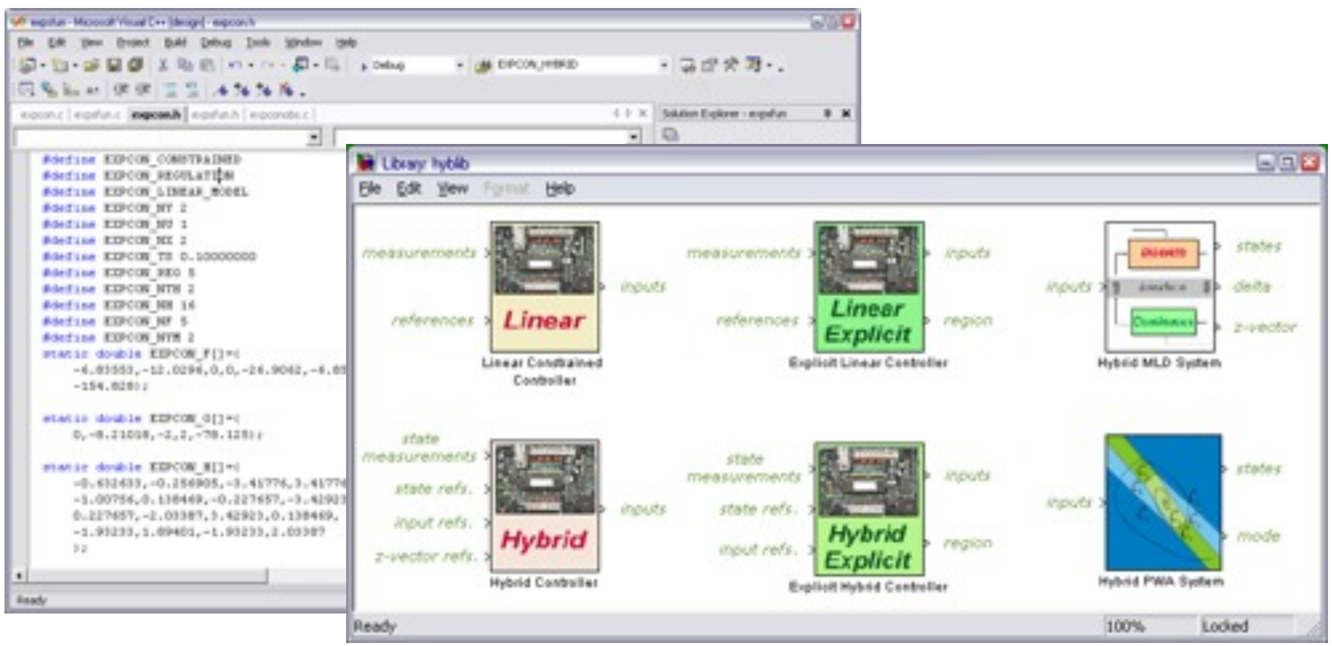
can be solved using mathematical programming

(However, all these problems are NP-hard !)

Hybrid Toolbox for Matlab

Features: (Bemporad, 2003)

- Hybrid model (MLD and PWA) design and simulation
- Control design for linear systems w/ constraints and hybrid systems (on-line optimization via QP/MILP/MIQP)
- Explicit control (via multiparametric programming)
- C-code generation
- Simulink



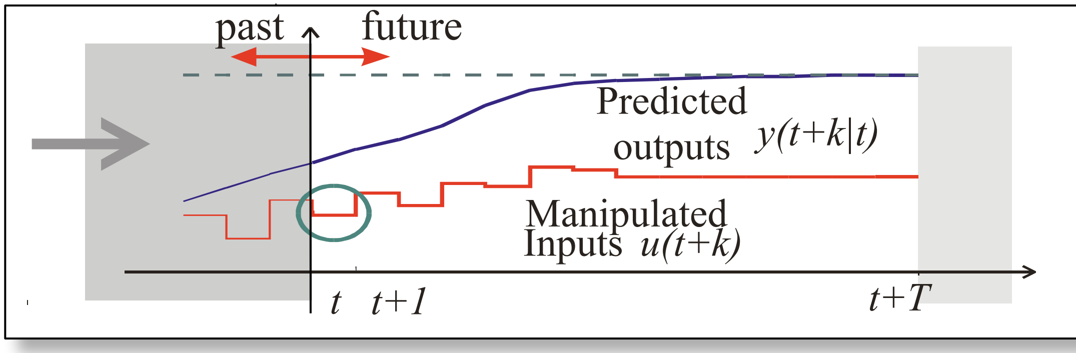
http://www.dii.unisi.it/hybrid/toolbox

Outline

- ✓ What is a "hybrid system" ?
- ✓ Models of hybrid systems
- Controller synthesis for hybrid systems
- Applications (automotive)

Controller Synthesis for Hybrid Systems

MPC for Hybrid Systems



Model
Predictive (MPC)
Control

- At time t solve with respect to $U \triangleq \{u(t), u(t+1), \ldots, u(t+T-1)\}$ the nite-horizon open-loop, optimal control problem:

$$\min_{u(t),\ldots,u(t+T-1)} \sum_{k=0}^{T-1} \|y(t+k|t) - r(t)\| + \rho\|u(t+k)\|$$
$$+\sigma(\|\delta(t+k) - \delta_r\| + \|z(t+k) - z_r\| + \|x(t+k|t) - x_r\|)$$

subject to hybrid model (MLD or PWA)

$$x(t|t) = x(t)$$
$$x(t+T|t) = x_r$$

- Apply only $u(t) = u^a(t)$ (discard the remaining optimal inputs)
- Repeat the whole optimization at time $t+1$

Closed-Loop Stability

Theorem 1 *Let $(x_r, u_r, \delta_r, z_r)$ be the equilibrium values corresponding to the set point r, and assume $x(0)$ is such that the MPC problem is feasible at time $t = 0$. Then $\forall Q, R \succ 0$, $\forall \sigma > 0$, the MPC controller stabilizes the MLD system*

$$\lim_{t\to\infty} y(t) = r$$
$$\lim_{t\to\infty} u(t) = u_r$$

$\lim_{t\to\infty} x(t) = x$ … $\mathrm{m}_{t\to\infty} z(t) = z_r$,
and all constrain …

More on this:
talk by M. Lazar
Session FA2

(Bemporad, Morari 1999)

Proof: Easily follows from standard Lyapunov arguments

Hybrid MPC - Example

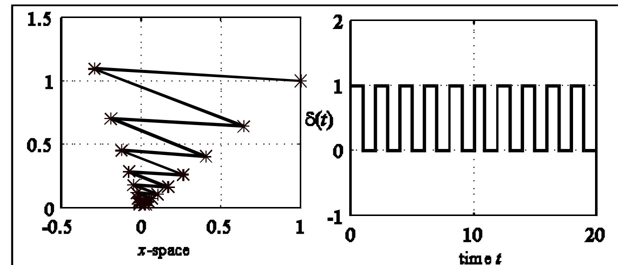
Switching System:

$$x(t+1) = 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \;\; 1]\, x(t)$$

$$\alpha(t) = \begin{cases} \frac{\pi}{3} & \text{if } [1 \;\; 0]x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if } [1 \;\; 0]x(t) < 0 \end{cases}$$

Open loop:

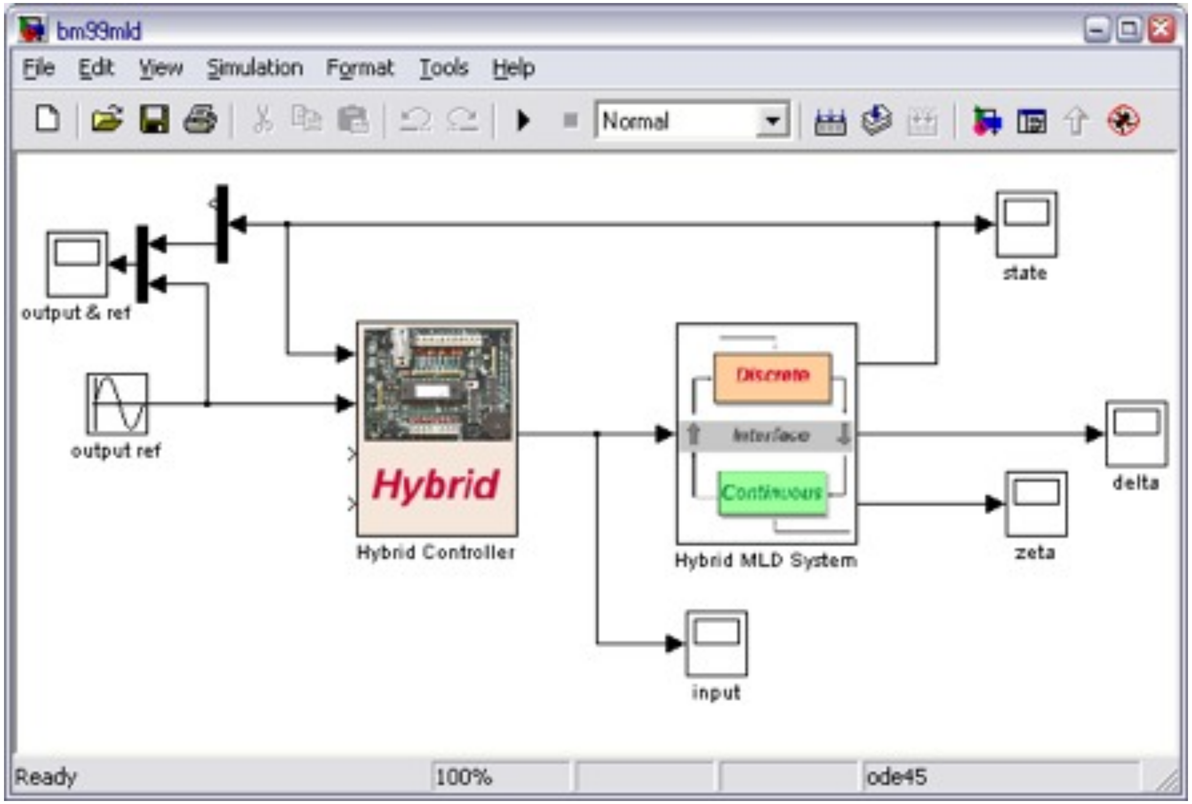


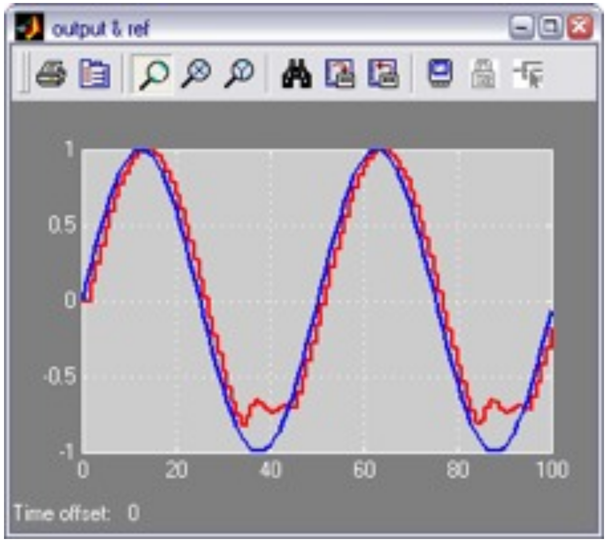
Constraint: $-1 \leq u(t) \leq 1$

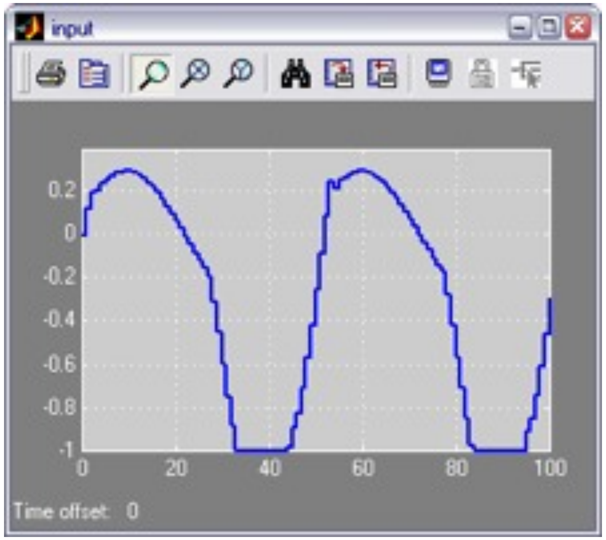
HybTbx: **/demos/hybrid/bm99sim.m**

Hybrid MPC - Example

Closed loop:







Optimal Control of Hybrid Systems: Computational Aspects

MIQP Formulation of MPC

(Bemporad, Morari, 1999)

$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{T-1} y_k' Q y_k + u_k' R u_k$$

$$\text{subject to} \begin{cases} x_{k+1} = Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k + B_5 \\ y_k = Cx_k + D_1 u_k + D_2 \delta_k + D_3 z_k + D_5 \\ E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5 \\ x_0 = x(t) \end{cases}$$

$$\xi = [u_0', \ldots, u_{T-1}', \delta_0', \ldots, \delta_{T-1}', z_0', \ldots z_{T-1}']'$$

$$\begin{aligned} \min_\xi \ & \frac{1}{2}\xi' H \xi + x(t)' F \xi + \cancel{\frac{1}{2}x'(t) Y x(t)} \\ \text{subj. to} \ & G\xi \leq W + Sx(t) \end{aligned}$$

Mixed Integer Quadratic Program (MIQP)

ξ has both real and $\{0,1\}$ components

MILP Formulation of MPC

(Bemporad, Borrelli, Morari, 2000)

$$
\min_{\xi} J(\xi, x(t)) = \sum_{k=0}^{T-1} \|Qy_k\|_\infty + \|Ru_k\|_\infty
$$

$$
\text{subject to} \begin{cases} x_{k+1} = Ax_k + B_1u_k + B_2\delta_k + B_3z_k + B_5 \\ y_k = Cx_k + D_1u_k + D_2\delta_k + D_3z_k + D_5 \\ E_2\delta_k + E_3z_k \leq E_4x_k + E_1u_k + E_5 \\ x_0 = x(t) \end{cases}
$$

- Basic trick: $\min |x|$ → $\min \epsilon$ s.t. $\epsilon \geq x$, $\epsilon \geq -x$

$\epsilon_k^x \geq \|Qy(t+k|t)\|_\infty$
$\epsilon_k^u \geq \|Ru(t+k)\|_\infty$

→

$$
\min_{\xi} \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u \quad \text{s.t. } G\xi \leq W + Sx(t)
$$

$\xi \triangleq [\epsilon_1^x, \ldots, \epsilon_{N_y}^x, \epsilon_1^u, \ldots, \epsilon_{T-1}, U, \delta, z]$

Mixed Integer Linear Program (MILP)

ξ has both real and $\{0, 1\}$ components

Mixed-Integer Program Solvers

- Mixed-Integer Programming is NP-hard

BUT

- Extremely rich literature in Operations Research (still very active)
- General purpose Branch & Bound/Branch & Cut solvers available for MILP and MIQP (CPLEX, Xpress-MP, BARON, GLPK, ...)

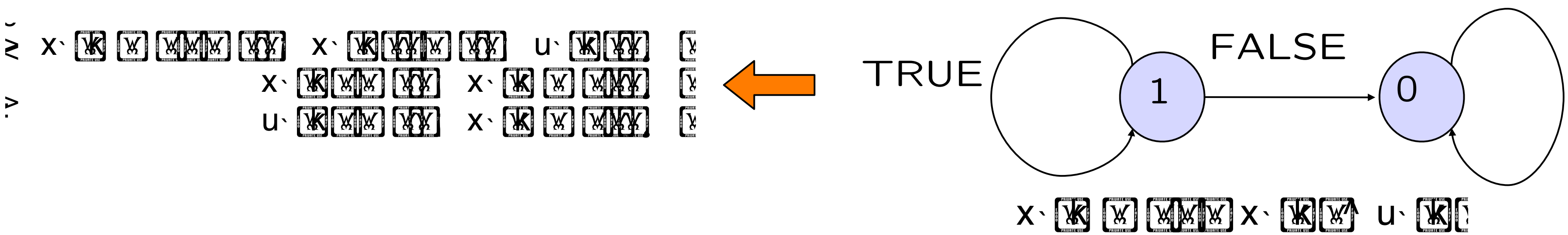
  More solvers and benchmarks: http://plato.la.asu.edu/bench.html

- No need to reach the global optimum for stability of MPC (see proof of the theorem), although performance deteriorates

Another Drawback of MIP

Main drawbacks when using Mixed-Integer Programming for implementing hybrid MPC control laws:

1. Loss of the original Boolean structure



Eciency of MIP solver usually not good when continuous LP/ QP relaxations are not tight

“Hybrid” Solvers

Combine MIP and Constraint Satisfaction (CSP) techniques to exploit the discrete structure of the problem

Why CSP ?

- More exible modeling than MIP
  (e.g.: constraint logic programming (CLP) and Satisability of Boolean formulas (SAT))
- Structure is kept and exploited to direct the search.

Why MIP ?

- Specialized techniques for highly structured problems (e.g. LP problems); Better for handling continuous vars
- A wide range of tight relaxations are available

Why a combined approach ?

Performance increase already shown in other application domains

(Harjunkoski, Jain, Grossmann, 2000)

SAT-Based Branch&Bound

The basic modeling framework has the following form:

(Bockmayr, Kasper,1998)

$$
\begin{aligned}
\min_z \ & f(z) \\
\text{s.t. } & Gz \le d, \ \overline{G}z = \overline{d} \\
& G'z + D'\mu \le E', \ \overline{G'}z + \overline{D'}\mu = \overline{E'} \\
& g(\nu, \mu) \\
& z \in \mathcal{R}^{n_z}, \ \mu \in \{0,1\}^{n_\mu}, \ \nu \in \{0,1\}^{n_\nu}
\end{aligned}
$$

Convex cost function

Continuous constraints

Mixed constraints

Purely logic constraints

optimization variables

SAT-based B&B "ingredients"

(Bemporad, Giorgetti, 2003)

A relaxed convex problem
(pure logic constraints dropped)

Convex solver (e.g.: LP/QP)

A SAT feasibility problem
(for logic constraints only)

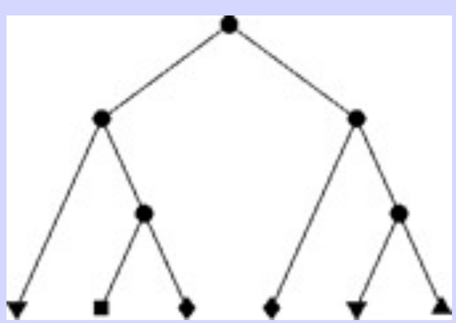
SAT solver

Computations: MILP vs. LB-B&B

T	Bool.	SATbB&B			Cplex 9.0		Pure B&B	
	Vars	(s)	LPs	SATs	(s)	LPs	(s)	LPs
5	82	0.09	5	6	0.03	18	0.48	23
10	157	0.18	5	6	0.13	79	3.7150	119
15	232	0.33	5	6	0.42	199	83.69	943
20	307	0.5110	6	8	0.5410	243	109.0870	2181
25	382	0.7620	8	10	0.8210	286	503.0030	3833
30	457	1.0520	9	12	1.0110	333	1072.3	6227
35	532	1.4420	10	13	1.7170	341	>1200	–
40	607	1.8630	13	16	2.5030	374	>1200	–
45	682	2.7740	15	20	3.8320	475	>1200	–

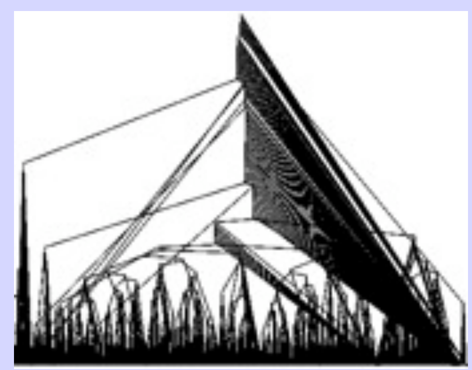
Computation time and # LP solved
for nding an optimal control sequence

SAT-based B&B



#nodes=11

Pure B&B



#nodes=6227

Pentium IV 1.8GHz SAT solver: zCHAFF 2003.07.22

MILP vs SAT

Uniform Random 3CNF benchmarks (from http://www.satlib.org)
All 3SAT instances are in the phase transition region.

		Sat instances		Unsat instances	
N. Vars	N. Cons	zCHAFF	CPLEX	zCHAFF	CPLEX
20	91	0	0.036	-	-
50	218	0	0.343	0	0.453
75	325	0	0.203	0	3.671
100	430	0	23.328	0	33.921
125	538	0.016	15.171	0.031	209.766
150	645	0.031	20.625	0.281	4949.58
175	753	0.031	> 1500	0.891	> 5000

Note: All clauses are passed to CPLEX 9.0 as logic constraints
(i.e.: not translated to linear integer inequalities)

Solvers:
CPLEX 9.0
zCHAFF 2003.12.04

PC: P4 2.8GHz + 1GB RAM

Another Drawback of MIP

Main drawbacks when using Mixed-Integer Programming for implementing hybrid MPC control laws:

1. **Loss of the original Boolean structure**

$$\begin{cases} x_\ell(k+1) + (1 - x_\ell(k)) + (1 - u_\ell(k)) \geq 1 \\ x_\ell(k) + (1 - x_\ell(k+1)) \geq 1 \\ u_\ell(k) + (1 - x_\ell(k+1)) \geq 1 \end{cases}$$

$$x_\ell(k+1) = x_\ell(k) \wedge u_\ell(k)$$

Eciency of MIP solver usually not good when continuous LP/ QP relaxations are not tight

2. **On-line combinatorial optimization**

Good for large sampling times (e.g., 1 h) / expensive hardware ...

... but not for fast sampling (e.g. 10 ms) / cheap hardware !

Explicit Hybrid Optimal Control

On-Line vs. O-Line Optimization

$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{T-1} \|Qy_k\|_\infty + \|Ru_k\|_\infty$$

$$\text{subject to} \begin{cases} x_{k+1} = Ax_k + B_1u_k + B_2\delta_k + B_3z_k + B_5 \\ y_k = Cx_k + D_1u_k + D_2\delta_k + D_3z_k + D_5 \\ E_2\delta_k + E_3z_k \leq E_4x_k + E_1u_k + E_5 \\ x_0 = x(t) \end{cases}$$

- On-line optimization: solve the problem *for each* given $x(t)$

Mixed-Integer Linear Program (MILP)

- O-line optimization: solve the MILP **for all** $x(t)$ in advance

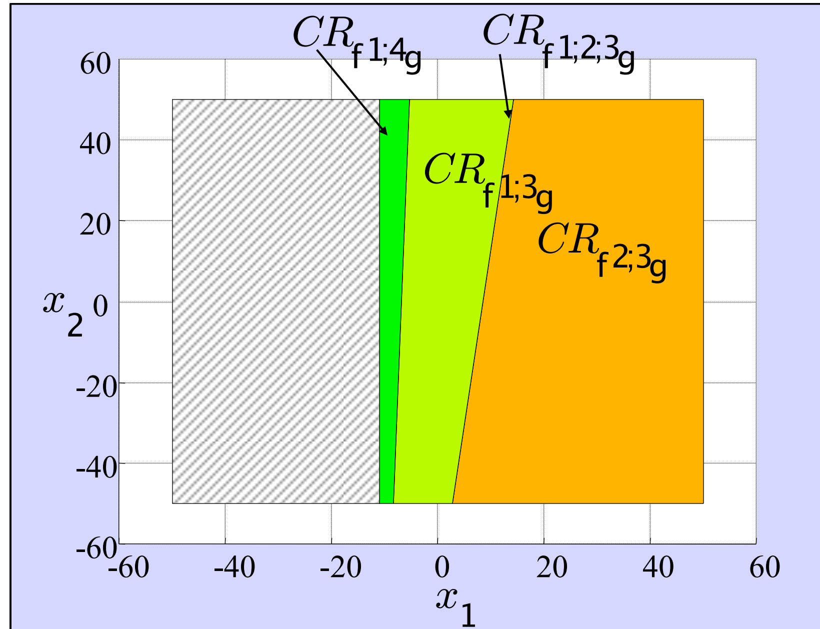
$$\begin{aligned} \min_\xi \quad & \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u \\ \text{s.t.} \quad & G\xi \leq W + Sx(t) \end{aligned}$$

multi-parametric Mixed Integer Linear Program (mp-MILP)

Example of Multiparametric Solution

Multiparametric LP (ø 2)R^2

$$
\begin{aligned}
\min_{\xi} \quad & -3\xi_1 - 8\xi_2 \\
\text{s.t.} \quad & \left\{
\begin{array}{lcl}
\xi_1 + \xi_2 & \leq & 13 + x_1 \\
5\xi_1 - 4\xi_2 & \leq & 20 \\
-8\xi_1 + 22\xi_2 & \leq & 121 + x_2 \\
-4\xi_1 - \xi_2 & \leq & -8 \\
-\xi_1 & \leq & 0 \\
-\xi_2 & \leq & 0
\end{array}
\right.
\end{aligned}
$$



$$
\xi(x) = \left\{
\begin{array}{llll}
\begin{bmatrix} 0.00 & 0.05 \\ 0 & 0.06 \end{bmatrix} x + \begin{bmatrix} 11.85 \\ 9.80 \end{bmatrix} & \text{if} & \begin{bmatrix} 0.02 & 0.00 \\ 0.00 & 0.02 \\ 0.00 & -0.02 \\ -0.12 & 0.01 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ -1.00 \end{bmatrix} & \mathrm{CR}_{\{2,3\}} \\[2em]
\begin{bmatrix} 0.73 & -0.03 \\ 0.27 & 0.03 \end{bmatrix} x + \begin{bmatrix} 5.50 \\ 7.50 \end{bmatrix} & \text{if} & \begin{bmatrix} 0.00 & 0.02 \\ 0.00 & -0.02 \\ 0.12 & -0.01 \\ -0.15 & 0.00 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} & \mathrm{CR}_{\{1,3\}} \\[2em]
\begin{bmatrix} -0.33 & 0.00 \\ 1.33 & 0 \end{bmatrix} x + \begin{bmatrix} -1.67 \\ 14.67 \end{bmatrix} & \text{if} & \begin{bmatrix} 0.00 & 0.02 \\ 0.00 & -0.02 \\ 0.15 & -0.00 \\ -0.09 & 0.00 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ -1.00 \\ 1.00 \end{bmatrix} & \mathrm{CR}_{\{1,4\}}
\end{array}
\right.
$$

Multiparametric MILP

$$\min_{\xi=\{\xi_c,\xi_d\}} f'\xi_c + d'\xi_d \qquad \xi_c \in \mathbb{R}^n$$
$$\text{s.t. } G\xi_c + E\xi_d \leq W + Fx \qquad \xi_d \in \{0,1\}^m$$

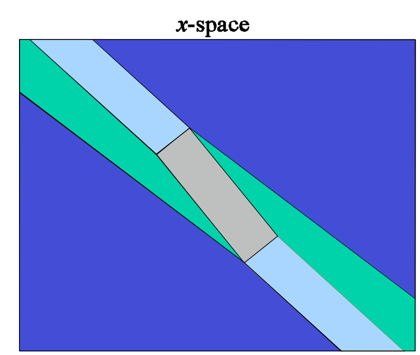
- mp-MILP can be solved (by alternating MILPs and mp-LPs)

(Dua, Pistikopoulos, 1999)

- Theorem: The multiparametric solution $\xi^*(x)$ is piecewise ane

- Corollary: The hybrid MPC controller is piecewise ane in x

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \leq K_N \end{cases}$$



More Ecient Approaches

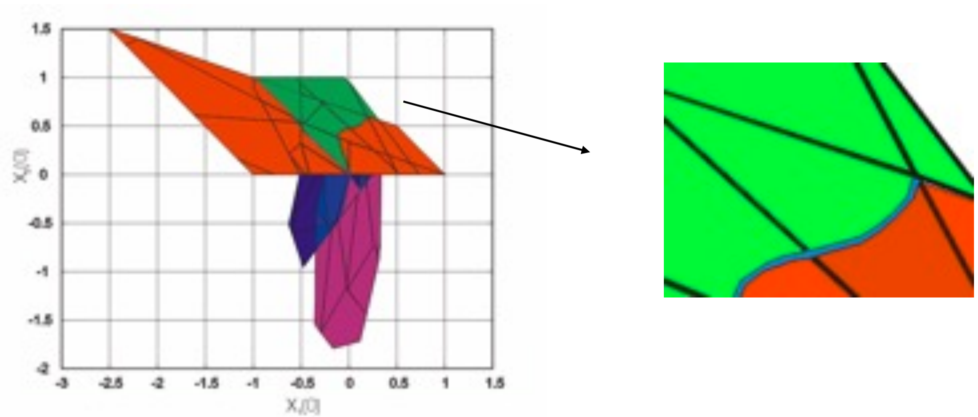
(Borrelli, Baotic, Bemporad, Morari, 2003)
(Mayne, ECC 2001)

- Explicit solutions to nite-time optimal control problems for PWA systems can be obtained using a combination of

  - Multiparametric linear (1-norm, ∞-norm), or quadratic (squared 2-norm) programming

  - Dynamic programming or enumeration of feasible mode sequences

Note: in the 2-norm case, the partition may not be polyhedral



Hybrid Control Example (Revisited)

Hybrid Control - Example

Switching System:

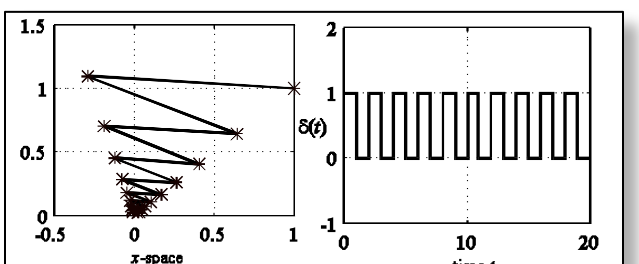
$$x(t+1) = 0.8\begin{bmatrix}\cos\theta(t) & -\sin\theta(t)\\ \sin\theta(t) & \cos\theta(t)\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t)$$

$$y(t) = [0 \quad 1]\,x(t)$$

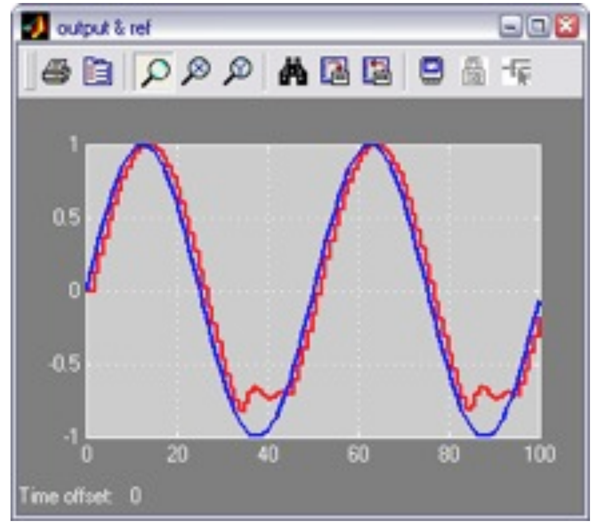
$$\theta(t) = \begin{cases}\frac{\pi}{3} & \text{if } [1\ \ 0]x(t) \geq 0\\ -\frac{\pi}{3} & \text{if } [1\ \ 0]x(t) < 0\end{cases}$$

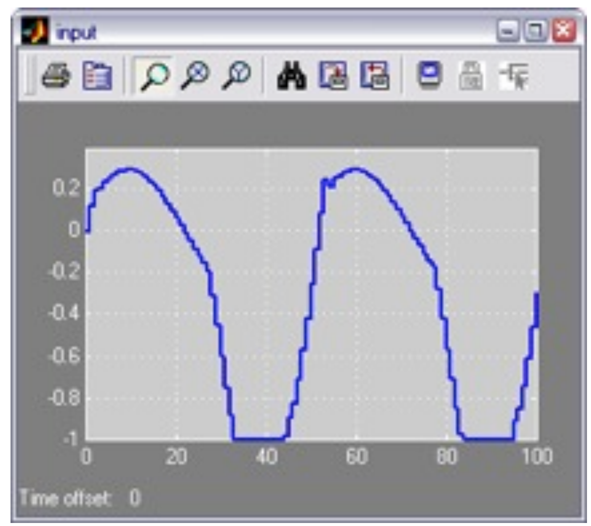
Constraint: $\boxed{-1 \leq u(t) \leq 1}$

Open loop:



Closed loop:

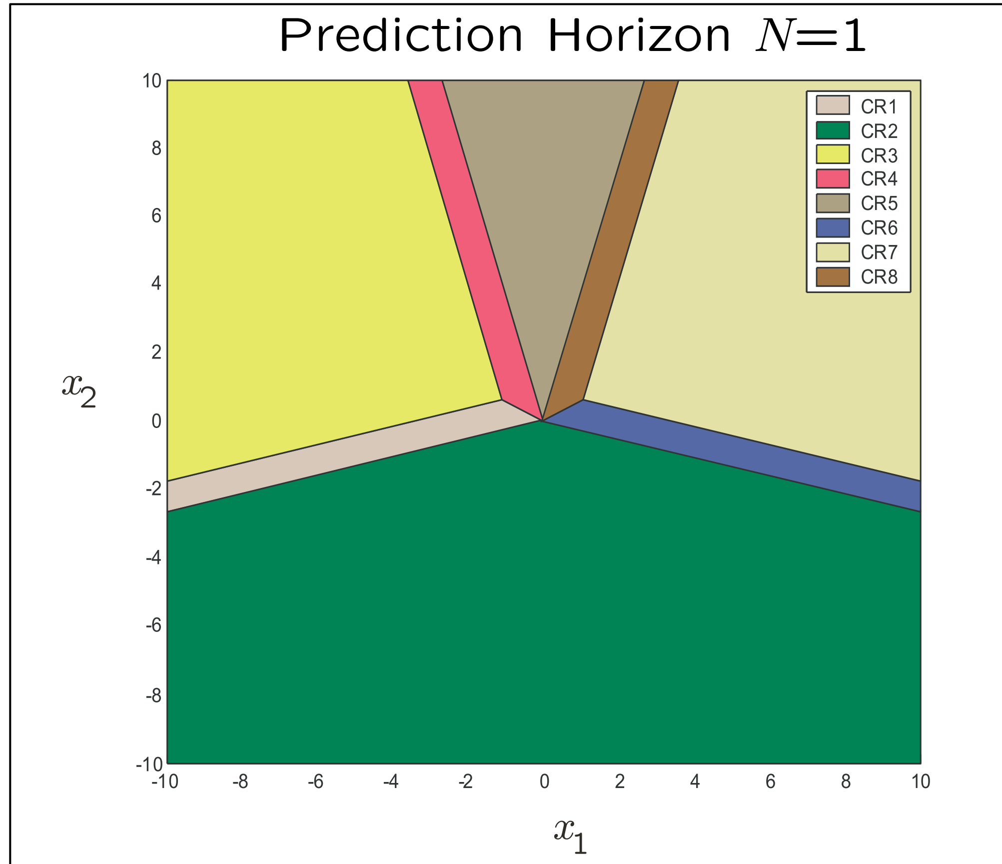




HybTbx: `/demos/hybrid/bm99sim.m`

Explicit PWA Controller

$$u(x) = \begin{cases} \begin{bmatrix} 0.2928 & -1.093 \end{bmatrix} x & \text{if } \begin{bmatrix} 0.5856 & -2.186 \\ -0.2928 & 1.093 \\ 8 & 13.86 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ & \text{(Region \#1)} \\ 0 & \text{if } \begin{bmatrix} -2.928 & 10.93 \\ 2.928 & 10.93 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ & \text{(Region \#2)} \\ -1 & \text{if } \begin{bmatrix} 1.093 & 0.2928 \\ 0.2928 & -1.093 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ & \text{(Region \#3)} \\ \begin{bmatrix} 1.093 & 0.2928 \end{bmatrix} x & \text{if } \begin{bmatrix} 2.186 & 0.5856 \\ -1.093 & -0.2928 \\ -8 & -13.86 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ & \text{(Region \#4)} \\ 0 & \text{if } \begin{bmatrix} -10.93 & -2.928 \\ 10.93 & -2.928 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ & \text{(Region \#5)} \\ \begin{bmatrix} -0.2928 & -1.093 \end{bmatrix} x & \text{if } \begin{bmatrix} -0.5856 & -2.186 \\ 0.2928 & 1.093 \\ -8 & 13.86 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ & \text{(Region \#6)} \\ -1 & \text{if } \begin{bmatrix} -0.2928 & -1.093 \\ -1.093 & 0.2928 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ & \text{(Region \#7)} \\ \begin{bmatrix} -1.093 & 0.2928 \end{bmatrix} x & \text{if } \begin{bmatrix} -2.186 & 0.5856 \\ 1.093 & -0.2928 \\ 8 & -13.86 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ & \text{(Region \#8)} \end{cases}$$



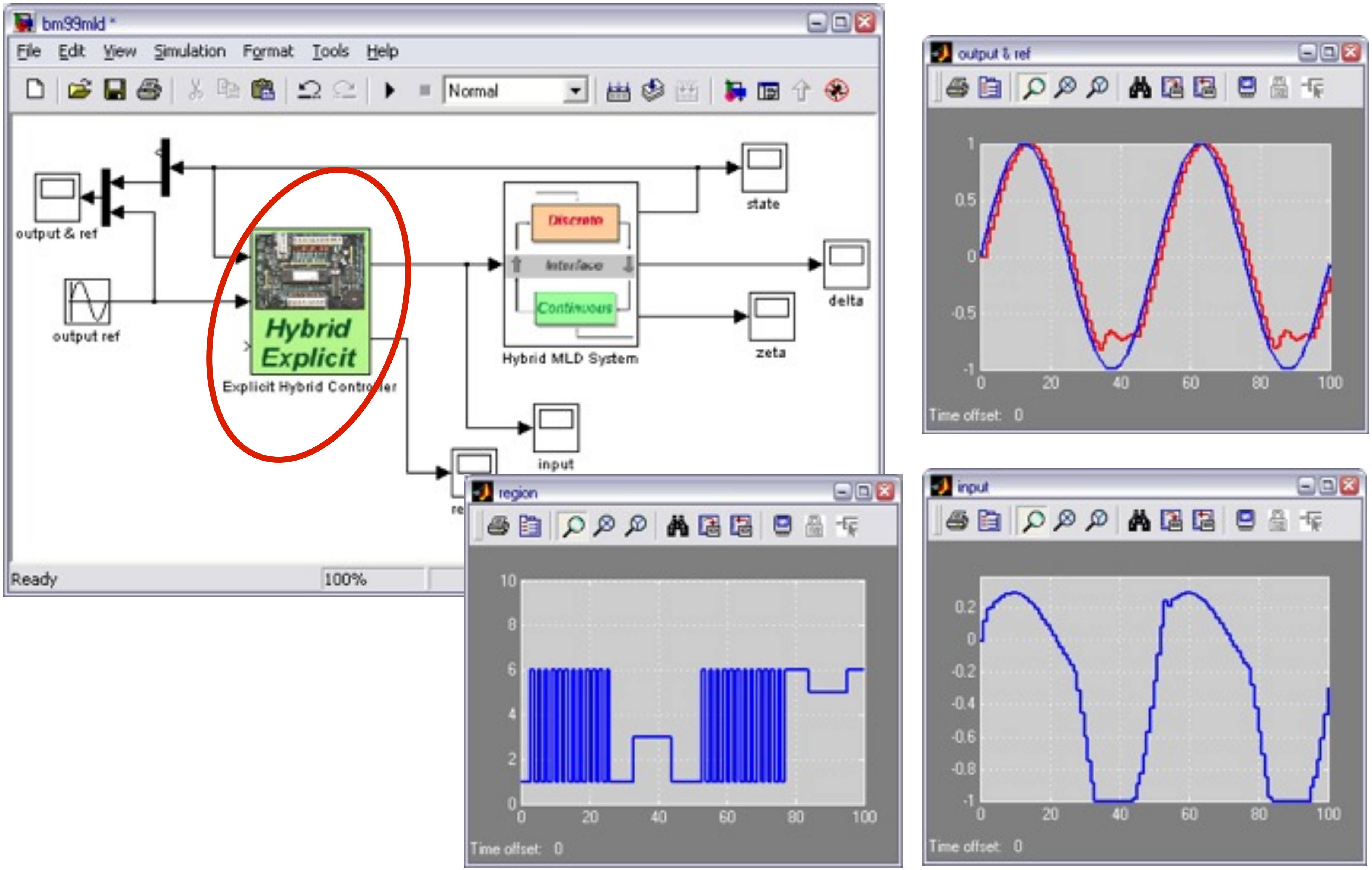
PWA law = MPC law

HybTbx: **/demos/hybrid/bm99benchmark.m**

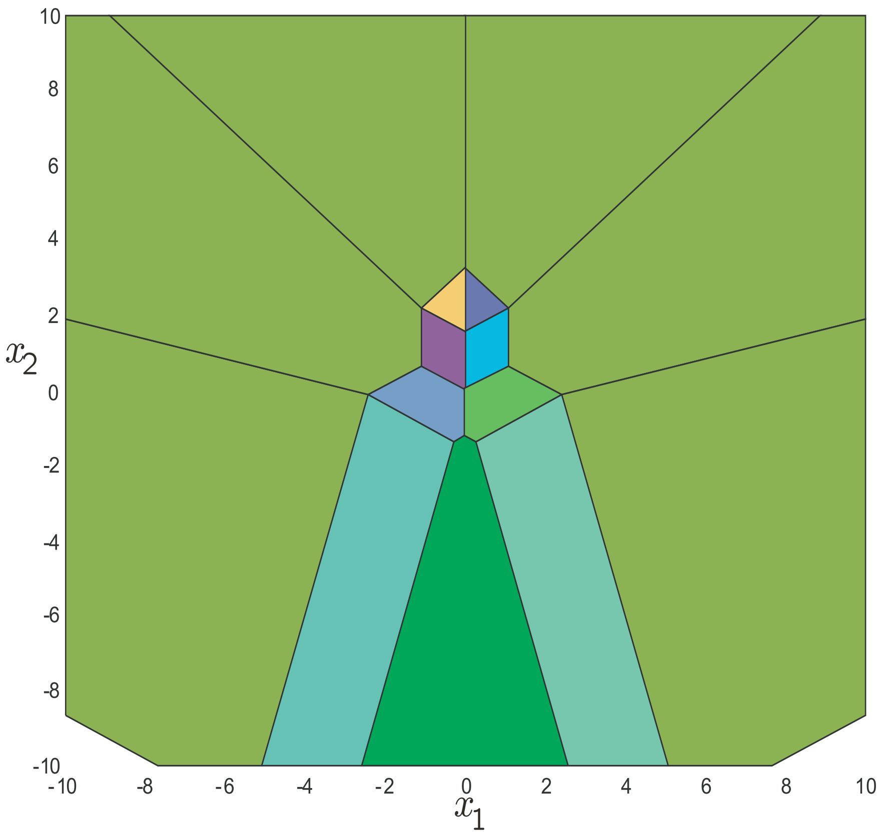
(CPU time: 1.44 s, Pentium III 800)

Hybrid MPC - Example

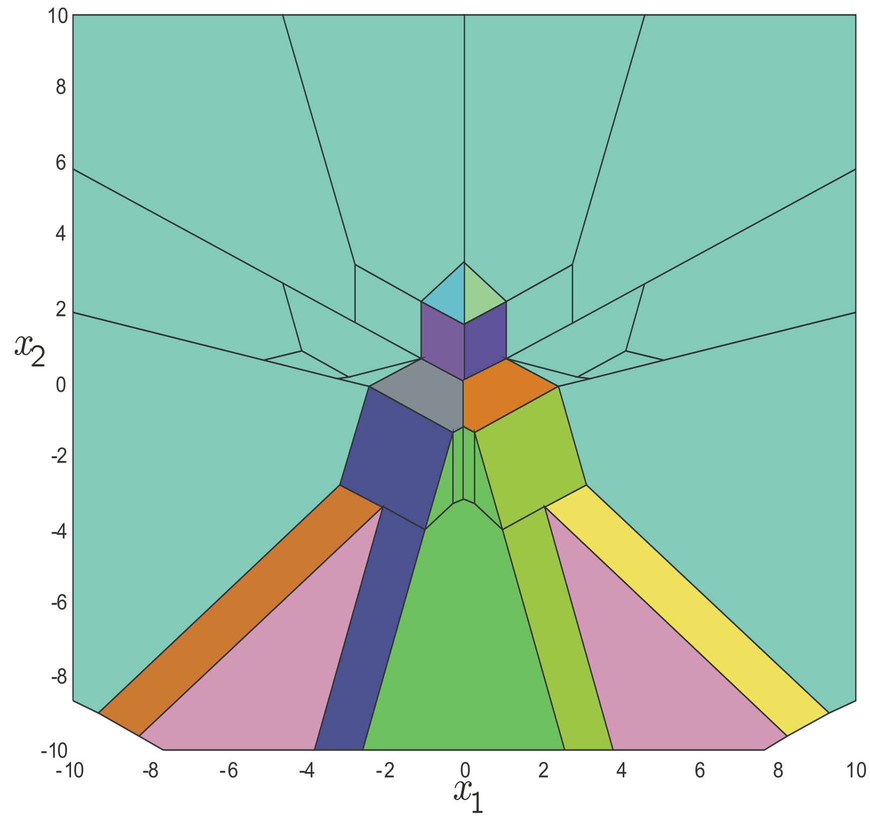
Closed loop:



Explicit PWA Controller



Prediction Horizon N=2



Prediction Horizon N=3

Comments on Explicit Solutions

- Alternative: either (1) solve an MIP on-line
  or (2) evaluate a PWA function

- For problems with many variables and/or long horizons: MIP may be preferable

- For simple problems:
  - time to evaluate the control law is shorter than MIP
  - control code is simpler (no complex solver must be included in the control software)
  - more insight in controller's behavior

Outline

- ✓ What is a “hybrid system” ?
- ✓ Models of hybrid systems
- ✓ Controller synthesis for hybrid systems
- Applications (automotive)

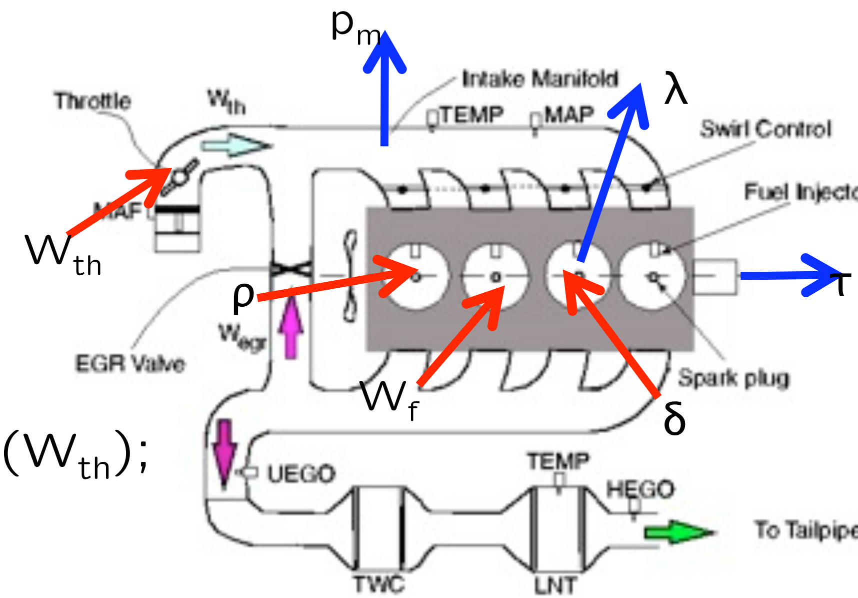
Hybrid Control of a DISC Engine

(joint work with N. Giorgetti, I. Kolmanovsky and D. Hrovat)

FORD RESEARCH LABORATORY

DISC Engine

- States/Controlled outputs:
  - Intake manifold pressure (p_m);
  - Air-to-fuel ratio (λ);
  - Engine brake torque (τ);



- Inputs (continuous):
  - Air Mass ow rate through throttle (W_{th});
  - Mass ow rate of fuel (W_f);
  - Spark timing (δ);
- Inputs (binary):
  - ρ = regime of combustion (homogeneous/stratied);
- Constraints on:
  - Air-to-Fuel ratio (due to engine roughness, misring, smoke emiss.)
  - Spark timing (to avoid excessive engine roughness)
  - Mass ow rate on intake manifold (constraints on throttle)

Dynamic equations are nonlinear
Dynamics and constraints depend on regime ρ !

DISC Engine - HYSDEL List

```
SYSTEM hysdisc{
   INTERFACE{
      STATE{
         REAL pm       [1, 101.325];
      }
      OUTPUT {
          REAL lambda;  /* [10,      50]; */
         REAL tau;      /* [0,      100]; */
      }
      INPUT{
         REAL Wth      [0,38.5218];
         REAL Wf       [0,         2];
         REAL delta    [0,        40];
         BOOL rho;
      }
      PARAMETER{
         REAL pm1, pm2;
         REAL l01, l02, l0c;
         REAL l11,l12,l1c;
         REAL t01,t02,t03,t04,t05;
         REAL t11,t12,t13,t14,t15;
      }
   }
IMPLEMENTATION{
      AUX{
          REAL lam;
          REAL taul;
          REAL lmin,lmax;
          REAL dmbt;   }
```

```
DA{
   lam={ IF rho THEN l11*pm+l12*Wf+l1c
                ELSE   l01*pm+l02*Wf+l0c};
  taul={IF rho THEN
            t11*pm+t12*Wf+t13*delta+t14*lam+t15
     ELSE   t01*pm+t02*Wf+t03*delta+t04*lam+t05 };
  lmin={IF rho THEN 13 ELSE 19}; /*rho=1 19 */
  lmax ={IF rho THEN 21 ELSE 38}; /*rho=0 21 */
  dmbt ={IF rho THEN -28.74+3.1845*lam
                ELSE 14.0877+0.2810*lam};
}
CONTINUOUS{
   pm=pm1*pm+pm2*Wth;
}
OUTPUT {
   lambda=lam;
   tau=taul;
}
MUST{
   lmin-lam     <=0;
   lam-lmax     <=0;
   delta-dmbt   <=0;
}
}
}
```

Optimal Control of DISC Engine

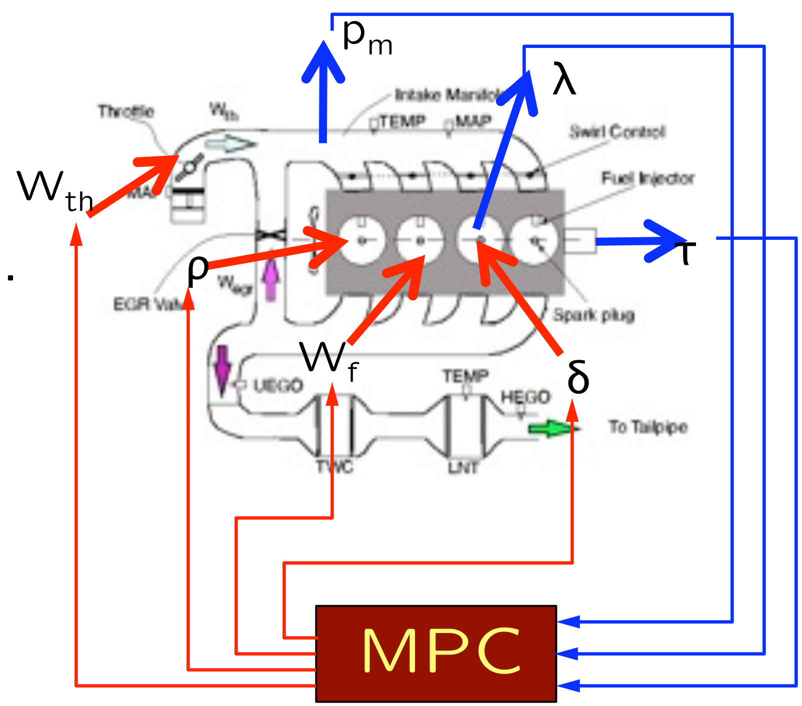
$$\min_{\xi} J(\xi, x(t)) = \sum_{k=0}^{N} |q_\tau(\tau_k - \tau_{\text{ref}})|_p + |q_\lambda(\lambda_k - \lambda_{\text{ref}})|_p$$
$$+|r_{W_{th}}(W_{th,k} - W_{th,\text{ref}})|_p + |r_{W_f}(W_{f,k} - W_{f,\text{ref}})|_p$$
$$+|r_\delta(\delta_k - \delta_{ref})|_p + |r_\rho(\rho_k - \rho(t-1))|_p$$
$$\text{subj. to} \left\{ \text{MLD model} \right. \qquad p = 2, \infty$$

- Two Sets of weights

  1. Torque control mode;
  2. Air-to-Fuel ratio control mode.

- Implementation:

  Receding horizon optimal control (MPC)



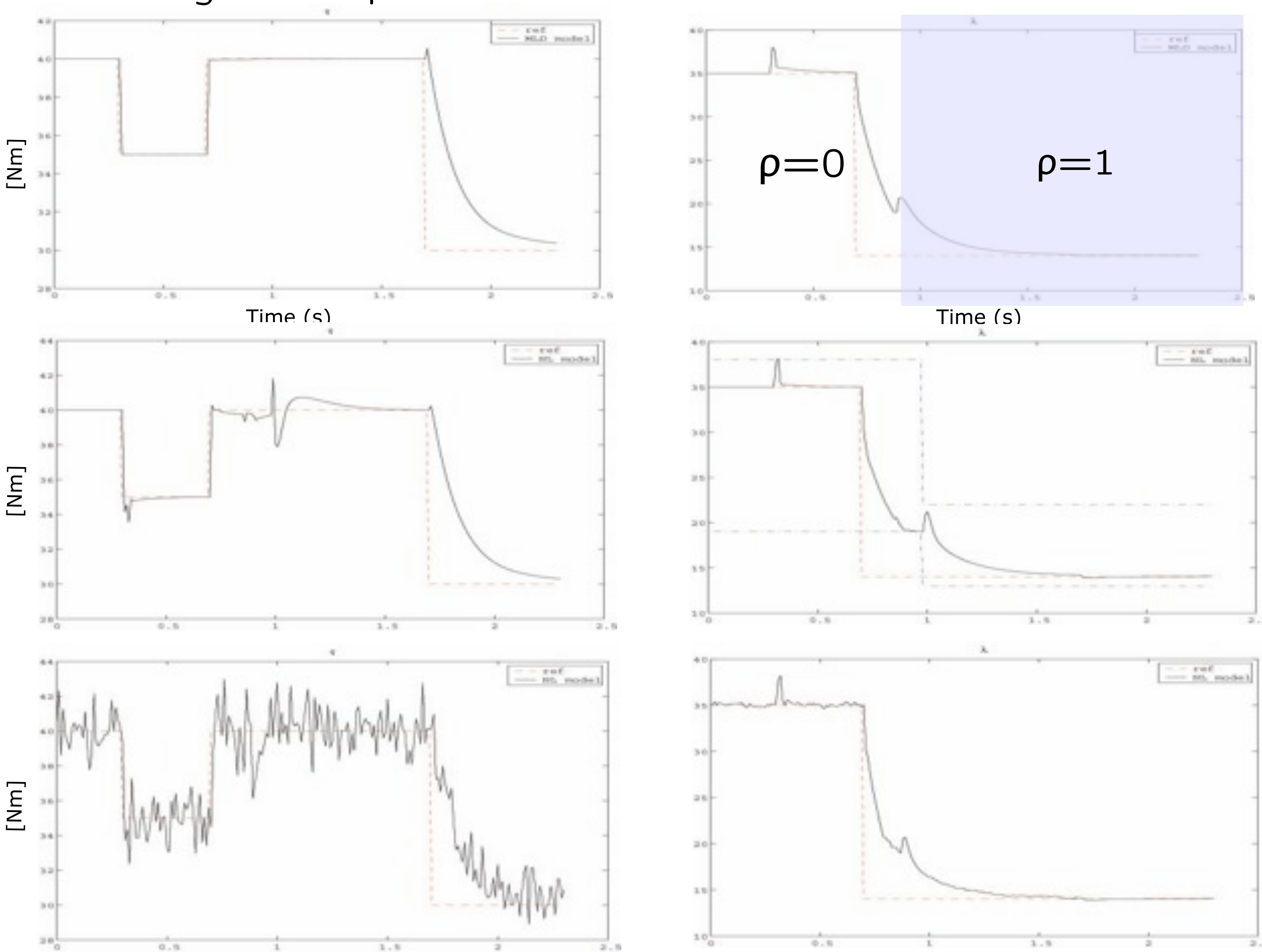
Simulation Results

Engine Torque | Air-to-Fuel Ratio

Nominal

Nonlinear

With noise

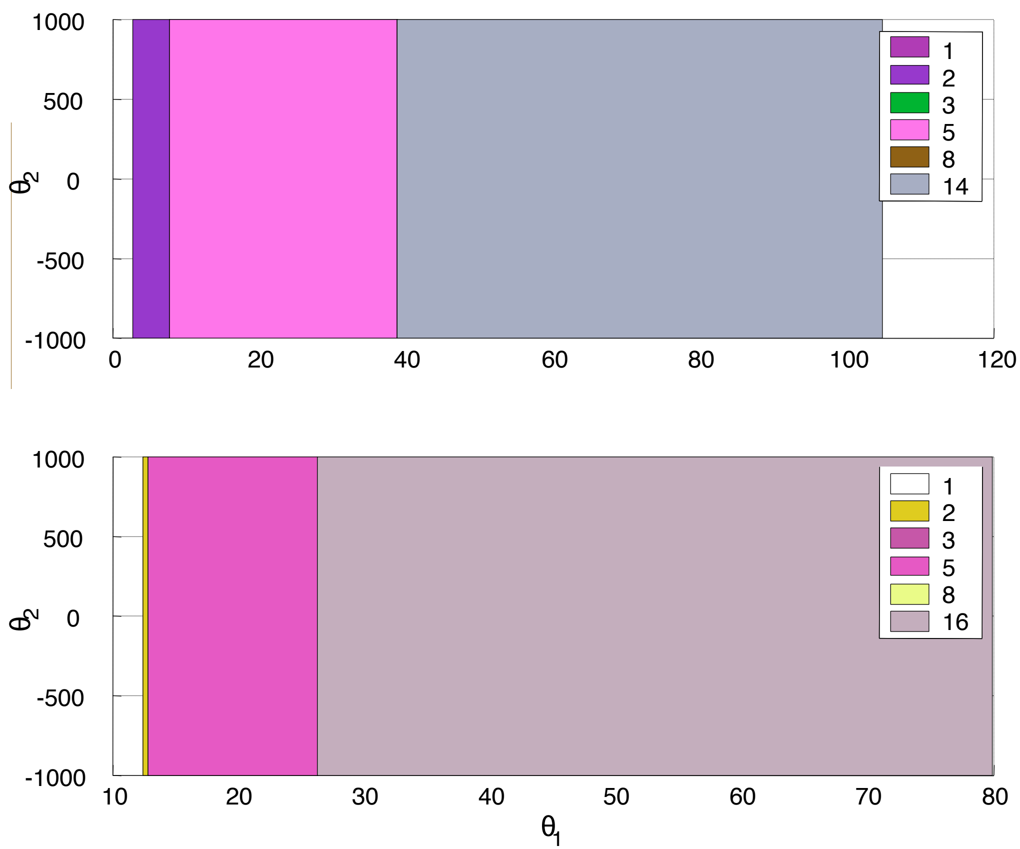


Explicit MPC Controller (2-Norm)

N=1 (control horizon)

7 parameters

33 partitions

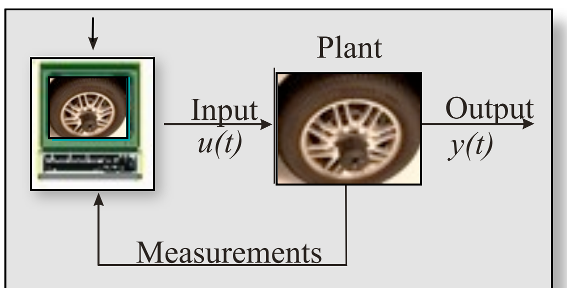


Conclusions

- Hybrid systems as a framework for new applications, where both logic and continuous dynamics are relevant

- Supervisory MPC controllers schemes can be synthesized via on-line mixed-integer programming (MILP/MIQP)

- Piecewise Linear MPC Controllers can be synthesized o-line via multiparametric programming for fast-sampling applications



Hybrid modeling and MPC design

Multiparametric programming

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \leq K_N \end{cases}$$

C-code download & testing

Conclusions

- Hybrid systems as a framework for new applications, where both logic and continuous dynamics are relevant

- Supervisory MPC controllers schemes can be synthesized via on-line mixed-integer programming (MILP/MIQP)

- Piecewise Linear MPC Controllers can be synthesized o-line via multiparametric programming for fast-sampling applications

- Current research activities
  - Control of Discrete Hybrid Stochastic Automata
  - DC-Programming for explicit hybrid control laws (2-norm)
  - Stochastic mixed-integer programming
  - Applications (Scheduling of Cement Mills, Automotive,...)

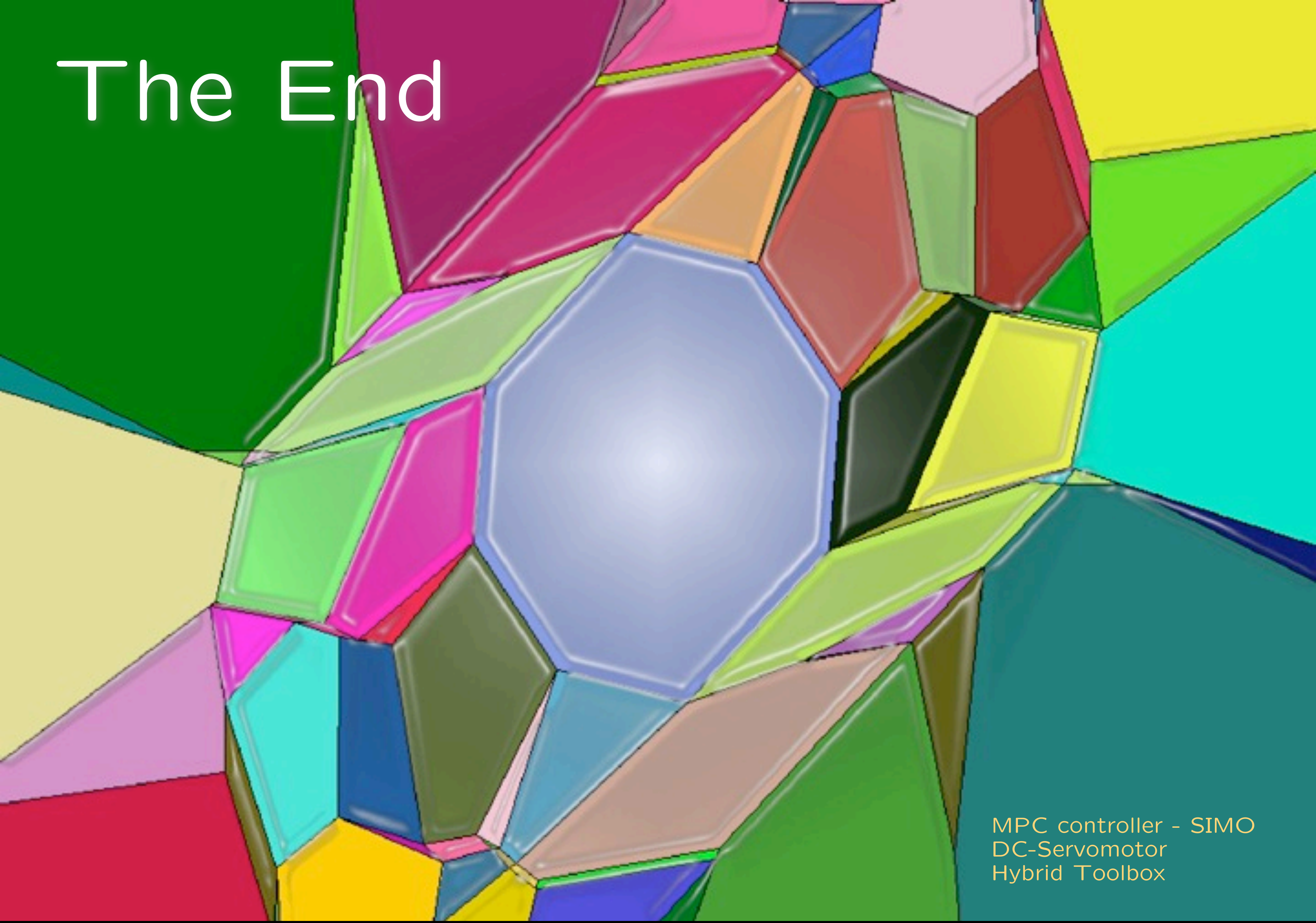
Acknowledgements

A. Alessio, M. Baotic, F. Borrelli, B. De Schutter,
S. Di Cairano, G. Ferrari-Trecate, K. Fukuda, N. Giorgetti,
M. Heemels, D. Hrovat, I. Kolmanovski, M. Lazar, L. Ljung,
D. Mignone, M. Morari, S. Paoletti, J. Roll, F.D. Torrisi

Announcements

- HYCON Network of Excellence (HYbrid CONtrol), FP6
- PhD School on Hybrid Systems, Siena, 2005

The End



MPC controller - SIMO
DC-Servomotor
Hybrid Toolbox

http://www.dii.unisi.it/hybrid/toolbox