Modeling, Control, and Reachability Analysis of Discrete-Time Hybrid Systems

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DISC Course on
Modeling and Control of Hybrid Systems

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Università degli Studi di Siena
(founded in 1240)

COHES Group
Control and Optimization of Hybrid and Embedded SystemS

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Outline

- What is a “hybrid system” ?
- Models for hybrid systems
- Control of hybrid systems
- Safety analysis of hybrid systems
- Examples (automotive)

Motivating Problem

GOAL:

command gear ratio, gas pedal, and brakes to **track** a desired speed and minimize consumptions

CHALLENGES:

- continuous **and** discrete inputs
- dynamics depends on gear
- nonlinear torque/speed maps

Hybrid Systems

(Witsenhausen, 1966)

Computer Science

Control Theory

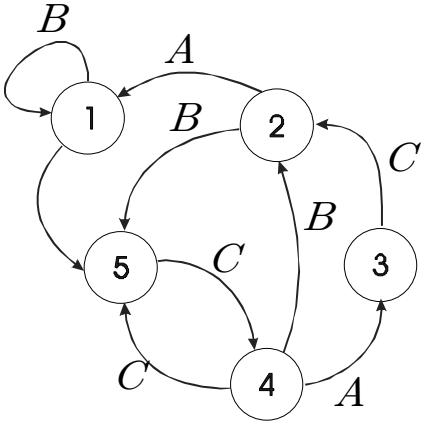
$X = \{1,2,3,4,5\}$
$U = \{A,B,C\}$

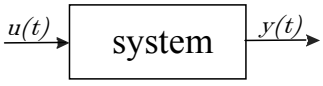
$x \in \mathbf{R}^n, u \in \mathbf{R}^m$
$y \in \mathbf{R}^p$

Finite state machines

Continuous dynamical systems

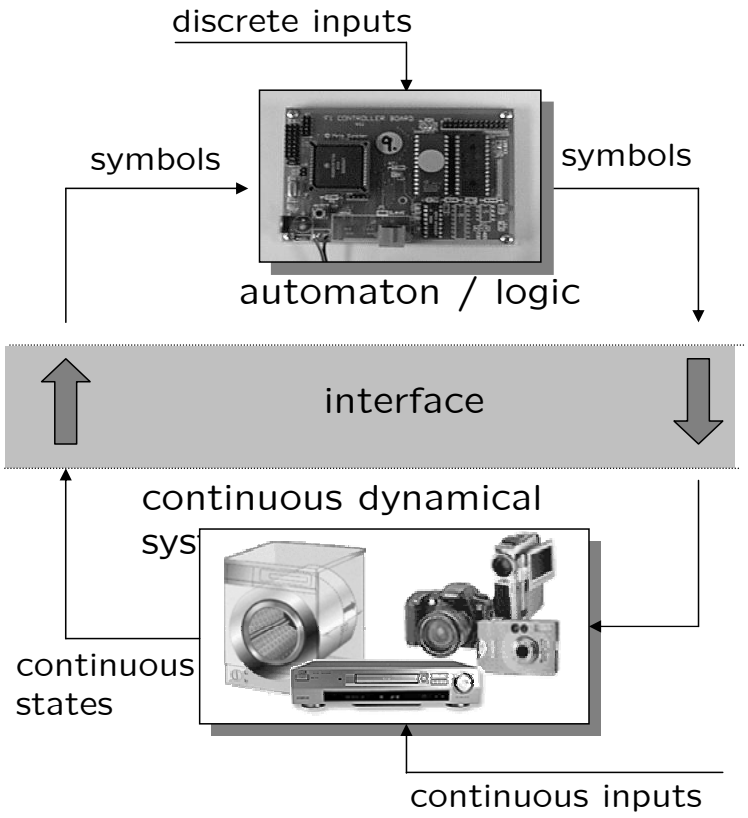
Hybrid systems





$$\begin{cases} \frac{dx(t)}{dt} = f(x(t),u(t)) \\ y(t) = g(x(t),u(t)) \end{cases}$$

Motivation: Embedded Systems

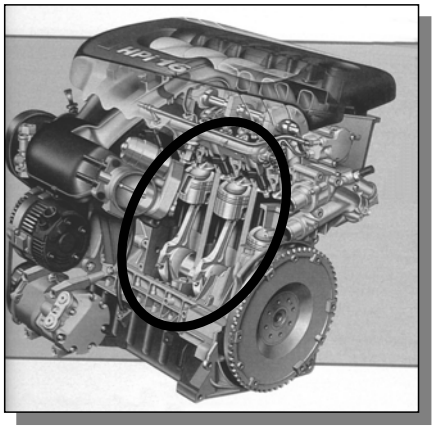


- Consumer electronics
- Home appliances
- Office automation
- Automobiles
- Industrial plants
- ...

Motivation: "Intrisically Hybrid"

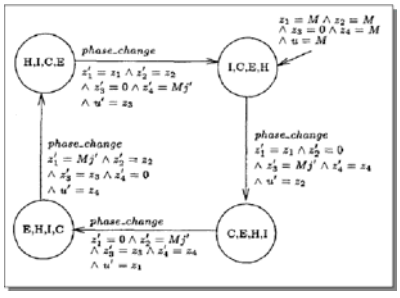
- Transmission

Discrete command (R,N,1,2,3,4,5) + Continuous dynamical variables (velocities, torques)



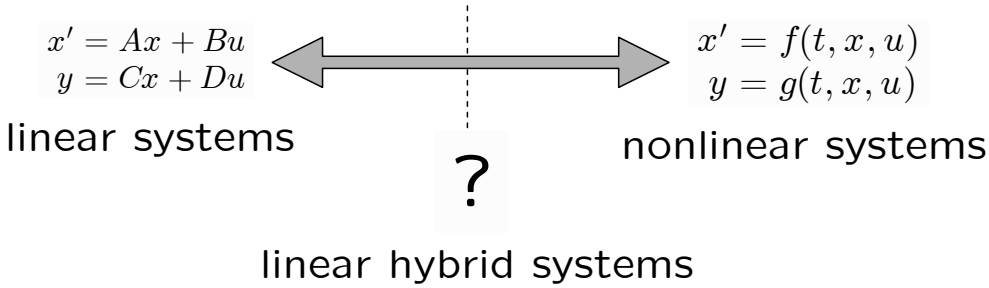
- Four-stroke engines

Automaton, dependent on power train motion →



Key Requirements for Hybrid Models

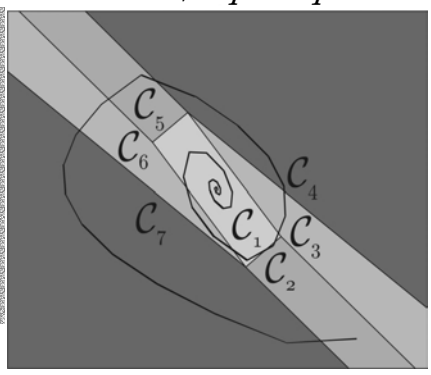
- **Descriptive** enough to capture the behavior of the system
  - continuous dynamics (physical laws)
  - logic components (switches, automata, software code)
  - interconnection between logic and dynamics
- **Simple** enough for solving *analysis* and *synthesis* problems



Piecewise Affine Systems *(Sontag 1981)*

$$
\begin{aligned}
x'(k) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\
y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\
\\
i(k) \quad &\text{s.t.} \quad H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}
\end{aligned}
$$

state+input space



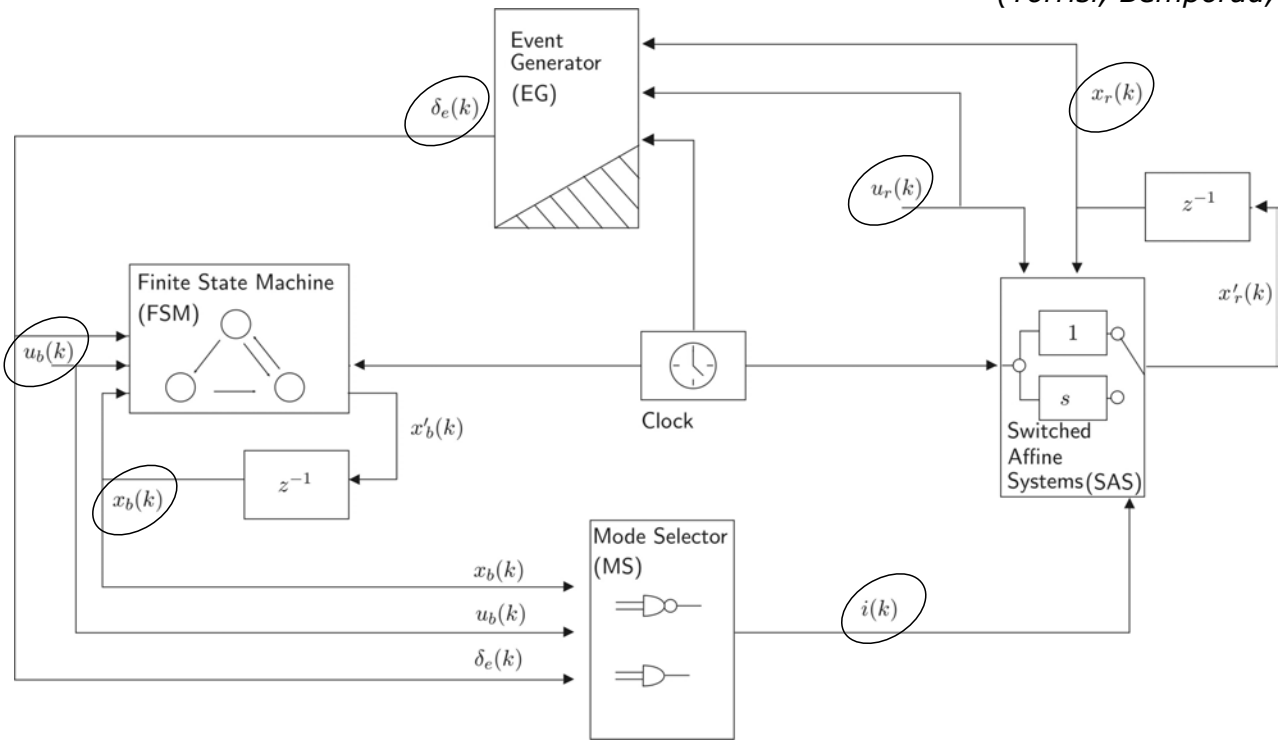
$x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^p$

$i(k) \in \{1, \ldots, s\}$

- Approximates nonlinear dynamics arbitrarily well
- Suitable for stability analysis, reachability analysis (verification), controller synthesis, ...

Discrete Hybrid Automata

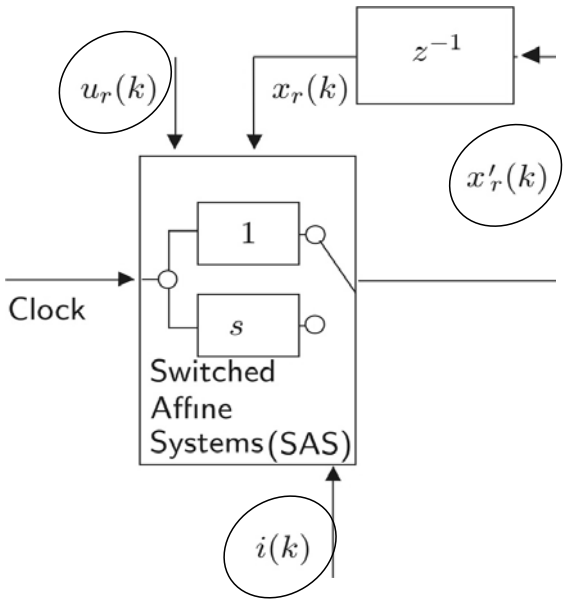
(Torrisi, Bemporad, 2003)



$x_r \in \mathbb{R}^{n_r}$ = continuous states
$x_b \in \{0,1\}^{n_b}$ = binary states
$i(k) \in \{1,\ldots,s\}$ = mode

$u_r \in \mathbb{R}^{m_r}$ = continuous inputs
$u_b \in \{0,1\}^{m_b}$ = binary inputs
$\delta_e \in \{0,1\}^{n_e}$ = event conditions

Switched Affine Systems



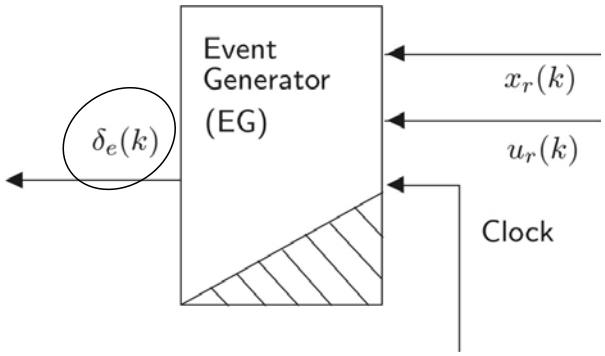
Linear affine dynamics depends upon the mode $i(k)$

$$x'_r(k) = A_{i(k)}x_r(k) + B_{i(k)}u_r(k) + f_{i(k)}$$

$$y_r(k) = C_{i(k)}x_r(k) + D_{i(k)}u_r(k) + g_{i(k)}$$

$x_r \in \mathbb{R}^{n_r},\ u_r \in \mathbb{R}^{m_r},\ y_r \in \mathbb{R}^{p_r}$

Event Generator

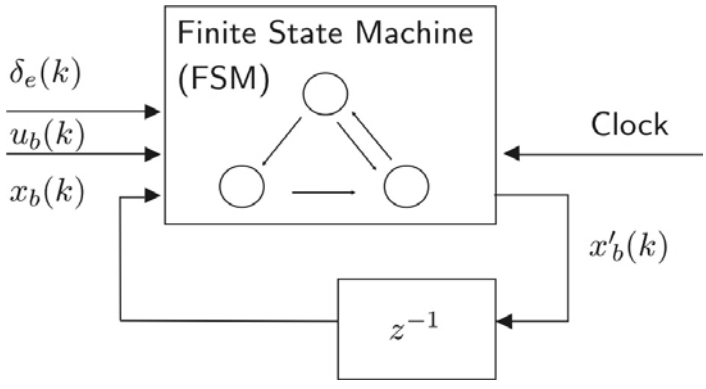


Generates a logic signal according to the satisfaction of a linear affine constraint

$$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_r(k) + K^i u_r(k) + J^i k \leq W^i], \;\; \delta_e^i \in \{0, 1\}$$

$$x_r \in \mathbb{R}^{n_r}, \;\; u_r \in \mathbb{R}^{m_r}, \;\; \delta_e^i \in \{0, 1\}$$

Finite State Machine

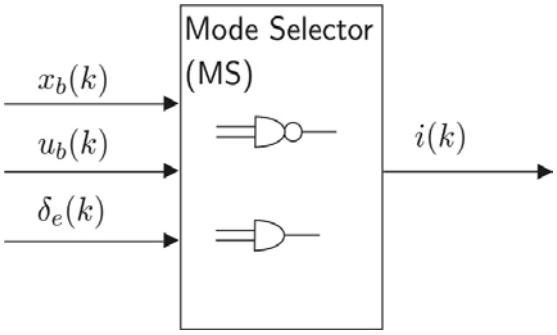


Discrete dynamics evolving according to a Boolean state update function

$$\begin{aligned} x'_b(k) &= f_{\mathsf{B}}(x_b(k), u_b(k), \delta_e(k)) \\ y_b(k) &= g_{\mathsf{B}}(x_b(k), u_b(k), \delta_e(k)) \end{aligned}$$

$$x_b \in \{0, 1\}^{n_b}, \;\; y_b \in \{0, 1\}^{p_b}$$

Mode Selector



a Boolean function selects the active mode $i(k)$ of the SAS

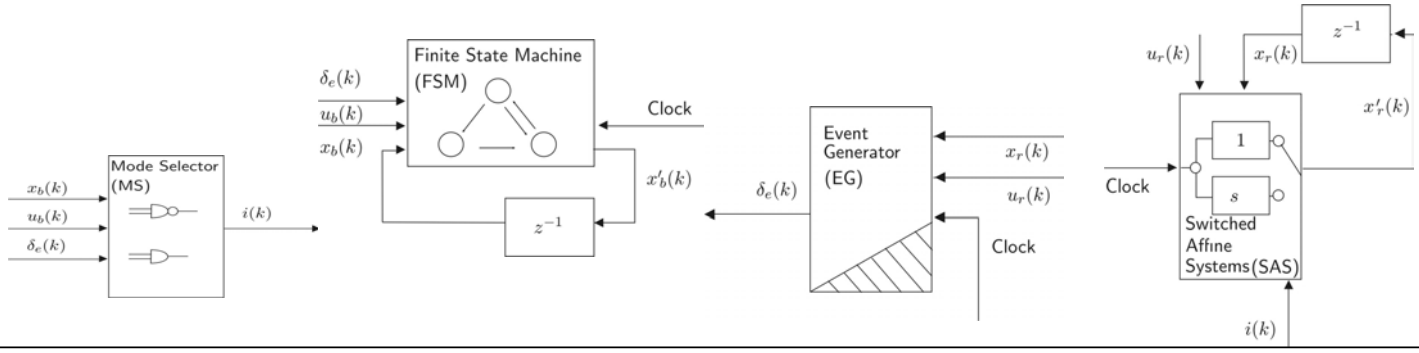
$$i(k) \;=\; f_{\mathsf{M}}(x_b(k), u_b(k), \delta_e(k))$$

$$i(k) \in \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \subset \{0,1\}^s$$

Logic and Inequalities

Glover 1975, Williams 1977

$X_1 \vee X_2$	$X_1 + X_2 \geq 1$
Any logic statement $f(X)$ =TRUE $\bigwedge_{j=1}^{m} \left(\vee_{i \in P_j} X_i \vee_{i \in N_j} \neg X_i \right)$ (CNF) $N_j, P_j \subseteq \{1, \ldots, n\}$	$A\delta \leq B$ $\begin{cases} 1 \leq \sum_{i \in P_1} X_i + \sum_{i \in N_1} (1 - X_i) \\ \vdots \\ 1 \leq \sum_{i \in P_m} X_i + \sum_{i \in N_m} (1 - X_i) \end{cases}$
$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_r(k) \leq W^i]$	$\begin{array}{rcl} H^i x_r(k) - W^i & \leq & M^i(1 - \delta_e^i) \\ H^i x_r(k) - W^i & > & m^i \delta_e^i \end{array}$
IF δ THEN $z = a_1^T x + b_1^T u + f_1$ ELSE $z = a_2^T x + b_2^T u + f_2$	$\begin{array}{rcl} (m_2 - M_1)\delta + z & \leq & a_2 x + b_2 u + f_2 \\ (m_1 - M_2)\delta - z & \leq & -a_2 x - b_2 u - f_2 \\ (m_1 - M_2)(1 - \delta) + z & \leq & a_1 x + b_1 u + f_1 \\ (m_2 - M_1)(1 - \delta) - z & \leq & -a_1 x - b_1 u - f_1 \end{array}$



Transformation of Logic into Linear Integer Inequalities

0. Given a Boolean statement $F(X_1, X_2, \ldots, X_n)$

1. Convert to Conjunctive Normal Form (CNF):

$$\bigwedge_{j=1}^{m} \left(\bigvee_{i \in P_j} X_i \bigvee_{i \in N_j} \bar{X}_i \right)$$

2. Transform into inequalities:

$$\begin{array}{l} 1 \leq \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) \\ \vdots \qquad \vdots \qquad \vdots \\ 1 \leq \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) \end{array}$$

$$A\delta \leq B, \quad \delta \in \{0,1\}$$

➡ Every logic proposition can be translated into linear integer inequalities

General Transformation into Linear Integer Inequalities

Geometric approach: The polytope $P \triangleq \{\delta : A\delta \leq B\}$ is the convex hull of the rows of the truth table T

T:

x_1	x_2	$\ldots$	x_{n-1}	$x_n = F(x_1, \ldots, x_{n-1})$
0	0		0	1
0	0		1	0
⋮	⋮		⋮	
1	1		1	1

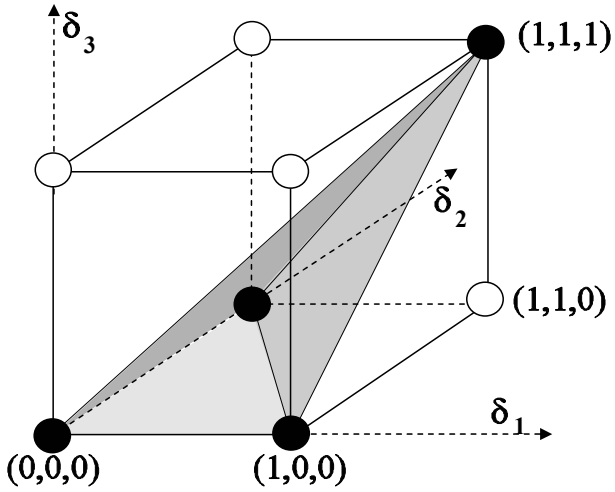
⇨ $A\delta \leq B, \delta \in \{0,1\}^n$

➡ Every logic proposition can be translated into linear integer inequalities

Truth Tables ➡ Linear Integer Inequalities

Example: logic "AND"

δ_1	δ_2	δ_3
0	0	0
0	1	0
1	0	0
1	1	1

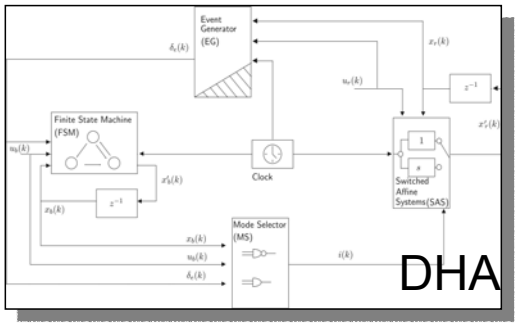


Key idea:
White points cannot be in the hull of black points

$$\text{conv}\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\left\{\delta:\begin{array}{r}-\delta_1+\delta_3\le 0\\-\delta_2+\delta_3\le 0\\\delta_1+\delta_2-\delta_3\le 1\end{array}\right\}$$

Convex hull algorithms: **`cdd`, `lrs`, `qhull`, `chD`, `Hull`, `Porto`**

Mixed Logical Dynamical Systems



Mixed Logical Dynamical (MLD) Systems

$$\begin{aligned}x(t+1) &= Ax(t)+B_1u(t)+B_2\delta(t)+B_3z(t)+B_5\\ y(t) &= Cx(t)+D_1u(t)+D_2\delta(t)+D_3z(t)+D_5\\ E_2\delta(t)+E_3z(t) &\le E_4x(t)+E_1u(t)+E_5\end{aligned}$$

(Bemporad, Morari 1999)

Continuous and binary variables

$x\in\mathbb{R}^{n_r}\times\{0,1\}^{n_b}$, $u\in\mathbb{R}^{m_r}\times\{0,1\}^{m_b}$
$y\in\mathbb{R}^{p_r}\times\{0,1\}^{p_b}$, $\delta\in\{0,1\}^{r_b}$, $z\in\mathbb{R}^{r_r}$

- Computationally oriented (Mixed Integer Programming)
- Suitable for optimal controller synthesis, verification, ...

A Simple Example

- System:

$$x(t+1) = \begin{cases} 0.8x(t) + u(t) & \text{if } x(t) \geq 0 \\ -0.8x(t) + u(t) & \text{if } x(t) < 0 \end{cases}$$

$$-10 \leq x(t) \leq 10, -1 \leq u(t) \leq 1$$

- Associate $[\delta(t) = 1] \leftrightarrow [x(t) \geq 0]$ and transform

$\longrightarrow$

$$\begin{aligned} -m\delta(t) &\leq x(t) - m \\ -(M+\epsilon)\delta(t) &\leq -x(t) - \epsilon \end{aligned} \qquad \begin{aligned} M &= -m = 10 \\ \epsilon &> 0 \quad \text{small} \end{aligned}$$

- Then $x(t+1) = 1.6\delta(t)x(t) - 0.8x(t) + u(t)$

$$z(t) = \delta(t)x(t) \longrightarrow \begin{aligned} z(t) &\leq M\delta(t) \\ z(t) &\geq m\delta(t) \\ z(t) &\leq x(t) - m(1-\delta(t)) \\ z(t) &\geq x(t) - M(1-\delta(t)) \end{aligned}$$

- Rewrite as linear equation

$\longrightarrow$

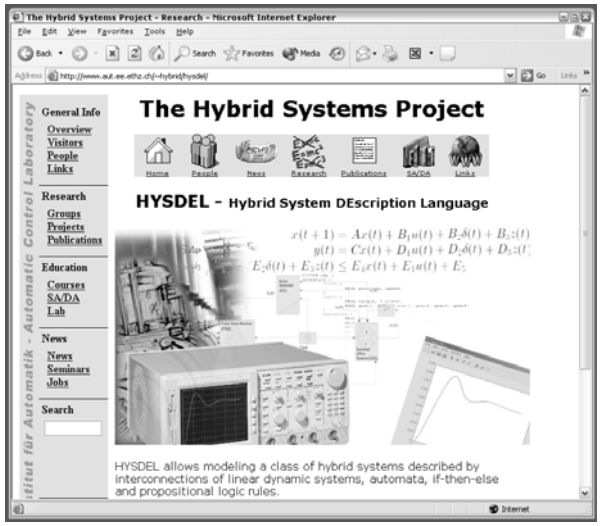
$$x(t+1) = 1.6z(t) - 0.8x(t) + u(t)$$

HYSDEL

(HYbrid Systems DEscription Language)

- Describe *hybrid systems:*

  - Automata
  - Logic
  - Lin. Dynamics
  - Interfaces
  - Constraints



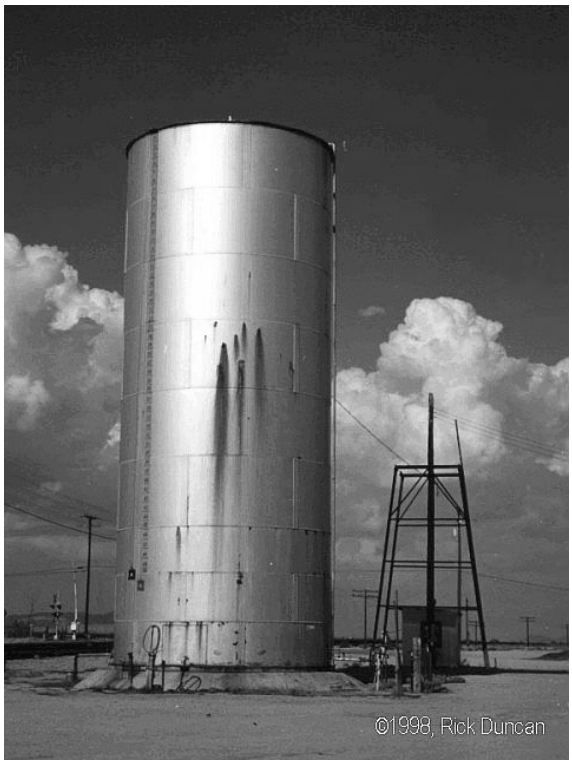
(Torrisi, Bemporad, 2003)

- Automatically generate MLD models in Matlab

- MLD model is not unique in terms of the number of auxiliary variables $\longrightarrow$ optimize model (minimize # binary variables !)

http://control.ethz.ch/~hybrid/hysdel

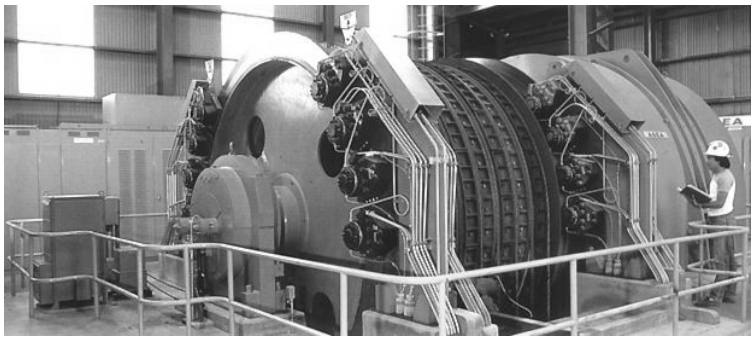
Example 1: AD section



$$[s = \mathsf{T}] \leftrightarrow [h \geq h_{\max}]$$

```
SYSTEM tank {
   INTERFACE {
      STATE {
         REAL h  [-0.3,0.3]; }
      INPUT {
         REAL Q  [-10,10]; }
      PARAMETER {
         REAL k    = 1;
         REAL e    = 1e-6; }
   } /* end interface */

   IMPLEMENTATION {
      AUX {
         BOOL s; }

      AD {
         s = hmax - h <= 0; }

      CONTINUOUS {
         h = h + k * Q; }
   } /* end implementation */
} /* end system */
```

Example 2: DA section



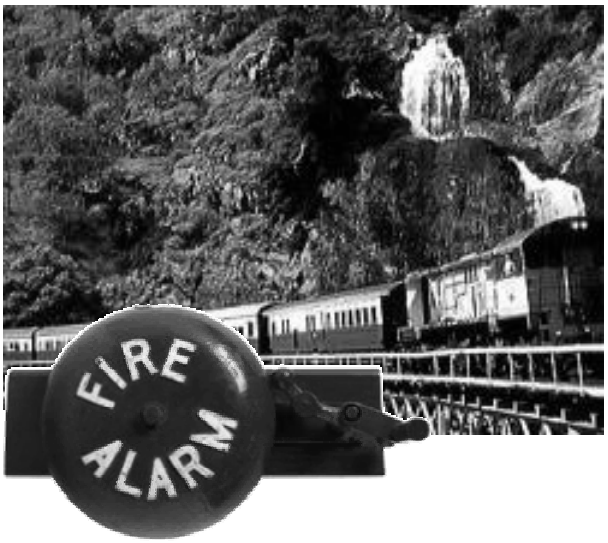
Nonlinear amplification unit

$$u_{comp} = \begin{cases} u & (u < u_t) \\ 2.3u - 1.3u_t & (u \geq u_t) \end{cases}$$

```
SYSTEM motor {
   INTERFACE {
      STATE {
         REAL ucomp; }
      INPUT {
         REAL u [0,10]; }
      PARAMETER {
         REAL ut   = 1;
         REAL e    = 1e-6;}
   } /* end interface */
```

```
IMPLEMENTATION {
      AUX {
         REAL unl;
         BOOL th; }
      AD {
         th = ut - u <= 0; }

      DA {
         unl = { IF th THEN 2.3*u - 1.3*ut
                       ELSE u };  }

      CONTINUOUS {
         ucomp = unl; }
   } /* end implementation */
} /* end system */
```

Example 3: LOGIC section

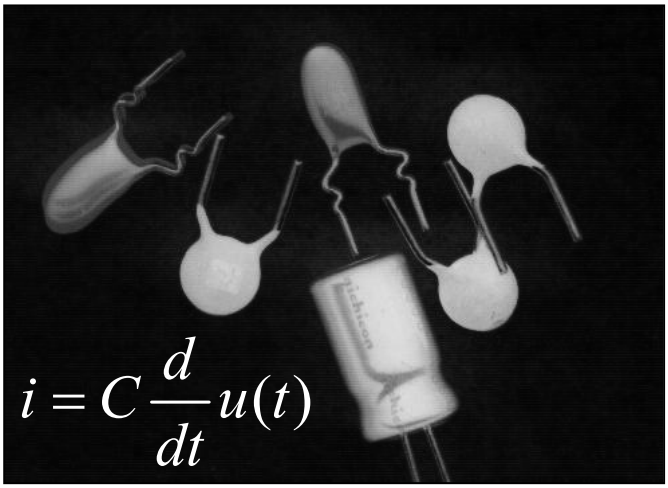


$$u_{brake} = u_{alarm} \wedge (\neg s_{tunnel} \vee \neg s_{fire})$$

$$s_{fire} \rightarrow u_{alarm}$$

```
SYSTEM train {
   INTERFACE {
      STATE {
         BOOL brake; }
      INPUT {
         BOOL alarm, tunnel, fire; }

   } /* end interface */


   IMPLEMENTATION {
      AUX {
         BOOL decision; }

      LOGIC {
         decision =
            alarm & (~tunnel | ~fire); }

      AUTOMATA {
         brake = decision; }
      MUST {
         fire -> alarm; }
   } /* end implementation */
} /* end system */
```

Example 4: CONTINUOUS section

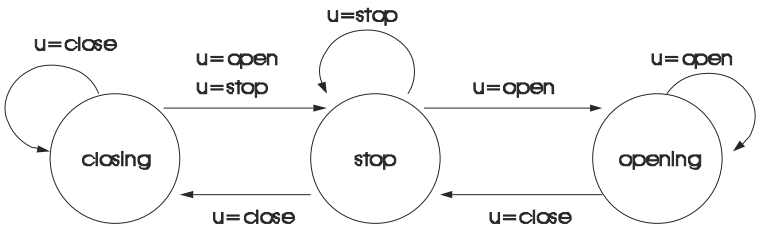


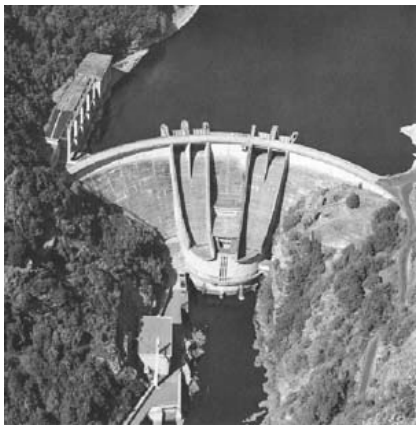
Apply forward difference rule:

$$u(k+1) = u(k) + \frac{T}{C} i(k)$$

```
SYSTEM capacitorD {
   INTERFACE {
      STATE {
         REAL u; }
      PARAMETER {
         REAL R = 1e4;
         REAL C = 1e-4;
         REAL T = 1e-1; }
   } /* end interface */


   IMPLEMENTATION {

      CONTINUOUS {
        u = u - T/C/R*i;  }

   } /* end implementation */
} /* end system */
```

Example 5: AUTOMATA section





```
SYSTEM outflow {
   INTERFACE {
      STATE {
         BOOL closing, stop, opening; }
      INPUT {
         BOOL uclose, uopen, ustop; }
   } /* end of interface */

   IMPLEMENTATION {
```

```
      AUTOMATA {
         closing = (uclose & closing) | (uclose & stop);
         stop    = ustop | (uopen & closing) | (uclose & opening);
         opening = (uopen & stop) | (uopen & opening);  }
```

```
      MUST {
         ~(uclose & uopen);
         ~(uclose & ustop);
         ~(uopen  & ustop); }
   } /* end implementation */
} /* end system */
```

Example 6: MUST section

$$0 \le h \le h_{\max}$$

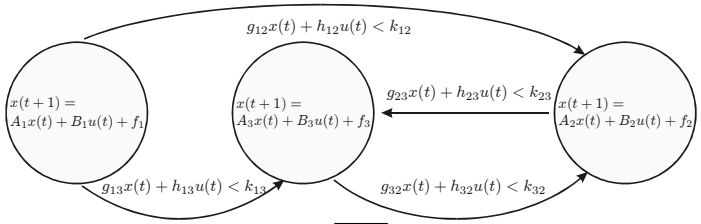
```
SYSTEM watertank {
   INTERFACE {
      STATE {
         REAL h; }
      INPUT {
         REAL Q; }
      PARAMETER {
         REAL hmax = 0.3;
         REAL k    = 1; }
   } /* end interface */


   IMPLEMENTATION {
      CONTINUOUS {
         h = h + k*Q; }
```

```
      MUST {
          h - hmax <= 0;
         -h        <= 0; }
```

```
   } /* end implementation */
} /* end system */
```

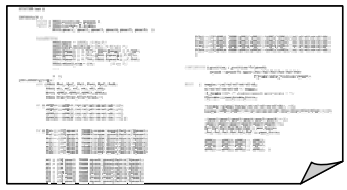
Modeling Flow

Process



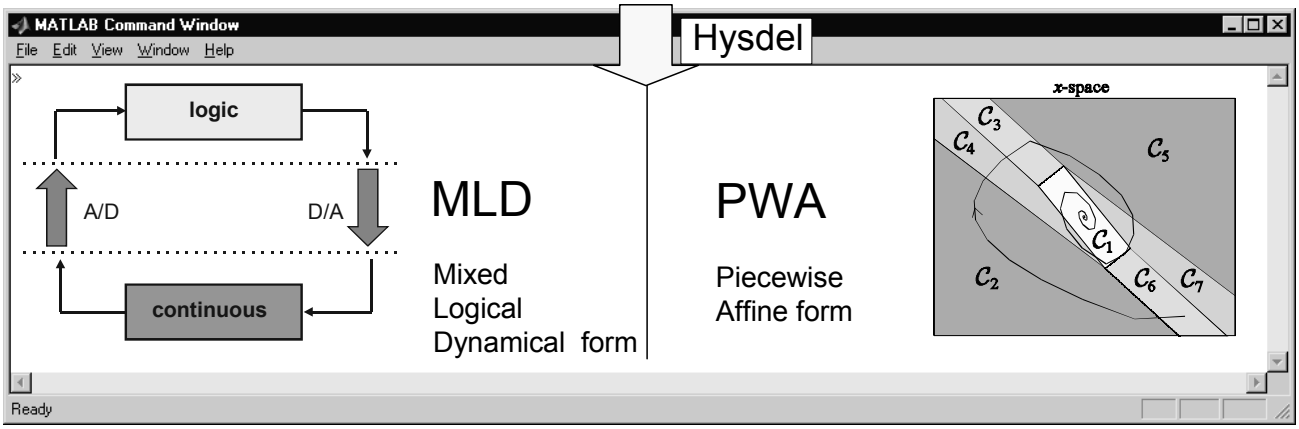
Designer

Text

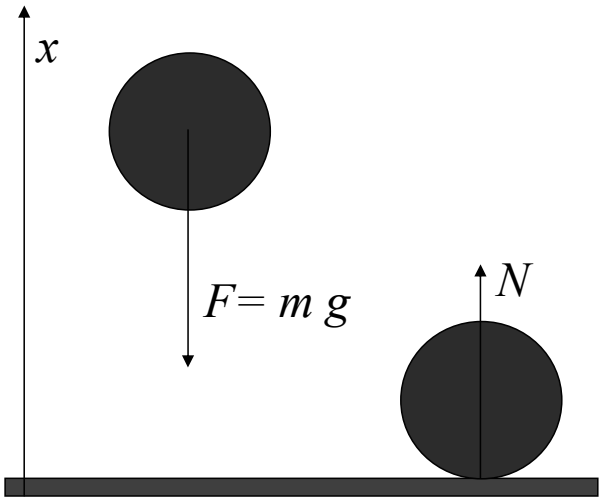


HYSDEL Source

Matlab



Example: Bouncing Ball



$$\ddot{x} = -g$$
$$x \leq 0 \Rightarrow \dot{x}(t^+) = -(1-\alpha)\dot{x}(t^-)$$

$$\alpha \in [0, 1]$$

How to model this system in MLD form?

HYSDEL - Bouncing Ball

```
SYSTEM bouncing_ball {
INTERFACE {
/* Description of variables and constants */
        STATE { REAL height [-10,10];
                REAL velocity [-100,100];   }

        PARAMETER {
                REAL g=9.8;
                REAL dissipation=.4;  /* 0=elastic, 1=completely anelastic */
                REAL Ts=.05; }
}
IMPLEMENTATION {
        AUX {   REAL z1;
                REAL z2;
                BOOL negative;   }

        AD {    negative = height <= 0; }

        DA {    z1 = {  IF negative THEN height-Ts*velocity
                        ELSE height+Ts*velocity-Ts*Ts*g};
                z2 = {  IF negative THEN -(1-dissipation)*velocity
                        ELSE velocity-Ts*g};  }
        CONTINUOUS {
                height = z1;
                velocity=z2;}
}}
```

System Theory for Hybrid Systems

- Analysis
  - Well-posedness
  - Realization & Transformation
  - Stability
  - Reachability (=Verification)
  - Observability
- Synthesis
  - Control
  - State estimation
  - Identification
  - Modeling language

Well-posedness

Are state and output trajectories defined ?
Uniquely defined ? Persistently defined ?

- MLD well-posedness :

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \\ y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \\ E_2\delta(t) + E_3z(t) &\leq E_1u(t) + E_4x(t) + E_5 \end{aligned}$$

$$\delta(t) = F(x(t), u(t))$$
$$z(t) = G(x(t), u(t))$$

$$\{x(t), u(t)\} \rightarrow \{x(t+1)\}$$
$$\{x(t), u(t)\} \rightarrow \{y(t)\}$$

are single valued

Definition 1 *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a set of input+state pairs. A hybrid MLD system is called* well-posed *on Ω, if for all pairs $(x(t), u(t)) \in \Omega$ there exists a solution $x(t+1), y(t), \delta(t), z(t)$ and moreover, $x(t+1), y(t)$ are uniquely determined.*

Numerical test based on mixed-integer programming available

(Bemporad, Morari, *Automatica,* 1999)

Realization and Transformation (State-Space Hybrid Models)

Other Existing Hybrid Models

- Linear complementarity (LC) systems (Heemels, 1999)

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u(t) + B_2w(t) \\ y(t) &= Cx(t) + D_1u(t) + D_2w(t) \\ v(t) &= E_1x(t) + E_2u(t) + E_3w(t) + e_4 \\ 0 &\le v(t) \perp w(t) \ge 0 \end{aligned}$$

Ex: mechanical systems
circuits with diodes etc.

- Extended linear complementarity (ELC) systems
  Generalization of LC systems (De Schutter, De Moor, 2000)

- Min-max-plus-scaling (MMPS) systems
  (De Schutter, Van den Boom, 2000)

$$\begin{aligned} x(t+1) &= M_x(x(t), u(t), d(t)) \\ y(t) &= M_y(x(t), u(t), d(t)) \\ 0 &\ge M_c(x(t), u(t), d(t)) \end{aligned}$$

MMPS function: defined by the grammar

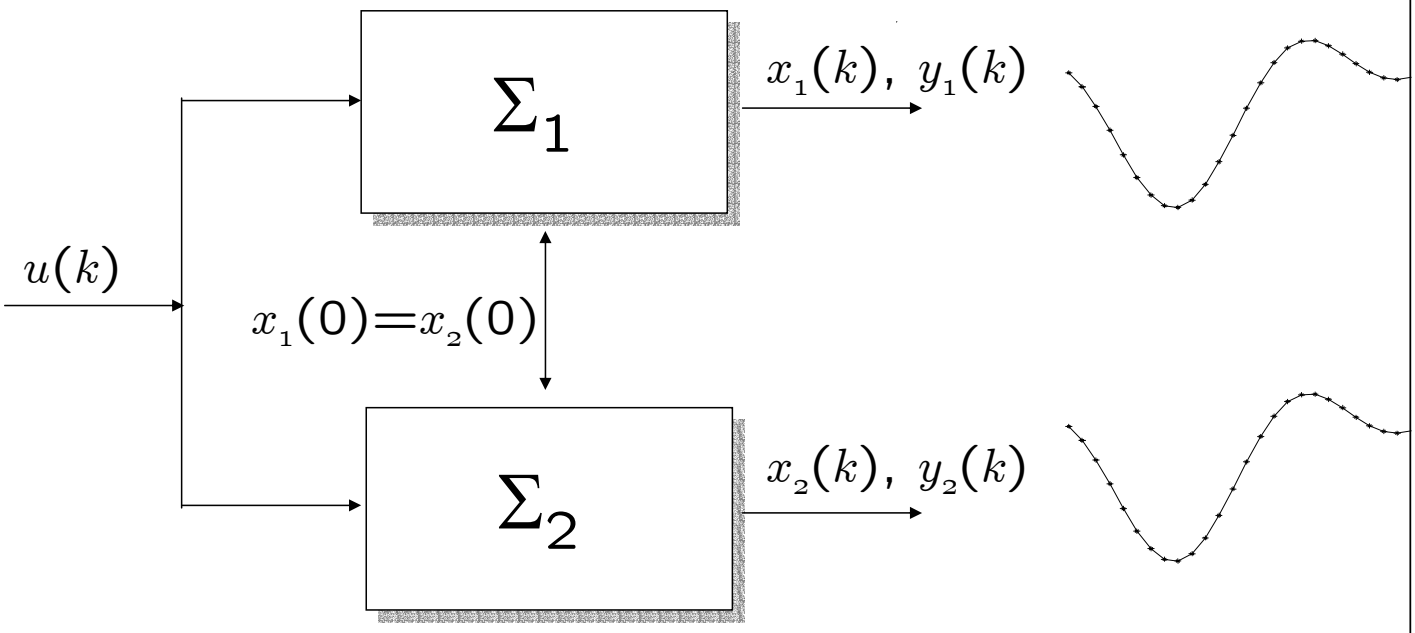
$$M := x_i | \alpha | \max(M_1, M_2) | \min(M_1, M_2) | M_1 + M_2 | \beta M_1$$

Example: $x(t+1) = 2\max(x(t), 0) + \min(-\frac{1}{2}u(t), 1)$

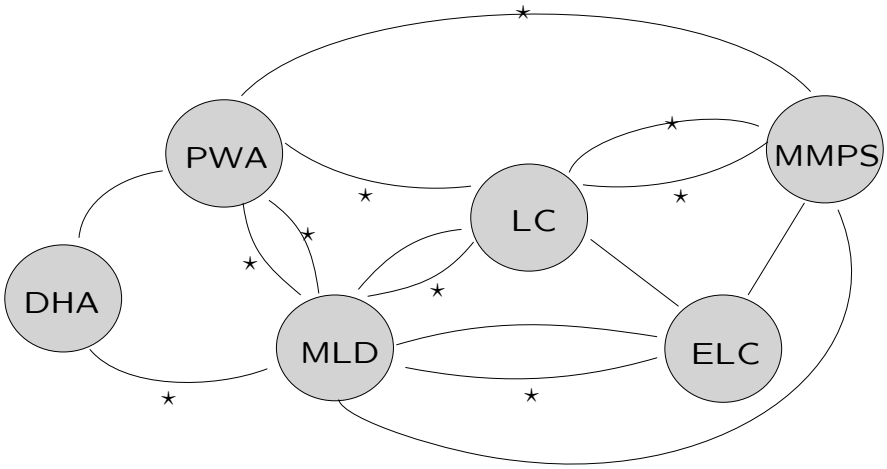
Used for modeling discrete-event systems (t=event counter)

Equivalences of Hybrid Models

Definition 1 *Two hybrid systems* Σ_1, Σ_2 *are* equivalent *if for all initial conditions* $x_1(0) = x_2(0)$ *and input* $\{u_1(k)\}_{k\in\mathbb{Z}_+} = \{u_2(k)\}_{k\in\mathbb{Z}_+}$ *then* $x_1(k) = x_2(k)$ *and* $y_1(k) = y_2(k)$, *for all* $k \in \mathbb{Z}_+$.



Equivalence Results



Theorem 1 *All the above six classes of discrete-time hybrid models are equivalent (possibly under some additional assumptions, such as boundedness of input and state variables)*

(Heemels, De Schutter, Bemporad, *Automatica,* 2001)
(Torrisi, Bemporad, *IEEE CST,* 2003)
(Bemporad and Morari, *Automatica*, 1999)
(Bemporad, Ferrari-T.,Morari, *IEEETAC*, 2000)

Theoretical properties and analysis/synthesis tools can be transferred from one class to another

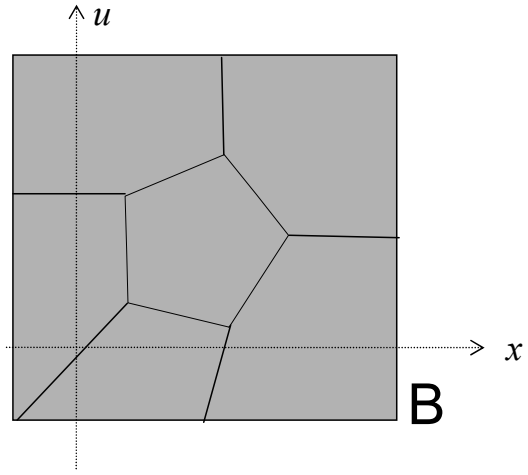
Efficient MLD to PWA Conversion

- Proof is constructive: given an MLD system it returns its equivalent PWA form
- Drawback: it needs the enumeration of all possible combinations of binary states, binary inputs, and δ variables
- Most of such combinations lead to empty regions Ω
- IDEA: use techniques from multiparametric programming to avoid such an enumeration

MLD to PWA Conversion Algorithm

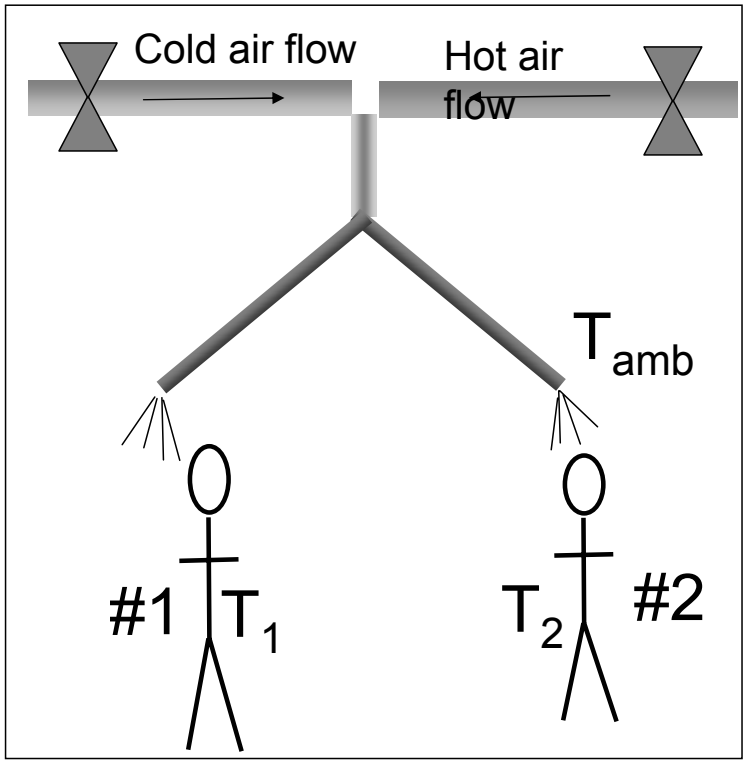
(Bemporad, 2002)

GOAL: split a given set B of states+inputs into polyhedral cells and find the affine dynamics in each cell, therefore determining the PWA system which is equivalent to the given MLD system



Note: the cells Ω_i are embedded in R^{n+m} by treating integer states and integer inputs as continuous vars:
$x_b, u_b \in [0,1] \rightarrow x_b, u_b \in [-1/2,1/2) \cup [1/2,3/2]$

Example: Heat and Cool



Rules of the game:

- #1 turns the heater (air conditioning) on whenever he is cold (hot)
- If #2 is cold he turns the heater on, unless #1 is hot
- If #2 is hot he turns the air conditioning on, unless #2 is cold
- Otherwise, heater and air conditioning are off

- $\dot{T}_1 = -\alpha_1(T_1 - T_{\text{amb}}) + k_1(u_{\text{hot}} - u_{\text{cold}})$ (body temperature dynamics of #1)
- $\dot{T}_2 = -\alpha_2(T_2 - T_{\text{amb}}) + k_2(u_{\text{hot}} - u_{\text{cold}})$ (body temperature dynamics of #2)

Hybrid HYSDEL Model

```
/* Heat and Cool example
   (C) 2002 by A. Bemporad, Las Vegas, December 9, 2002 */

SYSTEM heatcool {

INTERFACE {
    STATE { REAL T1 [-10,50];
            REAL T2 [-10,50];
        }
    INPUT { REAL Tamb [-10,50];
        }
    PARAMETER {
        REAL Ts = .5; /* sampling time, second
        REAL alpha1 = 1;
        REAL alpha2 = 0.5;
        REAL k1 = .8;
        REAL k2 = .4;
        REAL Thot1 = 30;
        REAL Tcold1 = 15;
        REAL Thot2 = 35;
        REAL Tcold2 = 10;
        REAL Uc = 2; /* cold water flow */
        REAL Uh = 2; /* hot water flow */
    }
}

IMPLEMENTATION {
        AUX { REAL uhot, ucold;
              BOOL hot1, hot2, cold1, cold2;
        }
        AD  { hot1 = T1>=Thot1;
              hot2 = T2>=Thot2;
              cold1 = T1<=Tcold1;
              cold2 = T2<=Tcold2;
        }
        DA  { uhot = (IF cold1 | (cold2 & ~hot1) THEN Uh ELSE 0);
              ucold = (IF hot1 | (hot2 & ~cold1) THEN Uc ELSE 0);
        }
        CONTINUOUS { T1 = T1+Ts*(-alpha1*(T1-Tamb)+k1*(uhot-ucold));
                     T2 = T2+Ts*(-alpha2*(T2-Tamb)+k2*(uhot-ucold));
        }
    }
}
```

Hybrid MLD Model

- MLD model

$$\begin{aligned} x(t+1) &= Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \\ y(t) &= Cx(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \\ E_2 \delta(t) + E_3 z(t) &\leq E_1 u(t) + E_4 x(t) + E_5 \end{aligned}$$

- 2 continuous states: (temperatures T_1,T_2)
- 1 continuous input: (room temperature T_{amb})
- 2 auxiliary continuous vars: (water flows u_{hot}, u_{cold})
- 6 auxiliary binary vars: (4 thresholds + 2 for OR condition)
- 20 mixed-integer inequalities

Possible combination of integer variables: $2^6 = 64$

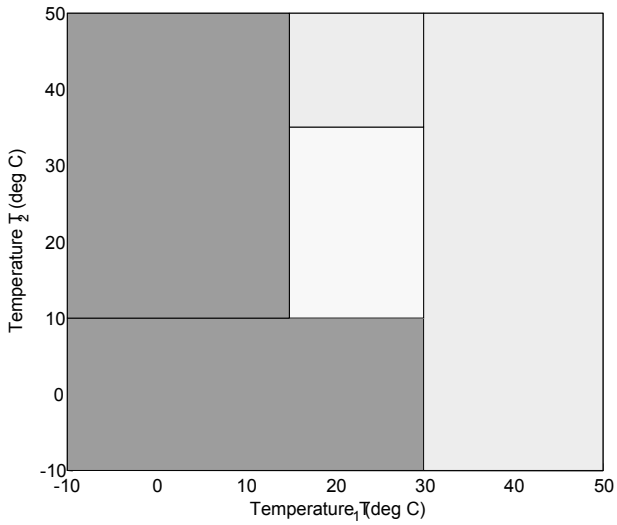
Hybrid PWA Model

- PWA model

$$\begin{array}{rcl} x'(k) & = & A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) & = & C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ i(k) & \text{s.t.} & H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)} \end{array}$$

- 2 continuous states:
  (temperatures T_1,T_2)
- 1 continuous input:
  (room temperature T_{amb})
- 5 polyhedral regions
  (partition does not depend on input)

Computation time: 0.72 s in Matlab

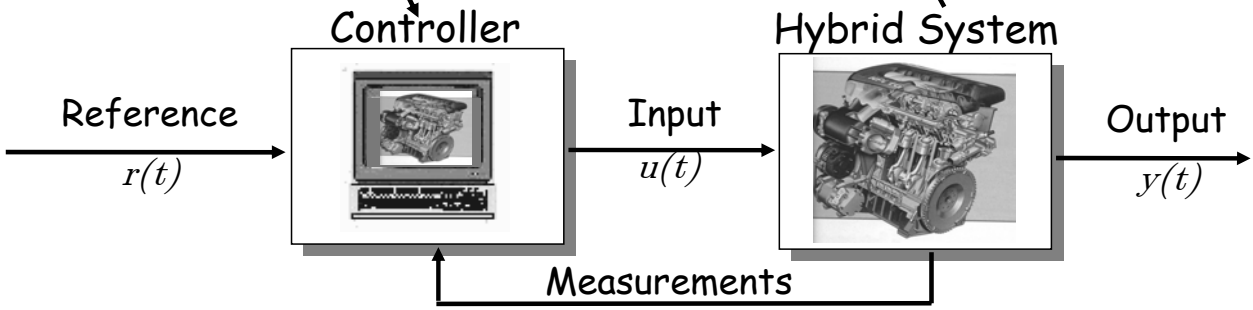


$$\begin{array}{rcl} u_{\text{hot}} & = & 0 \\ u_{\text{cold}} & = & 0 \end{array} \qquad \begin{array}{rcl} u_{\text{hot}} & = & 0 \\ u_{\text{cold}} & = & \bar{U}_C \end{array} \qquad \begin{array}{rcl} u_{\text{hot}} & = & \bar{U}_H \\ u_{\text{cold}} & = & 0 \end{array}$$

Controller Synthesis

Model Predictive Control of Hybrid Systems

$$x(t+1) = Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t)$$
$$y(t) = Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t)$$
$$E_2\delta(t) + E_3z(t) \leq E_4x(t) + E_1u(t) + E_5$$



- MODEL: a model of the plant is needed to predict the future behavior of the plant
- PREDICTIVE: optimization is based on the predicted future evolution of the plant
- CONTROL: control complex constrained multivariable plants

Receding Horizon Philosophy

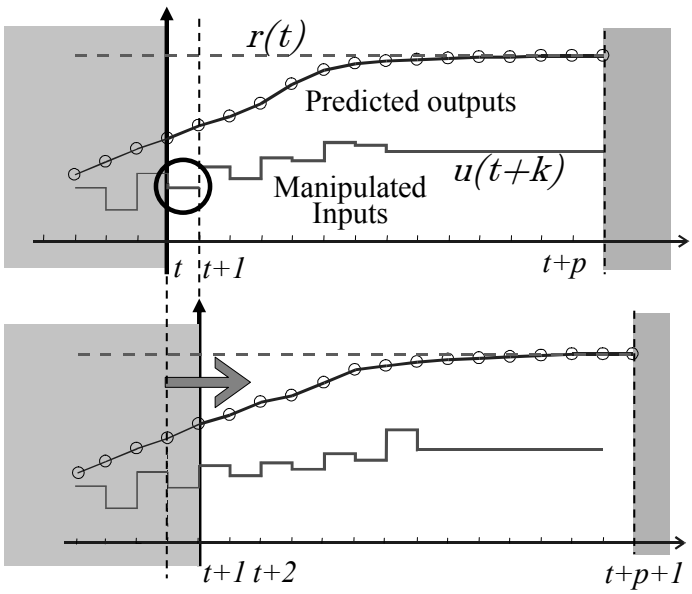
- At time t :

  Solve an optimal control problem over a finite future horizon p :

  - minimize $|y - r| + \rho|u|$

  – subject to constraints

$$u_{min} \leq u \leq u_{max}$$
$$y_{min} \leq y \leq y_{max}$$



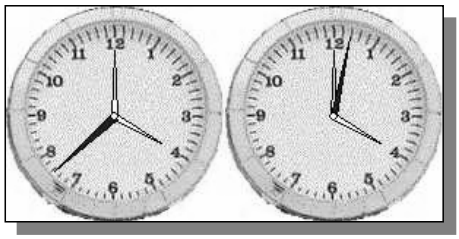
- Only apply the first optimal move $u^*(t)$

- Get new measurements, and repeat the optimization at time $t+1$

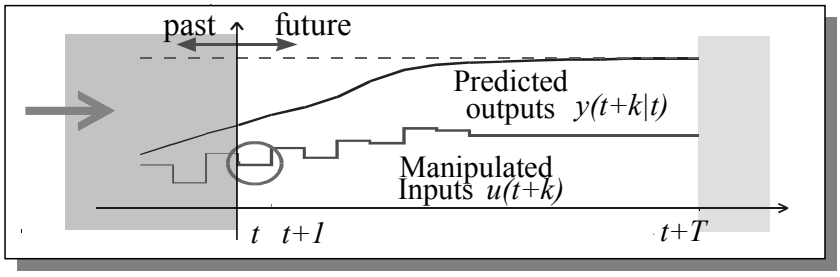
Advantage of on-line optimization: *FEEDBACK!*

Receding Horizon - Example

MPC is like playing chess !



MPC for Hybrid Systems



Model Predictive (MPC) Control

- At time t solve with respect to $U \triangleq \{u(t), u(t+1) \ldots, u(t+T-1)\}$ the finite-horizon open-loop, optimal control problem:

$$\min_U J(U, x(t), r) \triangleq \sum_{k=0}^{T-1} \|Q(y(t+k+1|t) - r)\| + \|R(u(t+k) - u_r)\|$$
$$+ \sigma \left(\|\delta(t+k|t) - \delta_r\| + \|z(t+k|t) - z_r\| + \|x(t+k|t) - x_r\| \right)$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \\ x(t+T|t) = x_r \end{cases}$$

- Apply only $u(t) = u^*(t)$ (discard the remaining optimal inputs)
- Repeat the whole optimization at time $t+1$

Closed-Loop Stability

Theorem 1 *Let $(x_r, u_r, \delta_r, z_r)$ be the equilibrium values corresponding to the set point r, and assume $x(0)$ is such that the MPC problem is feasible at time $t = 0$. Then $\forall Q, R \succ 0$, $\forall \sigma > 0$, the MPC controller stabilizes the MLD system*

$$\lim_{t\to\infty} y(t) = r$$
$$\lim_{t\to\infty} u(t) = u_r$$

$\lim_{t\to\infty} x(t) = x_r$, $\lim_{t\to\infty} \delta(t) = \delta_r$, $\lim_{t\to\infty} z(t) = z_r$, *and all constraints are fulfilled.*

(Bemporad, Morari 1999)

Proof: Easily follows from standard Lyapunov arguments

Stability Proof

- Assume we set the terminal constraint $x(T) = x_r$ in the optimal control problem
- Let $\mathcal{U}_t^*$ denote the optimal control sequence $\{u_t^*(0), \ldots, u_t^*(T-1)\}$
- Let $V(t) \triangleq J(\mathcal{U}_t^*, x(t))$=value function
- By construction, $\mathcal{U}_1 = \{u_t^*(1), \ldots, u_t^*(T-1), u_r\}$ is feasible @$t+1$
- Hence,

$$V(t+1) \le J(\mathcal{U}_1, x(t+1)) = V(t) - \|y(t) - r\|_Q - \|u(t) - u_r\|_R - \sigma(\|\delta(t) - \delta_r\| - \|z(t) - z_r\| - \|x(t) - x_r\|)$$

- Hence $V(t)$ is decreasing and lower-bounded by 0 $\Rightarrow \exists V_\infty = \lim_{t\to\infty} V(t)$ $\Rightarrow V(t+1) - V(t) \to 0$
- Hence, $\|y(t) - r\|_Q \to 0$,$\|u(t) - u_r\|_R \to 0$, $\ldots$, $\|x(t) - x_r\| \to 0$

Hybrid MPC - Example

Switching System:

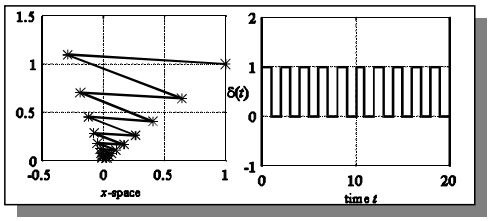
$$x(t+1) = 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [\,0 \quad 1\,]\, x(t)$$

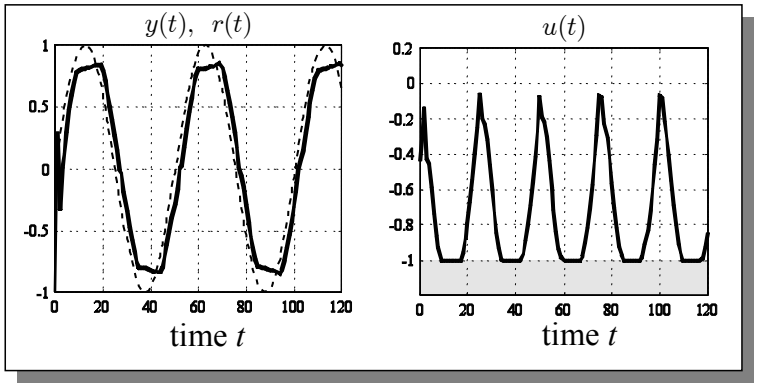
$$\alpha(t) = \begin{cases} \frac{\pi}{3} \text{ if } [1\ \ 0]x(t) \geq 0 \\ -\frac{\pi}{3} \text{ if } [1\ \ 0]x(t) < 0 \end{cases}$$

Constraint: $\quad -1 \leq u(t) \leq 1$

Open loop:



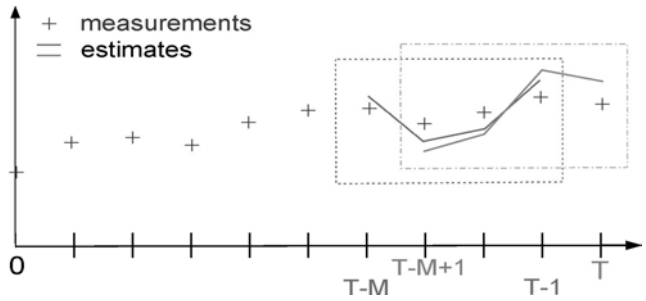
Closed loop:



State Estimation / Fault Detection

State Estimation / Fault Detection

- Problem: given past output measurements and inputs, estimate the current state/faults
- Solution: Use Moving Horizon Estimation for MLD systems (dual of MPC)



Augment the MLD model with:

- Input disturbances $\xi \in \mathbf{R}^n$
- Output disturbances $\zeta \in \mathbf{R}^p$

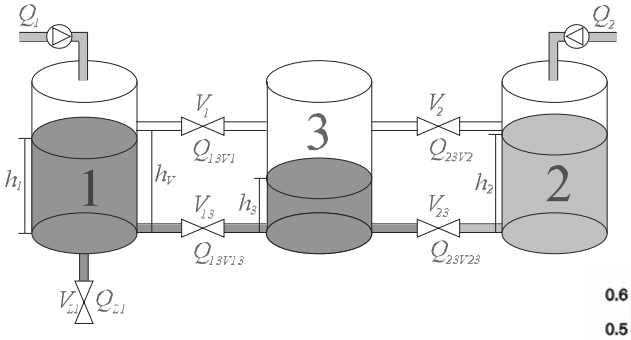
At each time t solve the optimization: $\min \sum_{k=1}^{T} \|\hat{y}(t-k|t) - y(t-k)\|^2 + \dots$ and get estimates $\hat{x}(t)$

➡ MHE optimization = MIQP (Bemporad, Mignone, Morari, ACC 1999)

➡ Convergence can be guaranteed (Ferrari-T., Mignone, Morari, 2002)

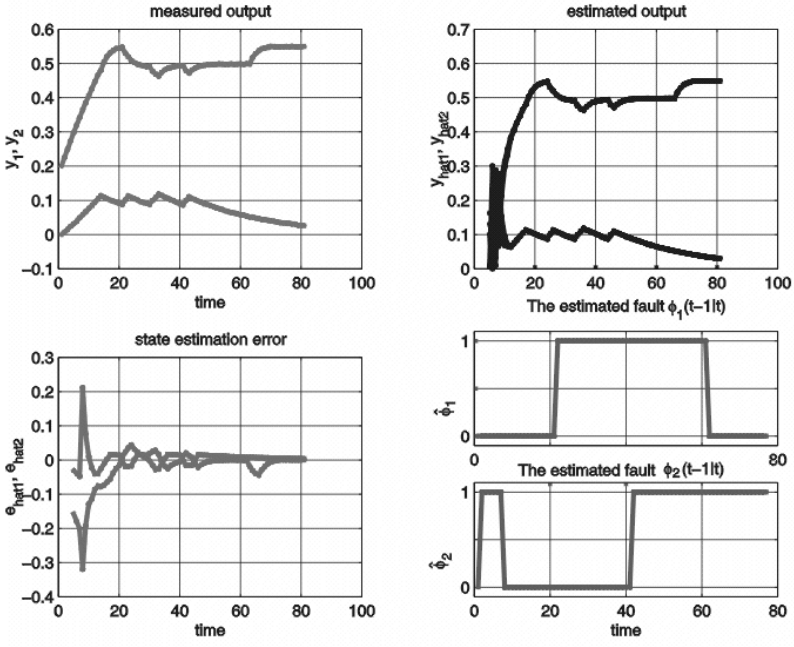
Fault detection: augment MLD with unknown **binary** distubances $\phi \in \{0,1\}^p$

Example: Three Tank System

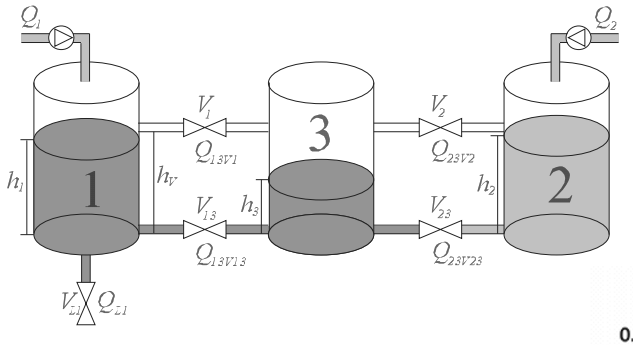


COSY Benchmark problem, ESF

- ϕ_1 : leak in tank 1 for $20s \le t \le 60s$
- ϕ_2 : valve V_1 blocked for $t \ge 40s$



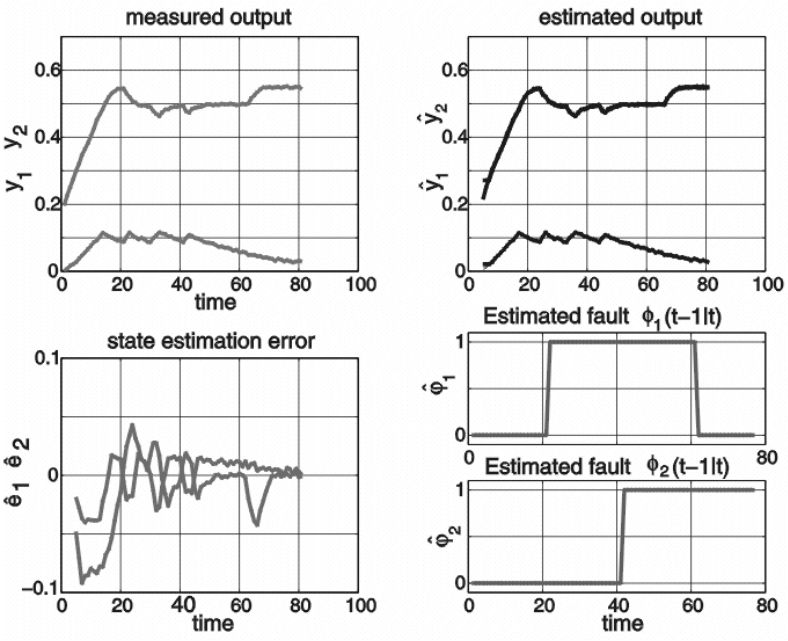
Example: Three Tank System



COSY Benchmark problem, ESF

- ϕ_1 : leak in tank 1 for $20s \leq t \leq 60s$
- ϕ_2 : valve V_1 blocked for $t \geq 40s$
- Add logic constraint

$$[h_1 \leq h_v] \Rightarrow \phi_2 = 0$$



Optimal Control of Hybrid Systems: Computational Aspects

MIQP Formulation of MPC

(Bemporad, Morari, 1999)

$$\min \sum_{k=0}^{T-1} \left\{ y'(t+k+1|t) Q y(t+k+1|t) + u'(t+k) R u(t+k) \right\}$$
$$\text{s.t.} \; \text{MLD dynamics}$$

- Set $\xi \triangleq [u(t), \ldots, u(t+T-1), \delta(t|t), \ldots, \delta(t+T-1|t), z(t|t), \ldots, z(t+T-1|t)]$

$$\min_{\xi} J(\xi, x(t)) = \frac{1}{2}\xi' H \xi + x'(t) F \xi + \frac{1}{2} x'(t) Y x(t)$$
$$\text{s.t.} \; G\xi \leq W + Sx(t)$$

Mixed Integer Quadratic Program (MIQP)

$$u \in \mathbf{R}^{n_u}, \quad \delta \in \{0,1\}^{n_\delta}, \quad z \in \mathbf{R}^{n_z} \quad \Rightarrow \quad \xi \in \mathbf{R}^{T(n_u+n_z)} \times \{0,1\}^{Tn_\delta}$$

MILP Formulation of MPC

(Bemporad, Borrelli, Morari, 2000)

$$\min \sum_{k=0}^{T-1} \|Q y(t+k+1|t)\|_\infty + \|R u(t+k)\|_\infty$$
$$\text{s.t.} \; \text{MLD dynamics}$$

- Introduce slack variables: $\min |x| \Rightarrow \begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \epsilon \geq x \\ & \epsilon \geq -x \end{array}$

$$\begin{array}{rcl} \epsilon_k^x & \geq & \|Qy(t+k|t)\|_\infty \\ \epsilon_k^u & \geq & \|Ru(t+k)\|_\infty \end{array} \Rightarrow \begin{array}{rcll} \epsilon_k^x & \geq & [Qy(t+k|t)]_i & i=1,\ldots,p, \; k=1,\ldots,T-1 \\ \epsilon_k^x & \geq & -[Qy(t+k|t)]_i & i=1,\ldots,p, \; k=1,\ldots,T-1 \\ \epsilon_k^u & \geq & [Ru(t+k))]_i & i=1,\ldots,m, \; k=0,\ldots,T-1 \\ \epsilon_k^u & \geq & -[Ru(t+k))]_i & i=1,\ldots,m, \; k=0,\ldots,T-1 \end{array}$$

- Set $\xi \triangleq [\epsilon_1^x, \ldots, \epsilon_{N_y}^x, \epsilon_1^u, \ldots, \epsilon_{T-1}, U, \delta, z]$

Mixed Integer Linear Program (MILP)

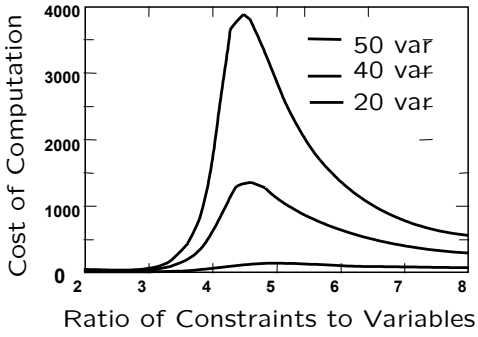
$$\min_{\xi} J(\xi, x(t)) = \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u$$
$$\text{s.t.} \; G\xi \leq W + Sx(t)$$

Mixed-Integer Program Solvers

- Mixed-Integer Programming is *NP*-hard

  Phase transitions have been found in computationally hard problems.

  BUT



(Monasson et al., *Nature*, 1999)

- General purpose Branch & Bound/Branch & Cut solvers available for MILP and MIQP (Ilog CPLEX, XPRESS-MP, Sahinidis, ...)
  More solvers and benchmarks: **http://plato.la.asu.edu/bench.html**

- No need to reach global optimum (see proof of the theorem), although performance deteriorates

Good for large sampling times (e.g., 1 h) / expensive hardware ...

... but not for fast sampling (e.g. 10 ms) / cheap hardware !

Hybrid Control Example: Heat Exchange

Hybrid Control - An Example

(Hedlund and Rantzer, CDC1999)

$$\begin{bmatrix}\dot{T}_1\\ \dot{T}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} + B_i u_0$$

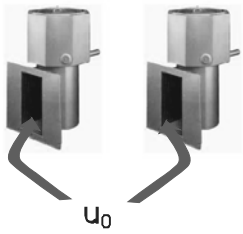
$$B_i = \begin{cases}\begin{bmatrix}1\\0\end{bmatrix} & \text{if first furnace heated}\\ \begin{bmatrix}0\\1\end{bmatrix} & \text{if second furnace heated}\\ \begin{bmatrix}0\\0\end{bmatrix} & \text{if no heating}\end{cases}$$

- Objective:
  - Control the temperature to a given set-point $T_1,\ T_2$
- Constraints:
  - Only three operation modes:
    1- Heat only the first furnace
    2- Heat only the second furnace
    3- Do not heat any furnaces

Amount of heating power u_o is constant

Alternate Heating of Two Furnaces

- HYSDEL model:




```
/* HEAT EXCHANGE model  - (C) 2001 by A. Bemporad, Zurich, March 13, 2001 */

SYSTEM furnaces {

INTERFACE {
        STATE {
                REAL t1,t2,u0; }
        INPUT {
                BOOL heat1,heat2; }
        PARAMETER {
                REAL Ts=0.08; /* sampling time, seconds */
                REAL Bd1=0.07688365361336; /* discretization of B matrix, heat 1 active */
                REAL Bd2=0.07392810551689; /* discretization of B matrix, heat 2 active */
                REAL A11=.9231; /* discretization of A matrix */
                REAL A22=.8521; /* discretization of A matrix */

                REAL u0max=10; /* upperbound on u0 */
                REAL e = 1e-6; /* precision for strict inequalities */  }
}

IMPLEMENTATION {
                AUX {REAL z1,z2; }

                DA {    z1 = {IF heat1 THEN Bd1*u0 [Bd1*u0max,0,e]};
                        z2 = {IF heat2 THEN Bd2*u0 [Bd2*u0max,0,e]}; }

                CONTINUOUS { t1 = A11*t1+z1;
                             t2 = A22*t2+z2;
                             u0 = u0; }

                MUST   {  ~heat1 | ~heat2;  /* heat1 and heat2 cannot be both active */ }
}
}
```

- MLD model:

$$x(t+1) = \begin{bmatrix}0.9231 & 0 & 0\\ 0 & 0.8521 & 0\\ 0 & 0 & 1.0000\end{bmatrix} x(t) + \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} z(t)$$

$$\begin{bmatrix}-1 & 0\\ 1 & 0\\ -1 & 0\\ 1 & 0\\ 0 & -1\\ 0 & 1\\ 0 & -1\\ 0 & 1\\ 0 & 0\end{bmatrix} z(t) \square \begin{bmatrix}-0.7688 & 0\\ 0 & 0\\ 0 & 0\\ 0.7688 & 0\\ 0 & -0.7393\\ 0 & 0\\ 0 & 0\\ 0 & 0.7393\\ -1.0000 & -1.0000\end{bmatrix} u(t) + \begin{bmatrix}-0.0769\\ 0.0769\\ 0\\ 0\\ -0.0739\\ 0.0739\\ 0\\ 0\\ 0\end{bmatrix} x_3(t) + \begin{bmatrix}0.7688\\ 0\\ 0\\ 0\\ 0.7393\\ 0\\ 0\\ 0\\ 1.0000\end{bmatrix}$$

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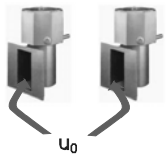


- Performance index

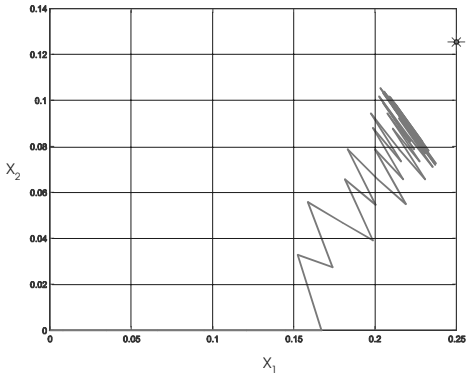
$$\min_{\{u(t),u(t+1),u(t+2)\}} \sum_{k=0}^{2} \|R(u(t+k+1)-u(t+k))\|_\infty + \|Q(x(t+k|t)-x_e)\|_\infty$$

- Constraints

$$\begin{array}{rcl} -1 & \leq x_1 \leq & 1 \\ -1 & \leq x_2 \leq & 1 \end{array}$$

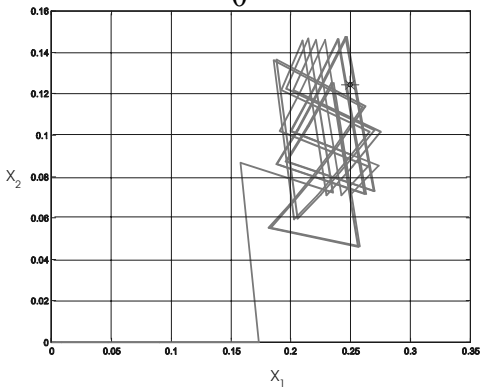
Closed-Loop Behavior

u_0=0.4



Set point cannot be reached

u_0=0.8



Set point is reached

- Computational complexity of on-line MILP

  (Bemporad, Borrelli, Morari, 2000)

Linear constraints	168
Continuous variables	33
Binary variables	6
Parameters	3
Time to solve MILP (av.)	1,09 s

On-Line vs. Off-Line Optimization

$$\min_U J(U, x(t)) \triangleq \sum_{k=0}^{T-1} \|Qy(t+k+1|t)\|_\infty + \|Ru(t+k)\|_\infty$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \\ x(t+T|t) = 0 \end{cases}$$

- On-line optimization: given $x(t)$, solve the problem at each time step t

Mixed-Integer Linear Program (MILP)

- Off-line optimization: solve the MILP for all $x(t)$

$$\min_\xi J(\xi, x(t)) \triangleq f'\xi$$
$$\text{s.t.} G\xi \leq W + Fx(t)$$

multi-parametric Mixed Integer Linear Program (mp-MILP)

Explicit Form of Model Predictive Control

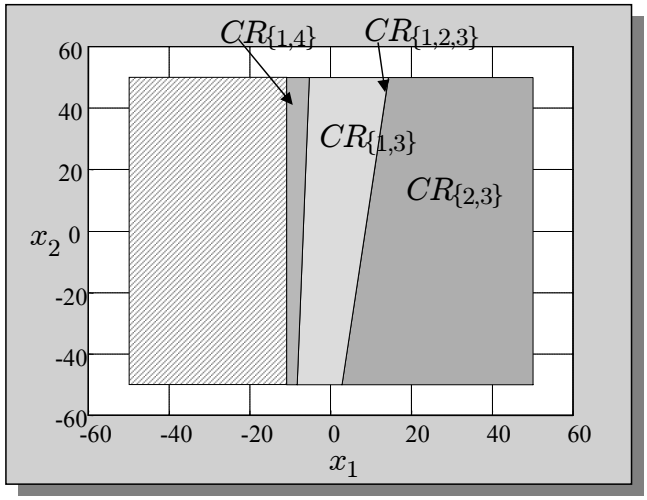
via Multiparametric Programming

Example of Multiparametric Solution

Multiparametric LP ($\xi \in \mathbf{R}^2$)

$$\min_{\xi} \quad -3\xi_1 - 8\xi_2$$

$$\text{s.t.} \quad \begin{cases} \xi_1 + \xi_2 & \leq & 13 + x_1 \\ 5\xi_1 - 4\xi_2 & \leq & 20 \\ -8\xi_1 + 22\xi_2 & \leq & 121 + x_2 \\ -4\xi_1 - \xi_2 & \leq & -8 \\ -\xi_1 & \leq & 0 \\ -\xi_2 & \leq & 0 \end{cases}$$



$$\xi(x) = \begin{cases} \begin{bmatrix} 1.00 & 0 \\ -33.00 & 55.00 \end{bmatrix} x + \begin{bmatrix} -13.00 \\ 13.00 \end{bmatrix} & \text{if} \quad \begin{bmatrix} 1.00 & 0.00 \\ -1.00 & 0.00 \\ 1.00 & -50.00 \\ 0.00 & 50.00 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ 60.00 \\ 0.00 \end{bmatrix} & \text{CR}^{\{1,4\}} \\ \begin{bmatrix} 1.00 & 0.00 \\ -1.00 & 21.00 \\ 1.00 & -50.00 \\ 0.00 & 50.00 \end{bmatrix} x \leq \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} & \text{CR}^{\{1,3\}} \\ \begin{bmatrix} 1.00 & 0.00 \\ -1.00 & 10.00 \\ 0.00 & -50.00 \\ 0.00 & 50.00 \end{bmatrix} x \leq \begin{bmatrix} -1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} & \text{CR}^{\{2,3\}} \end{cases}$$

Linear MPC

- Linear Model:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$\Delta u(t) = u(t) - u(t-1)$$

input increments
(for integral action)

- Prediction:

$$y(t+k|t) = CA^k x(t) + \sum_{i=0}^{k-1} CA^i B u(t+k-1-i)$$

$$u(t+k) = u(t-1) + \sum_{i=0}^{k-1} \Delta u(t+i)$$

Linear MPC

Quadratic Performance Index $\qquad U \triangleq \{\Delta u(t), \ldots, \Delta u(t+m-1)\}$

$$\min_U \sum_{k=0}^{p-1} \| W^y \left[y(t+k+1|t) - r(t) \right] \|^2 + \| W^{\Delta u} \Delta u(t+k) \|^2 + W^u \left[u(t+k) - u_{\text{target}}(t) \right] \|^2$$

$$\min_U J(U, x(t)) = \tfrac{1}{2} U' H U + [x'(t) \;\; r'(t)] F U + \tfrac{1}{2} [x'(t) \;\; r'(t)] Y \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}$$

$$\text{subject to } GU \leq W + K \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}$$

(convex) QUADRATIC PROGRAM (QP)

$$\begin{aligned} y_{min} \leq y(t+k|t) \leq y_{max}, \quad & k = 1, \ldots, p \\ u_{min} \leq u(t+k) \leq u_{max}, \quad & k = 0, 1, \ldots, m-1 \\ \Delta u_{min} \leq \Delta u(t+k) \leq \Delta u_{max}, \quad & k = 0, 1, \ldots, m-1 \end{aligned}$$

Constraints

Multiparametric Quadratic Programming

(Bemporad et al., 2002)

$$\begin{array}{ll} \min\limits_U & \frac{1}{2} U' H U + x F' U + \frac{1}{2} x Y x \\ \text{subj. to} & GU \leq W + Sx, \end{array}$$

$U \triangleq [u_0' \;\; \ldots \;\; u_{T-1}']'$

$U \in \mathbb{R}^r$, $r \triangleq n_u T$

$x \in \mathbb{R}^n$

- Objective: solve the QP **for all** $x \in X \subseteq \mathbb{R}^n$

Linearity of the Solution

$x_0 \in X$ ⇨ solve QP to find $U^*(x_0)$, $\lambda^*(x_0)$

⇨ identify active constraints at $U^*(x_0)$

⇨ form matrices $\tilde{G}$, $\tilde{W}$, $\tilde{S}$ by collecting active constraints $\tilde{G}U^*(x_0) - \tilde{W} - \tilde{S}x_0 = 0$

KKT optimality conditions:

(1) $HU + Fx + G'\lambda = 0$, (2) $\tilde{G}U - \tilde{W} - \tilde{S}x = 0$
(3) $\lambda_i(G^iU - W^i - s^ix) = 0$, (4) $\hat{G}U \leq \hat{W} + \hat{S}x = 0$
(5) $\tilde{\lambda}_i \geq 0$, $\hat{\lambda}_i = 0$

From (1) : $U = -H^{-1}(Fx + \tilde{G}'\tilde{\lambda})$

From (2) :

$$\tilde{\lambda}(x) = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + (\tilde{S} + \tilde{G}H^{-1}F)x).$$
$$U(x) = H^{-1}[\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W}+(\tilde{S}+\tilde{G}H^{-1}F)x)-Fx]$$

⇨ In some neighborhood of x_0, λ and U are explicit affine functions of x !

Determining a Critical Region

- Impose primal and dual feasibility:

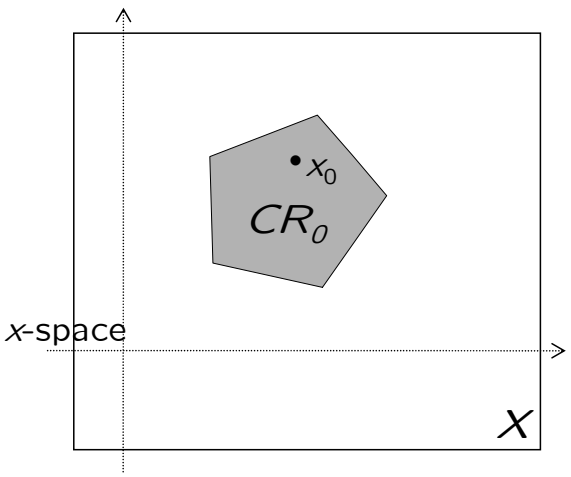
$$\hat{G}U(x) \leq \hat{W} + \hat{S}x$$
$$\tilde{\lambda}(x) \geq 0$$

⇨ linear inequalities in x !

- Remove redundant constraints

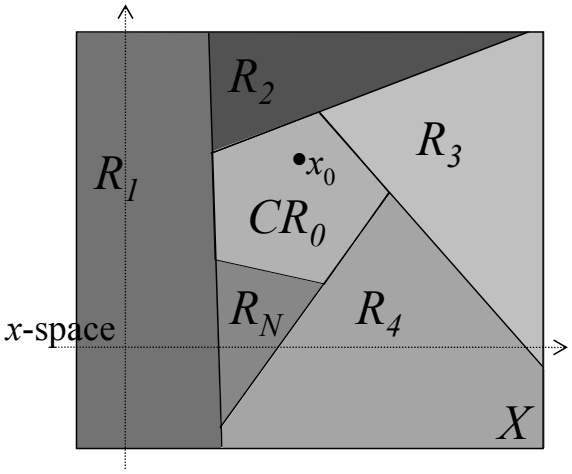
⇨ critical region CR_0

$$CR_0 = \{\mathcal{A}x \leq \mathcal{B}\}$$



- CR_0 is the set of all and only parameters x for which $\tilde{G}, \tilde{W}, \tilde{S}$ is the optimal combination of active constraints at the optimizer

Multiparametric QP



$$CR_0 = \{\mathcal{A}x \leq \mathcal{B}\}$$

$$R_i = \{x \in X : \ \mathcal{A}^i x > \mathcal{B}^i, \ \mathcal{A}^j z \leq \mathcal{B}^j, \ \forall j < i\}$$

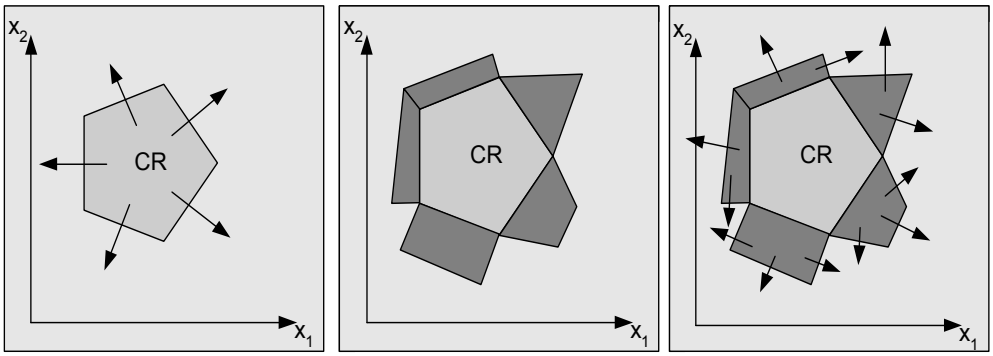
Theorem: $\{CR_0, R_1, \ldots, R_N\}$ is a partition of $X \subseteq \mathbf{R}^n$

Proceed iteratively: for each region R_i repeat the whole procedure with $X \leftarrow R_i$

The recursive algorithm terminates after a finite number of steps, because the number of combinations of active constraints is finite

Mp-QP – More efficient method

(Tøndel, Johansen, Bemporad, 2003)



The active set of a neighboring region is found by using the active set of the current region + knowledge of the type of hyperplane we are crossing:

$\widehat{G}^i U(x) \leq \widehat{W}^i + \widehat{S}^i x \Rightarrow$ The corresponding constraint is **added** to the active set

$\tilde{\lambda}_j(x) \geq 0 \quad \Rightarrow$ The corresponding constraint is **withdrawn** from the active set

Multiparametric QP

Theorem 1 *Consider a multi-parametric quadratic program with $H \succ 0$. The set X^* of parameters x for which the problem is feasible is a polyhedral set, the value function $J^* : X^* \mapsto \mathbb{R}$ is piecewise quadratic, convex and continuous and the optimizer $U^* : X^* \mapsto \mathbb{R}^r$ is piecewise affine and continuous.*

$$U^*(x) = \arg\min_U \ \tfrac{1}{2}U'HU + x'F'U \quad \text{subj. to} \quad Gz \leq W + Sx$$

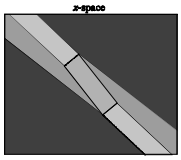
continuous, piecewise affine

$$V^*(x) = \tfrac{1}{2}x'Yx + \min_U \ \tfrac{1}{2}U'HU + x'F'U \quad \text{subj. to} \quad Gz \leq W + Sx$$

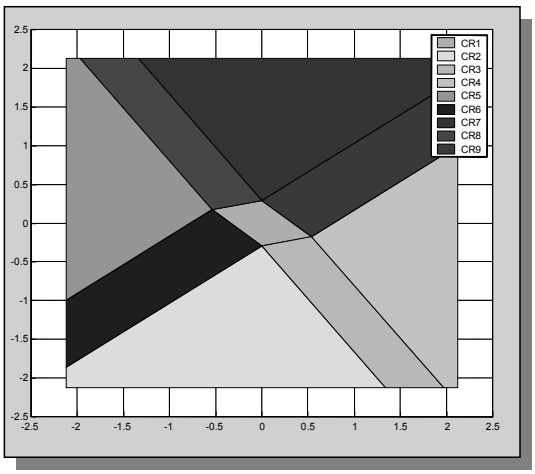
convex, continuous, piecewise quadratic, C^1 (if no degeneracy)

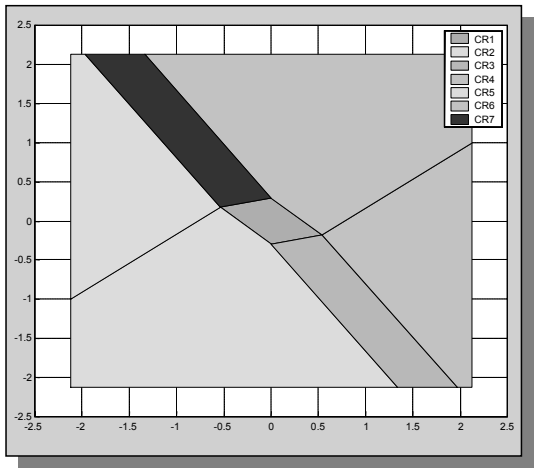
Corollary: The linear MPC controller is a continuous piecewise affine function of the state

$$u(x) = \begin{cases} F_1x + G_1 & \text{if} \quad H_1x \leq K_1 \\ \vdots & \vdots \\ F_Nx + G_N & \text{if} \quad H_Nx \leq K_N \end{cases}$$



Complexity Reduction





$$U(x(0)) \triangleq [u_0(x(0)), \ldots, u_{T-1}(x(0))]$$

Regions where the first component of the solution is the same can be joined (when their union is convex). (Bemporad, Fukuda, Torrisi, *Computational Geometry*, 2001)

Double Integrator Example

- System: $y(t) = \frac{1}{s^2}\ u(t)$ ⟹ (sampling + ZOH, *T=1* s)

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

- Constraints: $-1 \le u(t) \le 1$

- Control objective: minimize $\sum_{t=0}^{\infty} y'(t)y(t) + \frac{1}{100}u^2(t)$

$$u_{t+k} = K_{LQ}\ x(t+k|t)\ \ \forall k \ge N_u$$

- Optimization problem: for N_u=2

$$H = \begin{bmatrix} 0.8365 & 0.3603 \\ 0.3603 & 0.2059 \end{bmatrix},\ F = \begin{bmatrix} 0.4624 & 1.2852 \\ 0.1682 & 0.5285 \end{bmatrix}$$

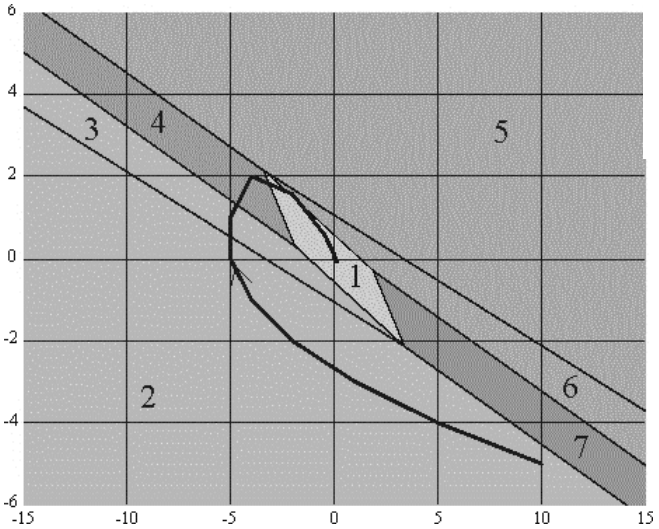
(cost function is normalized by max λ(H))

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix},\ W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

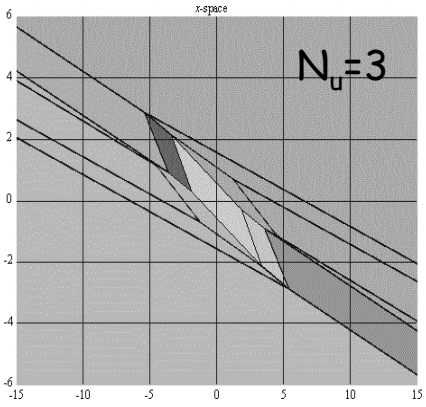
mp-QP solution

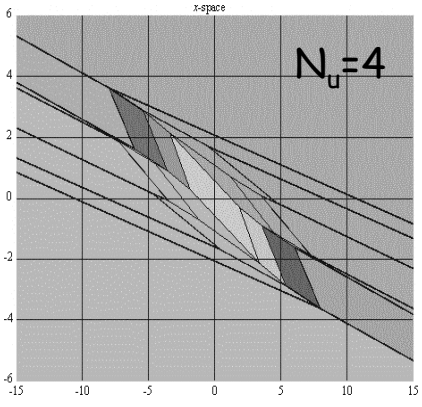
N_u=2

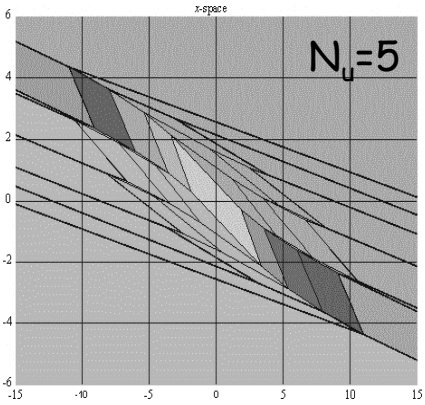
$$u(x) = \begin{cases} \begin{bmatrix} -0.8166 & -1.7499 \end{bmatrix} x & \text{if } \begin{bmatrix} -0.8166 & -1.7499 \\ 0.8166 & 1.7499 \\ 0.6124 & 0.4957 \\ -0.6124 & -0.4957 \end{bmatrix} x \le \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix} & \text{(Region \#1)} \\ 1.0000 & \text{if } \begin{bmatrix} 0.3864 & 1.0738 \\ 0.2970 & 0.9333 \end{bmatrix} x \le \begin{bmatrix} -1.0000 \\ -1.0000 \end{bmatrix} & \text{(Region \#2)} \\ 1.0000 & \text{if } \begin{bmatrix} 0.9712 & 2.6991 \\ -0.2970 & -0.9333 \\ 0.8166 & 1.7499 \end{bmatrix} x \le \begin{bmatrix} -1.0000 \\ 1.0000 \\ -1.0000 \end{bmatrix} & \text{(Region \#3)} \\ \begin{bmatrix} -0.5528 & -1.5364 \end{bmatrix} x + 0.4308 & \text{if } \begin{bmatrix} -0.9712 & -2.6991 \\ 0.3864 & 1.0738 \\ 0.6124 & 0.4957 \end{bmatrix} x \le \begin{bmatrix} 1.0000 \\ 1.0000 \\ -1.0000 \end{bmatrix} & \text{(Region \#4)} \\ -1.0000 & \text{if } \begin{bmatrix} -0.3864 & -1.0738 \\ -0.2970 & -0.9333 \end{bmatrix} x \le \begin{bmatrix} -1.0000 \\ -1.0000 \end{bmatrix} & \text{(Region \#5)} \\ -1.0000 & \text{if } \begin{bmatrix} -0.9712 & -2.6991 \\ 0.2970 & 0.9333 \\ -0.8166 & -1.7499 \end{bmatrix} x \le \begin{bmatrix} -1.0000 \\ 1.0000 \\ -1.0000 \end{bmatrix} & \text{(Region \#6)} \\ \begin{bmatrix} -0.5528 & -1.5364 \end{bmatrix} x - 0.4308 & \text{if } \begin{bmatrix} -0.3864 & -1.0738 \\ 0.9712 & 2.6991 \\ -0.6124 & -0.4957 \end{bmatrix} x \le \begin{bmatrix} 1.0000 \\ 1.0000 \\ -1.0000 \end{bmatrix} & \text{(Region \#7)} \end{cases}$$

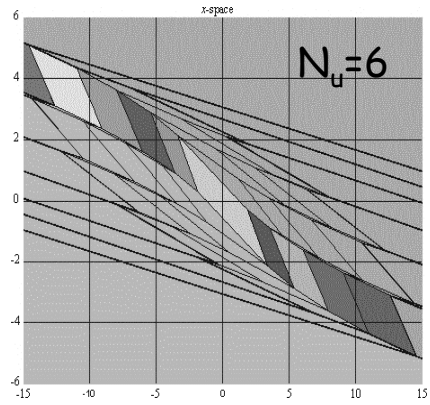


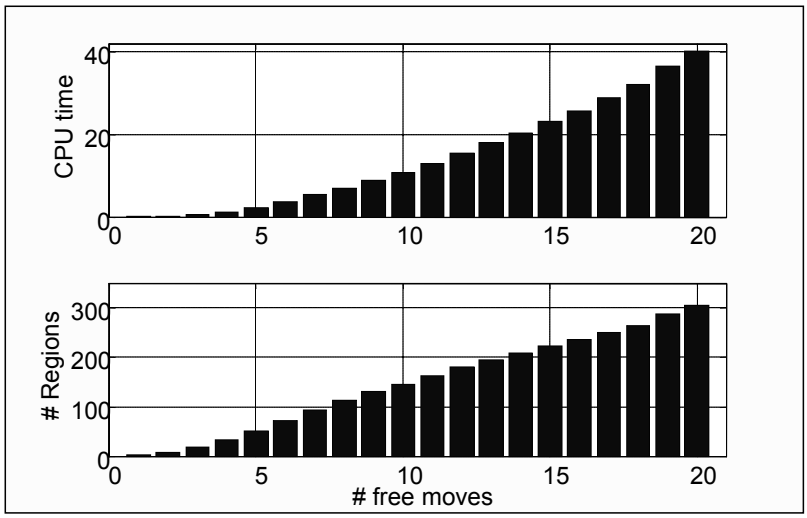
Complexity











Extensions

- Tracking of reference $r(t)$: $\delta u(t) = F(x(t), u(t-1), r(t))$

- Rejection of measured disturbance $v(t)$: $\delta u(t) = F(x(t), u(t-1), v(t))$

- Soft constraints: $u(t) = F(x(t))$

  $y_{\min} - \epsilon \leq y(t+k|t) \leq y_{\max} + \epsilon$

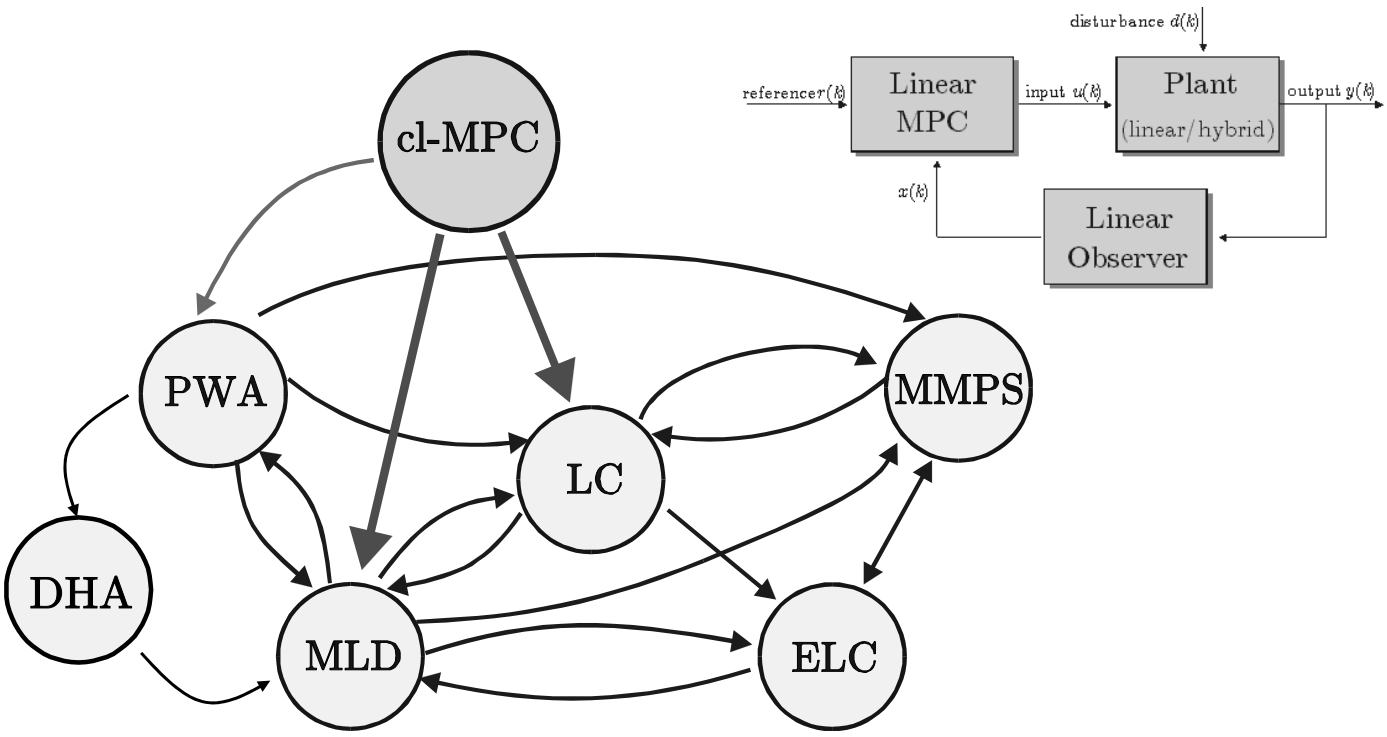
- Variable constraints: $u(t) = F(x(t), u_{\min}(t), \ldots, y_{\max}(t))$

  $u_{\min}(t) \leq u(t+k) \leq u_{\max}(t)$
  $y_{\min}(t) \leq y(t+k|t) \leq y_{\max}(t)$

- Linear norms: $\min_U J(U, x(t)) \triangleq \sum_{k=0}^{p} \|Qy(t+k|t)\|_\infty + \|Ru(t+k)\|_\infty$

  (Bemporad, Borrelli, Morari, IEEE TAC, 2002)

Closed-Loop MPC and Hybrid Systems



Motivation:

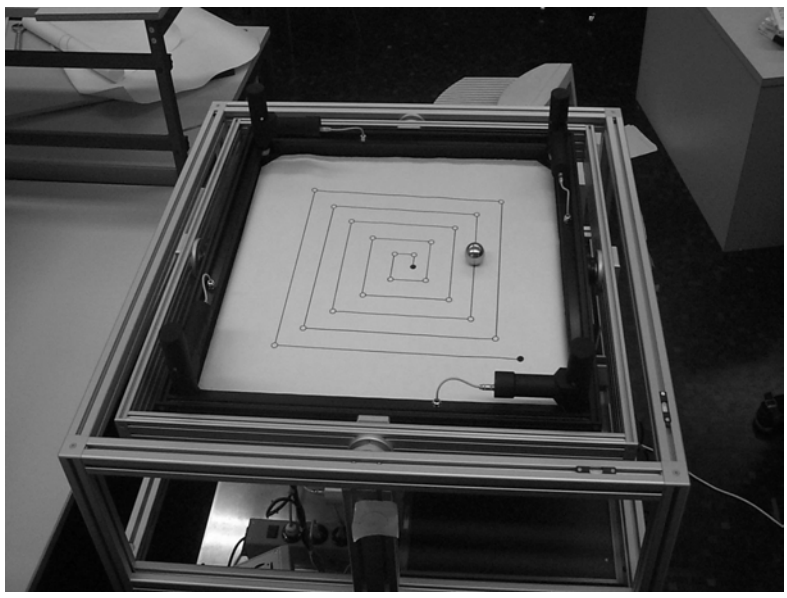
Use hybrid techniques to analyze closed-loop MPC systems !

(Bemporad, Heemels, De Schutter IEEE TAC, 2002)

MPC Regulation of a Ball on a Plate

Task:

- Tune an MPC controller by simulation, using the *MPC Simulink Toolbox*
- Get the *explicit solution* of the MPC controller.
- Validate the controller on *experiments*.



B&P Experimental Setup

Host

Ball & Plate System

TCP/IP Code Data

Ball & Plate

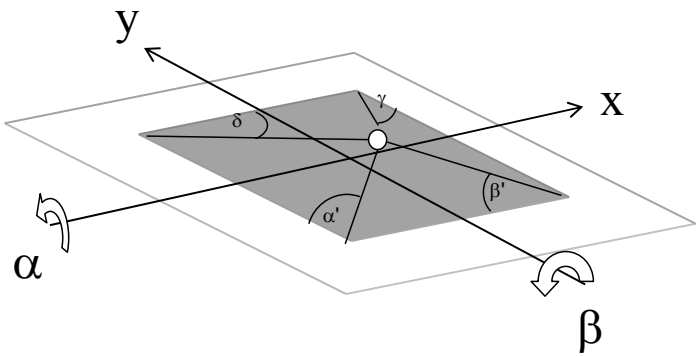
u_1, u_2

a,b

x, y

Target

Ball&Plate Experiment



- Specifications:

  Angle: -17 deg ... +17deg

  Plate: -30 cm ...+30 cm

  Input Voltage: -10 V... +10 V

  Computer: PENTIUM166

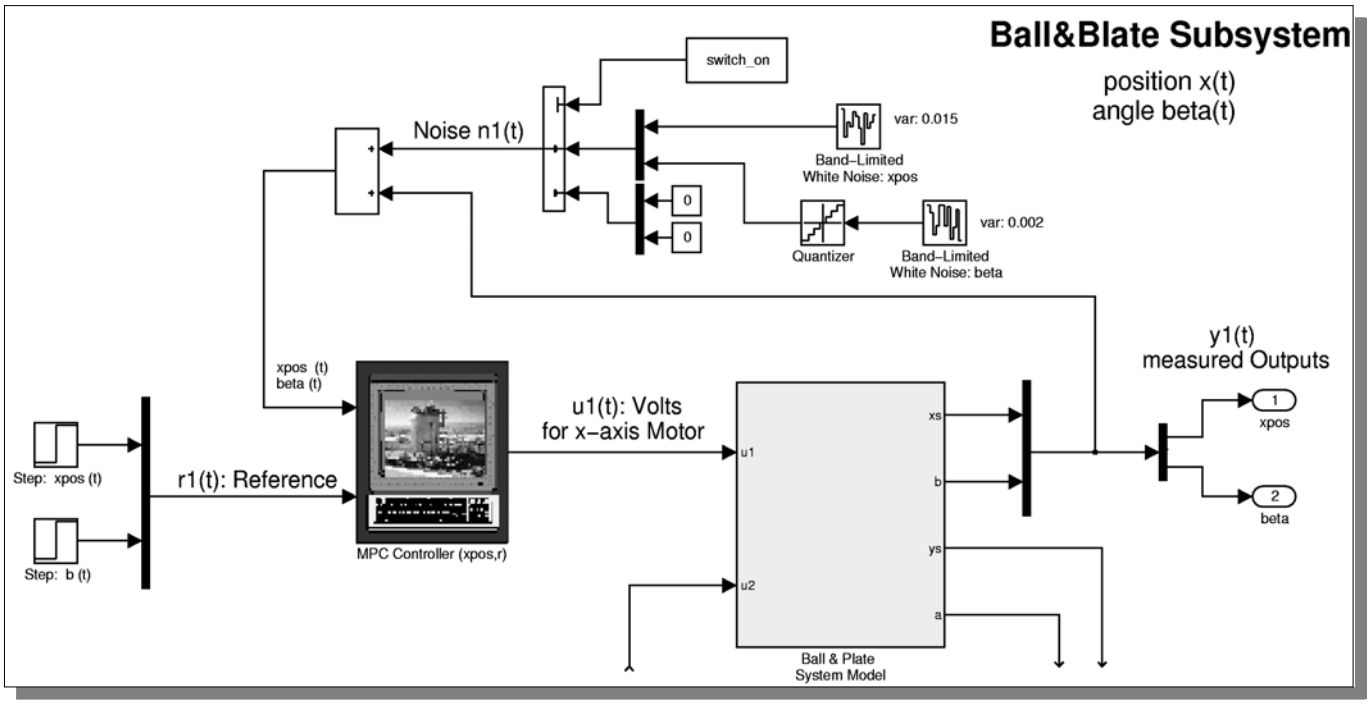
  Sampling Time:30 ms

- Model: LTI 14 states

  Constraints on inputs and states

General Philosophy: (1) MPC Design

- Step 1: Tune the MPC controller (in simulation)

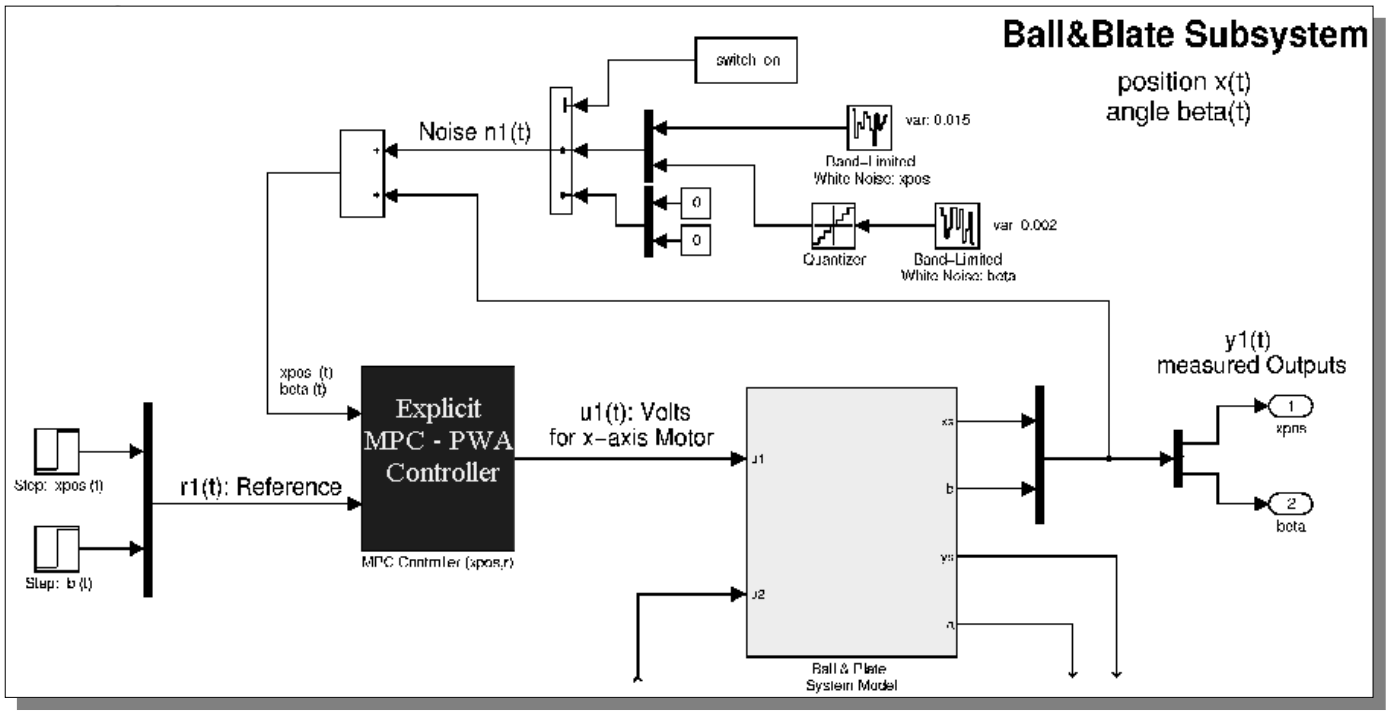


E.g: MPC Toolbox for Matlab

(Bemporad, Morari, Ricker, 2003)

General Philosophy: (2) Implementation

- Step 2: Solve mp-QP and implement Explicit

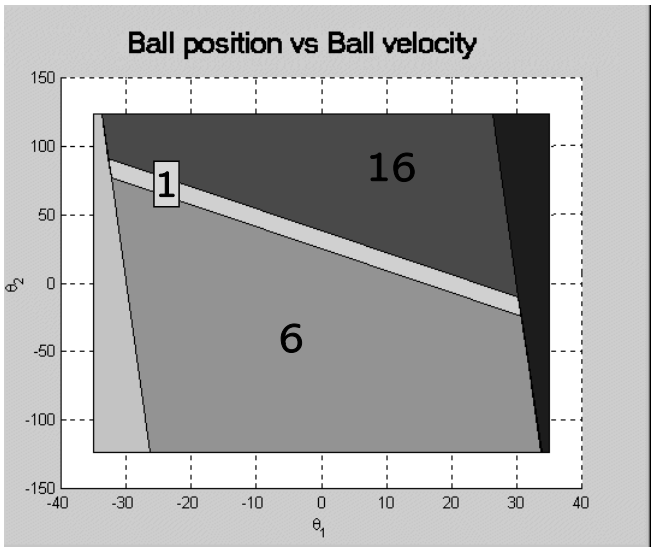


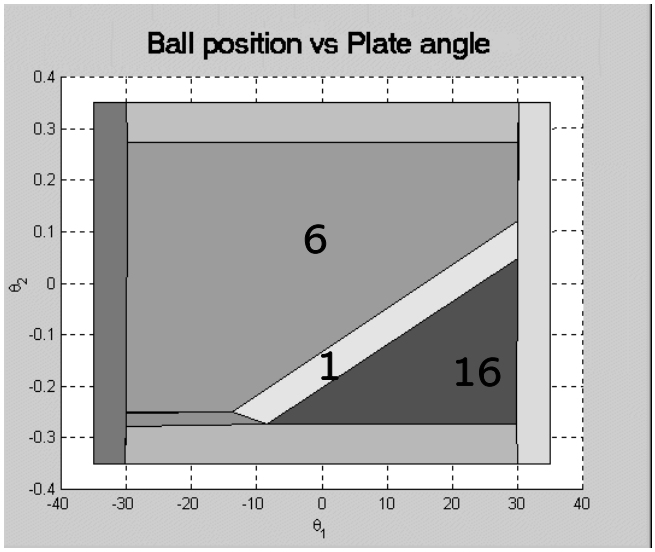
E.g: Real-Time Workshop + xPC Toolbox

Explicit MPC Solution

Controller: x: 22 Regions, y: 23 Regions

x-MPC: sections at $\alpha_x^{\circ}=0$, $\alpha_x=0$, $u_x=0$, $r_x=18$, $r_\alpha=0$





Region 1: LQR Controller (near Equilibrium)

Region 6: Saturation at -10

Region 16: Saturation at +10

MPC Regulation of a Ball on a Plate

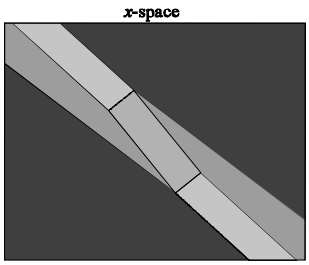
<u>Design Steps:</u>

- Tune an MPC controller by simulation, using the *MPC Simulink Toolbox* .
- Get the *explicit solution* of the MPC controller.
- ✓ Validate the controller on *experiments*.

Comments on Explicit MPC

- Multiparametric Quadratic Programs (mp-QP) can be solved efficiently
- Model Predictive Control (MPC) can be solved off-line via mp-QP
- Explicit solution of MPC controller $u = f(x)$ is Piecewise Affine

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if} \quad H_N x \leq K_N \end{cases}$$



⇒ Eliminate heavy on-line computation for MPC

⇒ Make MPC suitable for fast/small/cheap processes

MILP Formulation of MPC

(Bemporad, Borrelli, Morari, 2000)

$$\min \sum_{k=0}^{T-1} \|Qy(t+k+1|t)\|_\infty + \|Ru(t+k)\|_\infty$$
$$\text{s.t.} \text{MLD dynamics}$$

- Introduce slack variables: $\min |x|$ ⇒

$$\begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \epsilon \geq x \\ & \epsilon \geq -x \end{array}$$

$$\begin{array}{rcl} \epsilon_k^x & \geq & \|Qy(t+k|t)\|_\infty \\ \epsilon_k^u & \geq & \|Ru(t+k)\|_\infty \end{array} \Rightarrow$$

$$\begin{array}{rcll} \epsilon_k^x & \geq & [Qy(t+k|t)]_i & i=1,\ldots,p, \quad k=1,\ldots,T-1 \\ \epsilon_k^x & \geq & -[Qy(t+k|t)]_i & i=1,\ldots,p, \quad k=1,\ldots,T-1 \\ \epsilon_k^u & \geq & [Ru(t+k))]_i & i=1,\ldots,m, \quad k=0,\ldots,T-1 \\ \epsilon_k^u & \geq & -[Ru(t+k))]_i & i=1,\ldots,m, \quad k=0,\ldots,T-1 \end{array}$$

- Set $\xi \triangleq [\epsilon_1^x, \ldots, \epsilon_{N_y}^x, \epsilon_1^u, \ldots, \epsilon_{T-1}, U, \delta, z]$

Mixed Integer Linear Program (MILP)

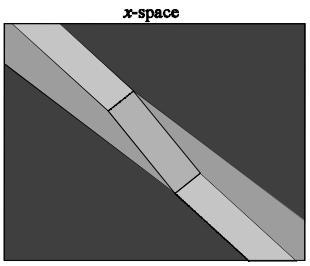
$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u$$
$$\text{s.t.} G\xi \leq W + Sx(t)$$

Multiparametric MILP

$$\begin{aligned} \min_{\xi=\{\xi_c,\xi_d\}} \quad & f'\xi_c + d'\xi_d \\ \text{s.t.} \quad & G\xi_c + E\xi_d \leq W + Fx \end{aligned} \qquad \begin{aligned} \xi_c &\in \mathbf{R}^n \\ \xi_d &\in \{0,1\}^m \end{aligned}$$

- mp-MILP can be solved (by alternating MILPs and mp-LPs)
  (Dua, Pistikopoulos, 1999)
- **Theorem:** The multiparametric solution $\xi^*(x)$ is piecewise affine
- **Corollary: The MPC controller is piecewise affine in** x

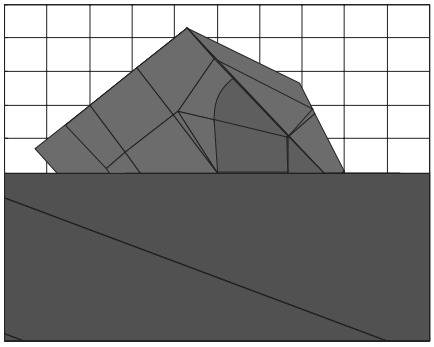
$$u(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if} \quad H_N x \leq K_N \end{cases}$$



Solutions via Dynamic Programming

(Borrelli, Bemporad, Baotic, Morari, 2003)
(Mayne, ECC 2001)

- Explicit solutions to finite-time optimal control problems for PWA systems can be obtained using a combination of
  - Dynamic Programming
  - Multiparametric Linear (1-norm, ∞-norm), or or Quadratic (squared 2-norm) programming



Note: in the 2-norm case,
the partition may not be polyhedral

Hybrid Control Examples (Revisited)

Hybrid Control - Example

Switching System:

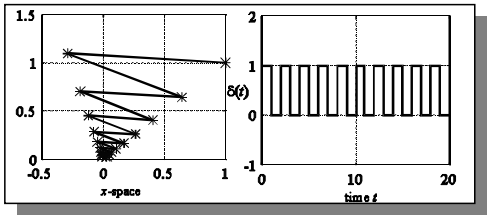
$$x(t+1) = 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [\,0 \quad 1\,]\, x(t)$$

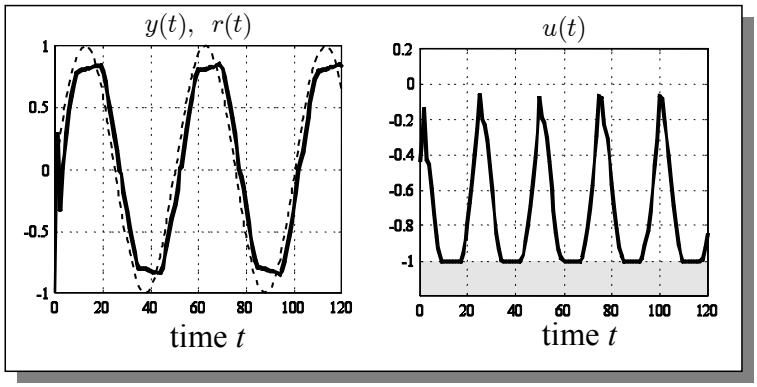
$$\alpha(t) = \begin{cases} \frac{\pi}{3} \text{ if } [1\ \ 0]x(t) \geq 0 \\ -\frac{\pi}{3} \text{ if } [1\ \ 0]x(t) < 0 \end{cases}$$

Constraint: $\boxed{-1 \leq u(t) \leq 1}$

Open loop:



Closed loop:



Hybrid MPC - Example

- MLD system

State x(t)	2 variables
Input u(t)	1 variables
Aux. binary vector δ(t)	1 variables
Aux. continuous vector z(t)	4 variables

- mp-MILP optimization problem

$$\min_{\left\{v_0^1\right\}} J(v_0^1, x(t)) \triangleq \sum_{k=0}^{1} \|Q_1(v(k)-u_e)\|_\infty + \|Q_2(\delta(k|t)-\delta_e)\|_\infty + \|Q_3(z(k|t)-z_e)\|_\infty + \|Q_4(x(k|t)-x_e)\|_\infty$$

subject to constraints

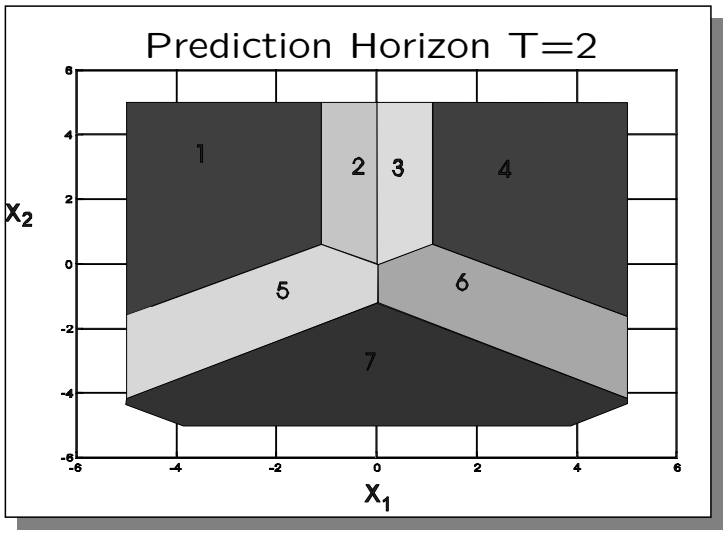
to be solved in the region $\begin{array}{rcl} -5 & \leq x_1 \leq & 5 \\ -5 & \leq x_2 \leq & 5 \end{array}$

- Computational complexity of mp-MILP

Linear constraints	84
Continuous variables	20
Binary variables	2
Parameters	2
Time to solve mp-MILP	3 min
Number of regions	7

mp-MILP Solution

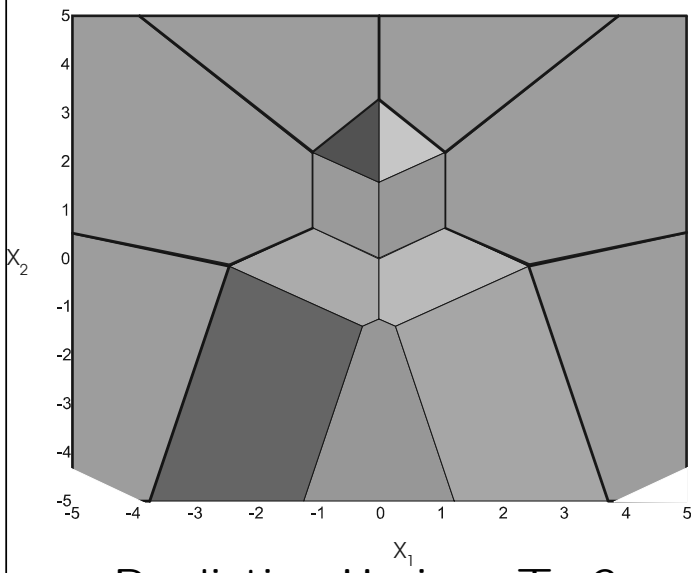
$$u = \begin{cases} -1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.7296 \\ -1.0000 & 0.0000 \\ 0.0000 & 1.0000 \\ 94.5067 & -1.0000 \end{bmatrix} x \leq \begin{bmatrix} -2.1680 \\ 5.0000 \\ 5.0000 \\ -105.1385 \end{bmatrix} \\ \text{(Region \#1)} \\ \begin{bmatrix} 0.9238 & 0.0000 \end{bmatrix} x & \text{if } \begin{bmatrix} -188.1504 & 1.0000 \\ -20.3158 & -34.1997 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \leq \begin{bmatrix} 204.3621 \\ 1.0000 \\ 0.0000 \\ 5.0000 \end{bmatrix} \\ \text{(Region \#2)} \\ \begin{bmatrix} -0.9238 & -0.0000 \end{bmatrix} x & \text{if } \begin{bmatrix} -217.00 & 1.0000 \\ 38.1919 & -64.29 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \leq \begin{bmatrix} 4.9708 \\ 1.0000 \\ 1.0000 \\ 5.0000 \end{bmatrix} \\ \text{(Region \#3)} \\ -1.0000 & \text{if } \begin{bmatrix} -188.15 & -1.0000 \\ -1.0000 & -1.6858 \\ 1.0000 & 0.0000 \\ 1.0000 & 133.68 \end{bmatrix} x \leq \begin{bmatrix} -204.39 \\ -2.1673 \\ 5.0000 \\ 665.60 \end{bmatrix} \\ \text{(Region \#4)} \\ \begin{bmatrix} 0.4619 & -0.8000 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.0000 & -1.7193 \\ -1.0000 & 1.7295 \\ -1.0000 & 0.0000 \\ 13.4029 & 23.6383 \\ 43.4009 & 0.0000 \end{bmatrix} x \leq \begin{bmatrix} 2.1387 \\ 2.1174 \\ 5.0000 \\ -1.0000 \\ -1.0000 \end{bmatrix} \\ \text{(Region \#5)} \\ \begin{bmatrix} -0.4619 & -0.8000 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.0000 & 1.7175 \\ -20.2527 & 34.1997 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.7369 \\ 106.3247 & 1.0000 \end{bmatrix} x \leq \begin{bmatrix} 2.0413 \\ -1.0000 \\ 0.0000 \\ 2.2345 \\ 525.0259 \end{bmatrix} \\ \text{(Region \#6)} \\ 1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.6446 \\ -1.0000 & 1.7449 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.6788 \\ 1.0000 & -265.137 \\ 1.0000 & 1.7369 \\ 1.0000 & 0.0000 \end{bmatrix} x \leq \begin{bmatrix} 12.0940 \\ -2.2448 \\ 5.0000 \\ 12.3141 \\ 1329.55 \\ -2.2345 \\ 5.0000 \end{bmatrix} \\ \text{(Region \#7)} \end{cases}$$



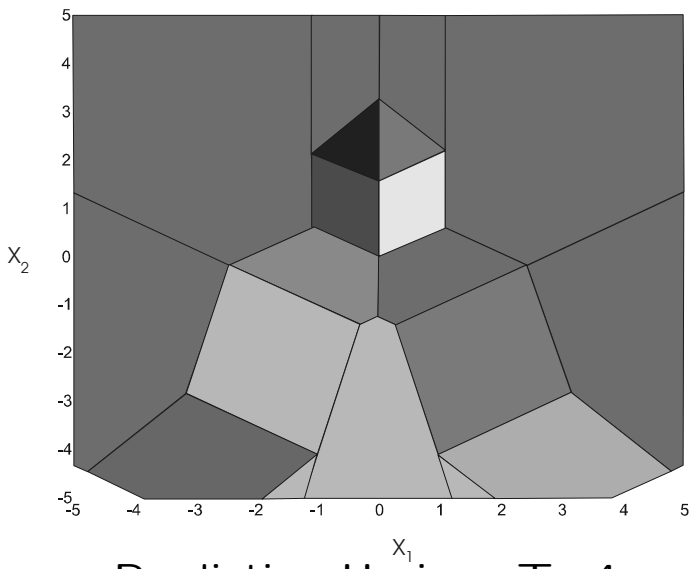
PWA law $\equiv$ MPC law

Linear constraints	84
Continuous variables	20
Binary variables	2
Parameters	2
Time to solve mp-MILP	3 min
Number of regions	7

mp-MILP Solution



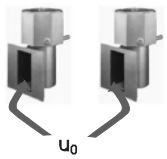
Prediction Horizon T=3



Prediction Horizon T=4

Alternate Heating of Two Furnaces

(Bemporad, Borrelli, Morari, 2000)



- mp-MILP optimization problem

$$\min_{\{u(t),u(t+1),u(t+2)\}} \sum_{k=0}^{2} \|R(u(t+k+1)-u(t+k))\|_\infty + \|Q(x(t+k|t)-x_e)\|_\infty$$

- Parameter set

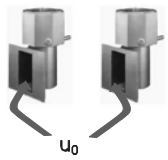
$$\begin{array}{rcl} -1 & \leq x_1 \leq & 1 \\ -1 & \leq x_2 \leq & 1 \\ 0 & \leq u_0 \leq & 1 \end{array}$$

parameterized !

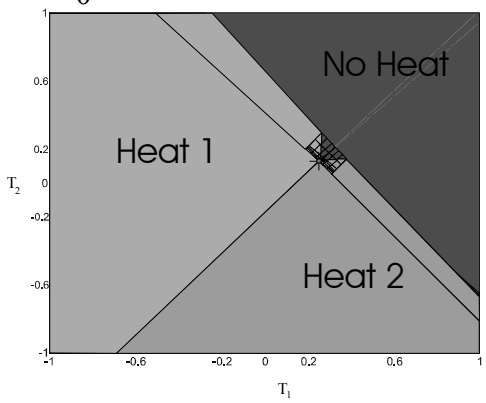
- Computational complexity of mp-MILP

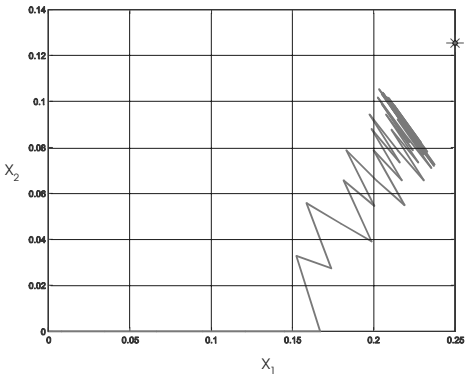
Linear constraints	168
Continuous variables	33
Binary variables	6
Parameters	3
Time to solve mp-MILP	5 min
Number of regions	105

mp-MILP Solution



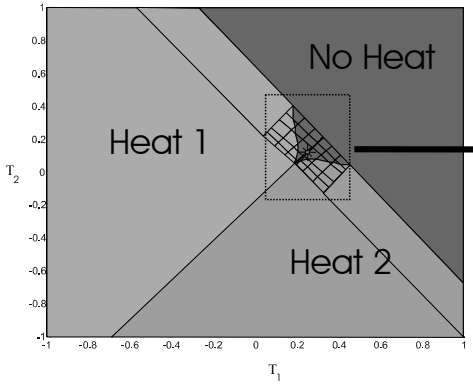
u_0=0.4

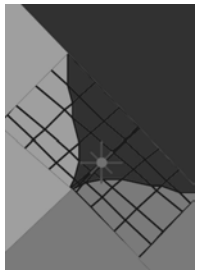


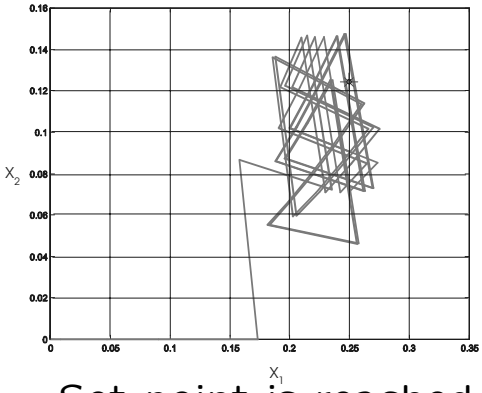


Set point cannot be reached

u_0=0.8







Set point is reached

NOTE: The control action of explicit and of implicit MPC are totally equal !

Hybrid Control Example: Traction Control System

Vehicle Traction Control

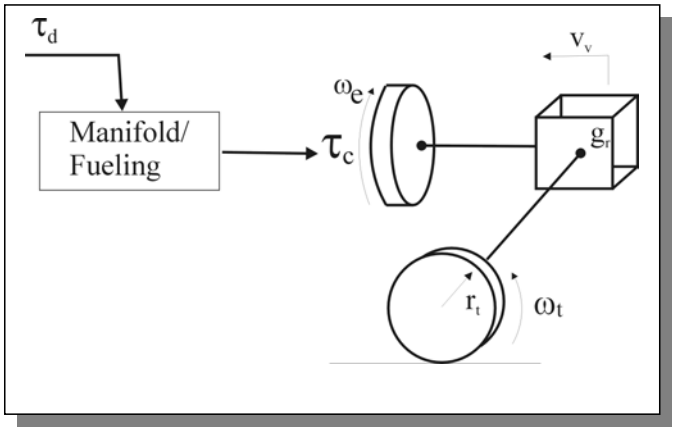
Improve driver's ability to control a vehicle under adverse external conditions (wet or icy roads)

Model
nonlinear, uncertain, constraints

Controller
suitable for real-time implementation

MLD hybrid framework + optimization-based control strategy

Simple Traction Model

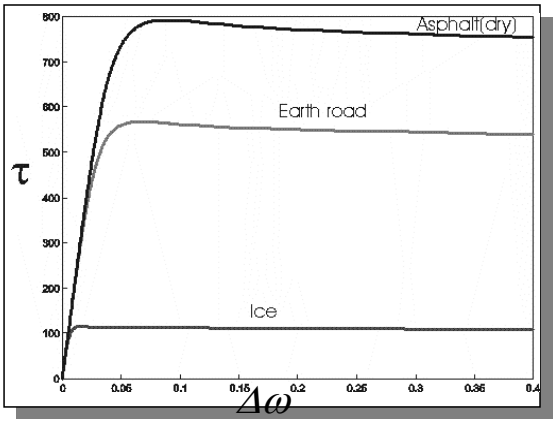


- Mechanical system

$$\dot{\omega}_e = \frac{1}{J_e}\left(\tau_c - b_e\omega_e - \frac{\tau_t}{g_r}\right)$$
$$\dot{v}_v = \frac{\tau_t}{m_v r_t}$$

- Manifold/fueling dynamics

$$\tau_c = b_i\tau_d(t - \tau_f)$$

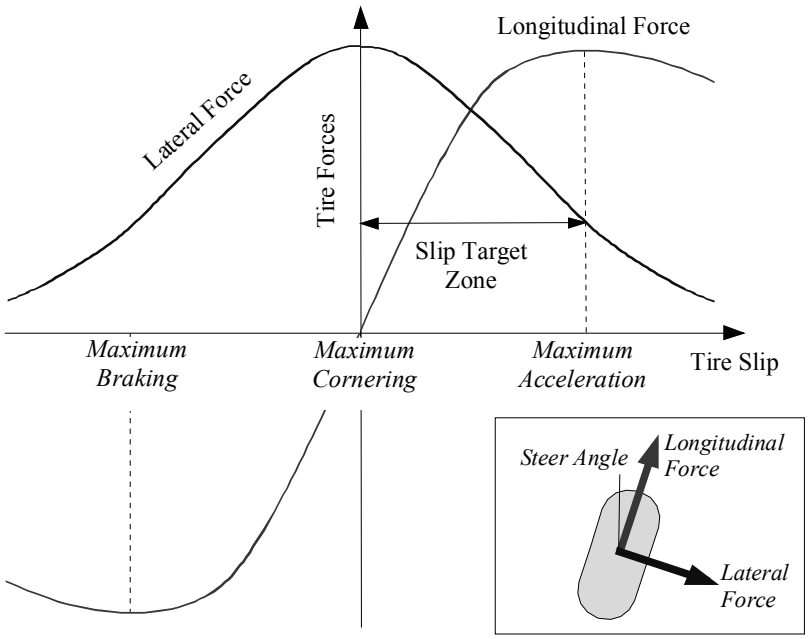


- Tire torque τ_t is a function of slip $\Delta\omega$ and road surface adhesion coefficient μ

$$\Delta\omega = \frac{\omega_e}{g_r} - \frac{v_v}{r_t}$$

Wheel slip

Tire Force Characteristics



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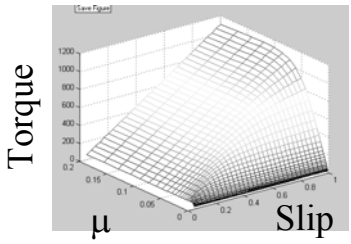
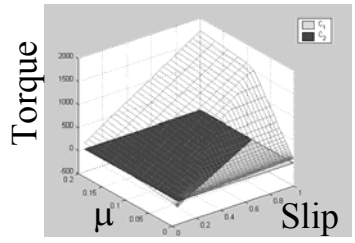
Hybrid Model

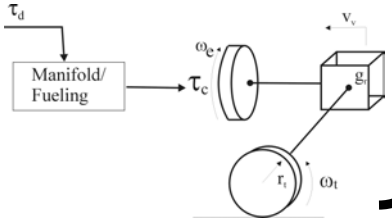


Nonlinear tire torque $\tau_t = f(\Delta\omega, \mu)$

PWA Approximation

(PWL Toolbox, Julian, 1999)





HYSDEL
(Hybrid Systems Description Language)

Mixed-Logical Dynamical (MLD) Hybrid Model (discrete time)

MLD Model

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\ y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\ E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5 \end{aligned}$$

State x(t)	9 variables
Input u(t)	1 variable
Aux. Binary vars δ(t)	3 variables
Aux. Continuous vars z(t)	4 variables

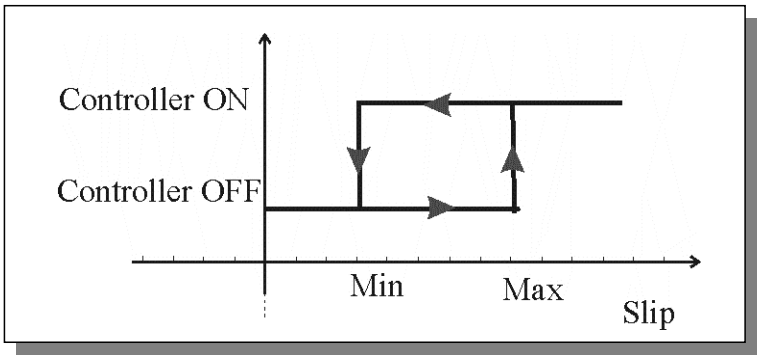
➡ The MLD matrices are automatically generated in Matlab format by HYSDEL

Performance and Constraints

- Control objective:

$$\begin{aligned} &\min \sum_{k=0}^{N} \quad |\Delta\omega(k|t) - \Delta\omega_{des}| \\ &\text{subj. to.} \quad \text{MLD Dynamics} \end{aligned}$$

- Constraints:
  - Limits on the engine torque: $-20Nm \leq \tau_d \leq 176Nm$
- Logic Constraint:
  - Hysteresis



Experimental Apparatus

Experimental Apparatus

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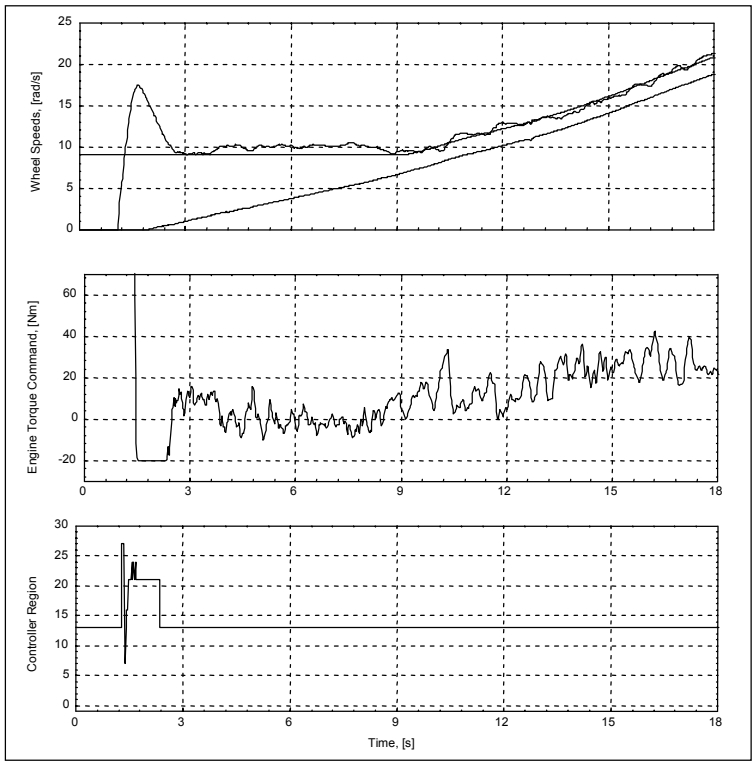
Experimental Results



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Experiment

- >500 regions
- 20ms sampling time
- Pentium 266Mhz + Labview

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Hybrid Control Example: Cruise Control System

Hybrid Control Problem

Renault Clio 1.9 DTI RXE

GOAL:

command gear ratio, gas pedal, and brakes to **track** a desired speed and minimize consumption

Hybrid Model

- Vehicle dynamics

$$m\ddot{x} = F_e - F_b - \beta\dot{x}$$

$\dot{x}$ = vehicle speed

F_e = traction force

F_b = brake force

$\Longrightarrow$ discretized with sampling time $T_s = 0.5$ s

- Transmission kinematics

$$\omega = \frac{R_g(i)}{k_s}\dot{x}$$

$$F_e = \frac{R_g(i)}{k_s}M$$

ω = engine speed

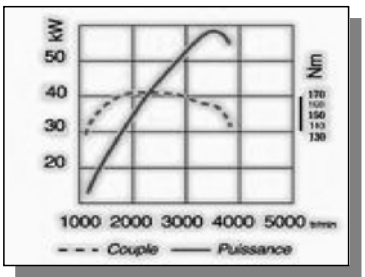
M = engine torque

i = gear

Hybrid Model

- Engine torque $-C_e^-(\omega) \leq M \leq C_e^+(\omega)$

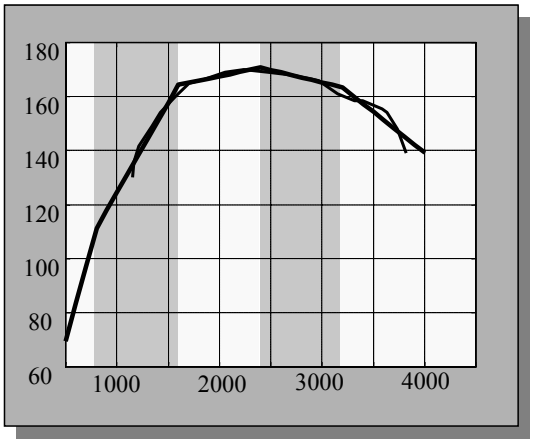
- Max engine torque $C_e^+(\omega)$



http://www.renault.fr

Piecewise-linearization:
(PWL Toolbox, Julián, 1999)

$\Longrightarrow$



requires: 4 binary aux variables
4 continuous aux variables

- Min engine torque $C_e^-(\omega) = \alpha_1 + \beta_1\omega$

Hybrid Model

- Gear selection: for each gear #i,

  define a binary input $\boxed{g_i \in \{0,1\}}$

- Gear selection (traction force):

  $\boxed{F_e = \frac{R_g(i)}{k_s} M}$ depends on gear #i

  define auxiliary continuous variables:

  $$\boxed{\text{IF } \; g_i = 1 \; \text{ THEN } \; F_{ei} = \frac{R_g(i)}{k_s} M \; \text{ ELSE } \; 0}$$

  $$\Longrightarrow \boxed{F_e = F_{eR} + F_{e1} + F_{e2} + F_{e3} + F_{e4} + F_{e5}}$$

- Gear selection (engine/vehicle speed):

  $\boxed{\omega = \frac{R_g(i)}{k_s} \dot{x}}$ similarly, also requires 6 auxiliary continuous variables

Hybrid Model

- MLD model

$$\begin{aligned} x(t+1) &= Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \\ y(t) &= Cx(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \\ E_2 \delta(t) + E_3 z(t) &\leq E_4 x(t) + E_1 u(t) + E_5 \end{aligned}$$

- 2 continuous states: x, v (vehicle position and speed)
- 2 continuous inputs: M, F_b (engine torque, brake force)
- 6 binary inputs: $g_R, g_1, g_2, g_3, g_4, g_5$ (gears)
- 1 continuous output: v (vehicle speed)
- 16 auxiliary continuous vars: (6 traction force, 6 engine speed, 4 PWL max engine torque)
- 4 auxiliary binary vars: (PWL max engine torque breakpoints)
- 96 mixed-integer inequalities

Hysdel Model

```
SYSTEM car {

INTERFACE {
        STATE { REAL position, speed; }
        INPUT { REAL torque, F_brake;
                BOOL gear1, gear2, gear3, gear4, gear5, gearR; }

        PARAMETER{
                REAL mass = 1020; /* kg */
                REAL beta_friction = 25; /* N/m*s */
                REAL Rgear1 = 3.7271; REAL Rgear2 = 2.048;
                REAL Rgear3 = 1.321; REAL Rgear4 = 0.971;
                REAL Rgear5 = 0.756; REAL RgearR = -3.545;
                REAL wheel_rim = 14;        /* in */
                ...
                } }
IMPLEMENTATION {
        AUX {REAL Fe1, Fe2, Fe3, Fe4, Fe5, FeR;
             REAL w1, w2, w3, w4, w5, wR;
             BOOL dPWL1,dPWL2,dPWL3,dPWL4;
             REAL DCe1,DCe2,DCe3,DCe4; }

        AD { dPWL1 = wPWL1-(w1+w2+w3+w4+w5+wR)<=0;
             dPWL2 = wPWL2-(w1+w2+w3+w4+w5+wR)<=0;
             dPWL3 = wPWL3-(w1+w2+w3+w4+w5+wR)<=0;
             dPWL4 = wPWL4-(w1+w2+w3+w4+w5+wR)<=0; }

        DA { Fe1 = {IF gear1  THEN torque/speed_factor*Rgear1;
             Fe2 = {IF gear2  THEN torque/speed_factor*Rgear2;
             Fe3 = {IF gear3  THEN torque/speed_factor*Rgear3;
             Fe4 = {IF gear4  THEN torque/speed_factor*Rgear4;
             Fe5 = {IF gear5  THEN torque/speed_factor*Rgear5;
             FeR = {IF gearR  THEN torque/speed_factor*RgearR;

             w1 = {IF gear1  THEN speed/speed_factor*Rgear1;
             w2 = {IF gear2  THEN speed/speed_factor*Rgear2;
             w3 = {IF gear3  THEN speed/speed_factor*Rgear3;
             w4 = {IF gear4  THEN speed/speed_factor*Rgear4;
             w5 = {IF gear5  THEN speed/speed_factor*Rgear5;
             wR = {IF gearR  THEN speed/speed_factor*RgearR;

             DCe1 = {IF dPWL1 THEN (aPWL2-aPWL1)+(bPWL2-bPWL1)*(w1+w2+w3+w4+w5+wR);
             DCe2 = {IF dPWL2 THEN (aPWL3-aPWL2)+(bPWL3-bPWL2)*(w1+w2+w3+w4+w5+wR);
             DCe3 = {IF dPWL3 THEN (aPWL4-aPWL3)+(bPWL4-bPWL3)*(w1+w2+w3+w4+w5+wR);
             DCe4 = {IF dPWL4 THEN (aPWL5-aPWL4)+(bPWL5-bPWL4)*(w1+w2+w3+w4+w5+wR);
             }

        CONTINUOUS { position = position+Ts*speed;
                     speed = speed+Ts/mass*(Fe1+Fe2+Fe3+Fe4+Fe5+FeR-
                                            F_brake-beta_friction*speed); }

        MUST    {  wemin <= w1+w2+w3+w4+w5+wR;
                   w1+w2+w3+w4+w5+wR <= wemax;
                   -F_brake <=0; /* brakes cannot accelerate ! */
                   F_brake <= max_brake_force;

                   -torque-(alpha1+beta1*(w1+w2+w3+w4+w5+wR)) <=0;
                   torque-(aPWL1+bPWL1*(w1+w2+w3+w4+w5+wR)+DCe1+DCe2+DCe3+DCe4)-1<=0;

                   -(gear1+gear2+gear3+gear4+gear5+gearR)<=-1;
                   (gear1+gear2+gear3+gear4+gear5+gearR)<=1;
                   Fe1+Fe2+Fe3+Fe4+Fe5+FeR <= max_force;
                   -Fe1-Fe2-Fe3-Fe4-Fe5-FeR <= -max_force;

                   dPWL4 -> dPWL3; dPWL4 -> dPWL2;
                   dPWL4 -> dPWL1; dPWL3 -> dPWL2;
                   dPWL3 -> dPWL1; dPWL2 -> dPWL1;  }
```

http://control.ethz.ch/~hybrid/hysdel

Hybrid Controller

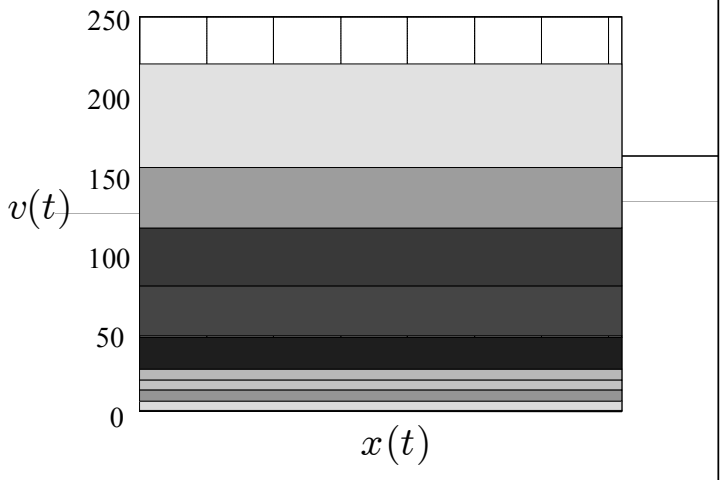
- Max-speed controller

$$\max_{u_t} J(u_t, x(t)) \triangleq v(t+1|t)$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

MILP optimization problem

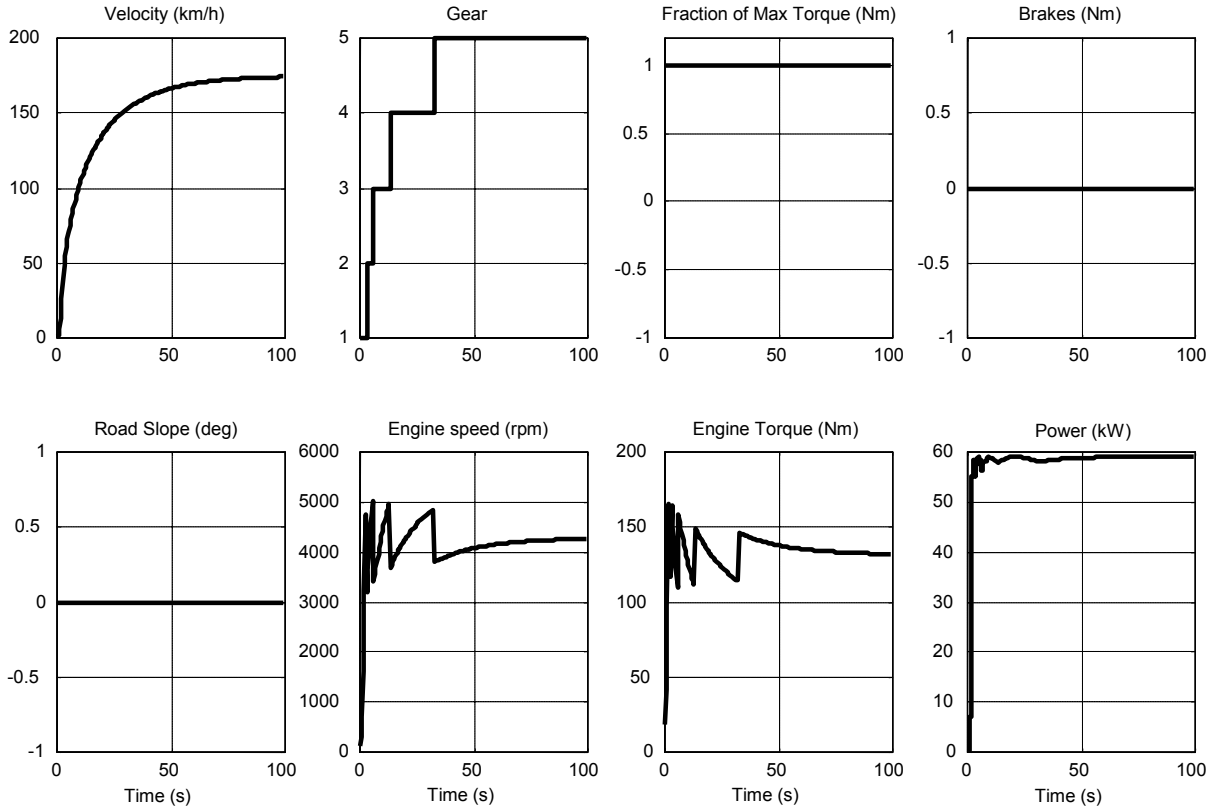
Linear constraints	96
Continuous variables	18
Binary variables	10
Parameters	1
Time to solve mp-MILP (Sun Ultra 10)	45 s
Number of regions	**11**



($x(t)$ is irrelevant)

Hybrid Controller

- Max-speed controller



Hybrid Controller

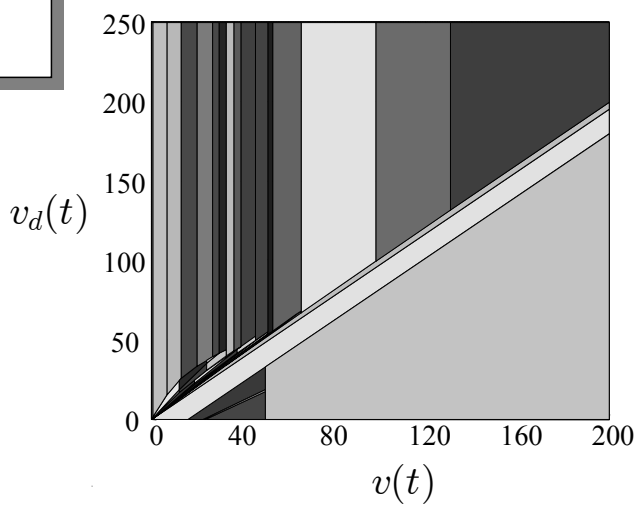
- Tracking controller

$$\min_{u_t} J(u_t, x(t)) \triangleq |v(t+1|t) - v_d(t)| + \rho|\omega|$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

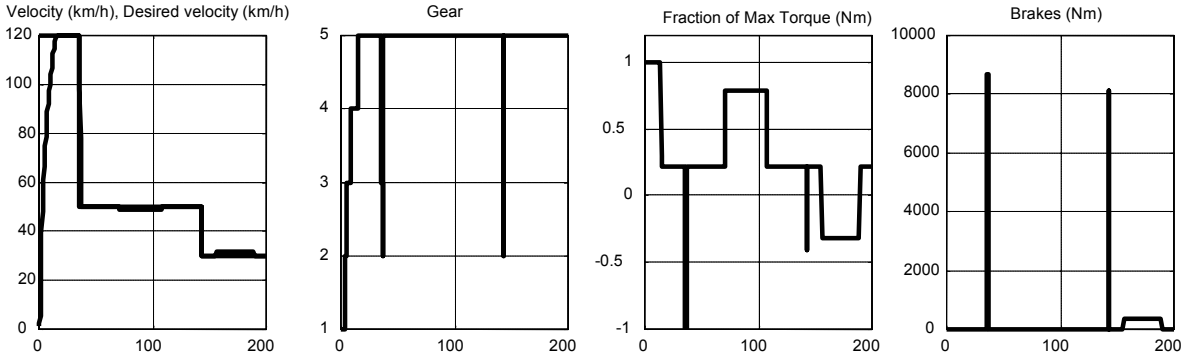
MILP optimization problem

Linear constraints	98
Continuous variables	19
Binary variables	10
Parameters	2
Time to solve mp-MILP (Sun Ultra 10)	27 m
Number of regions	**49**

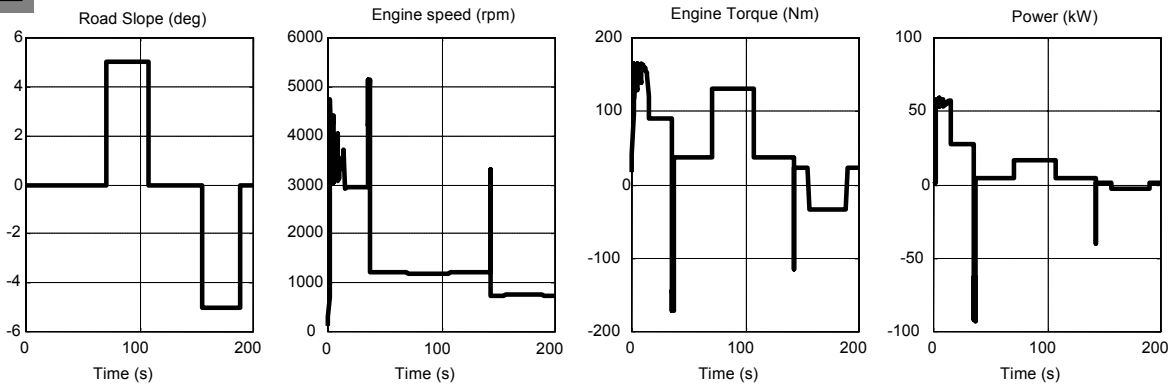


Hybrid Controller

- Tracking controller $\min_{u_t} J(u_t, x(t)) \triangleq |v(t+1|t) - v_d(t)| + \rho|\omega|$



$\rho = 0.001$



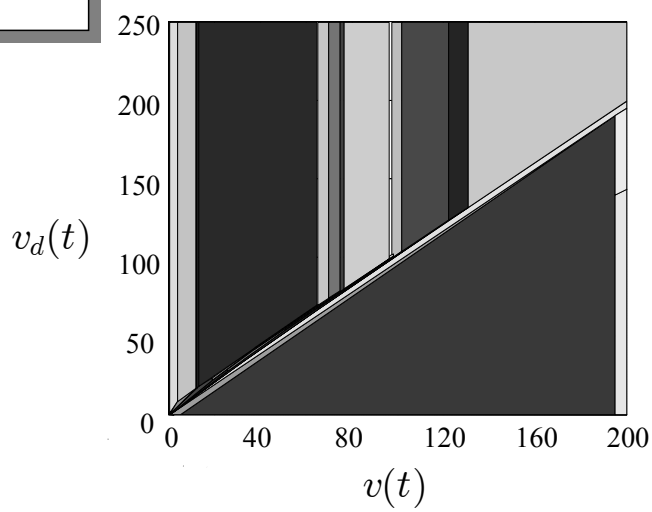
Hybrid Controller

- Smoother tracking controller

$$\min_{u_t} J(u_t, x(t)) \triangleq |v(t+1|t) - v_d(t)| + \rho|\omega|$$
$$\text{subj. to} \begin{cases} |v(t+1|t) - v(t)| < T_s a_{\max} \\ \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

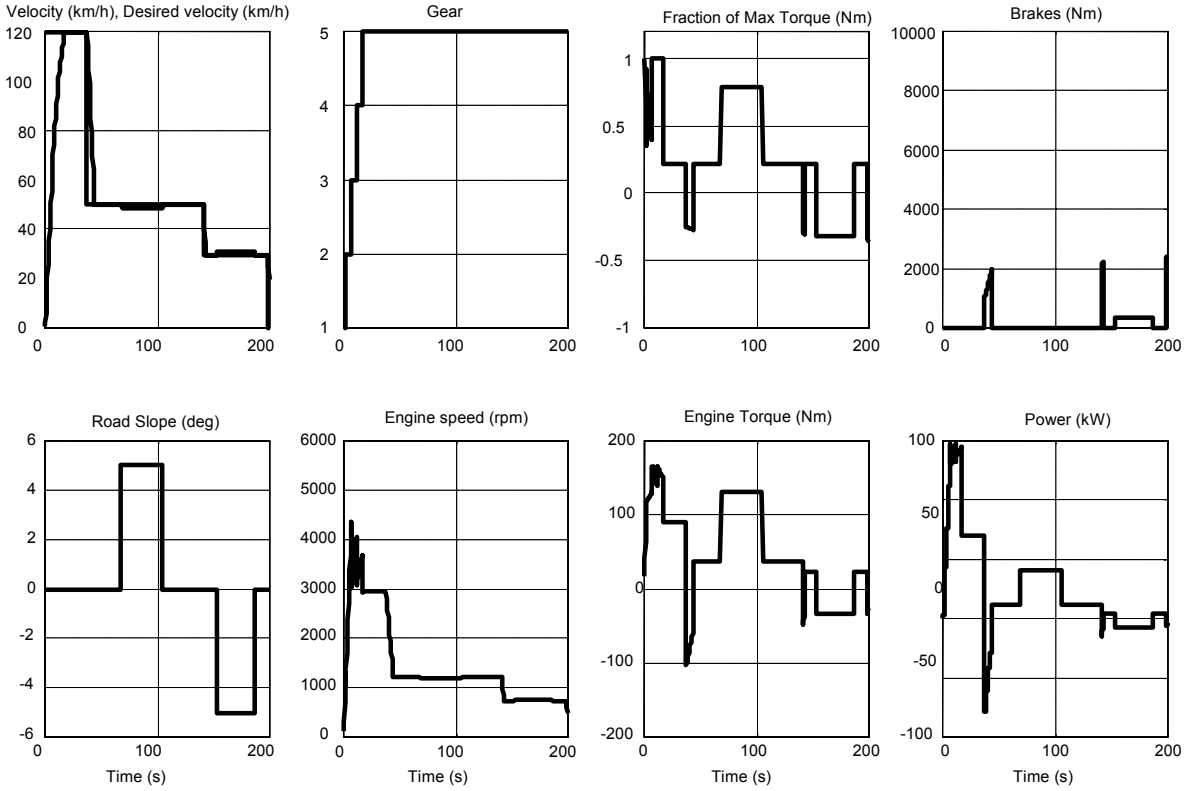
MILP optimization problem

Linear constraints	100
Continuous variables	19
Binary variables	10
Parameters	2
Time to solve mp-MILP (Sun Ultra 10)	28 m
Number of regions	**54**



Hybrid Controller

- Smoother tracking controller



Reachability Analysis (Verification)

Verification

- **GIVEN**: an embedded system (continuous dynamical system + logic controller)

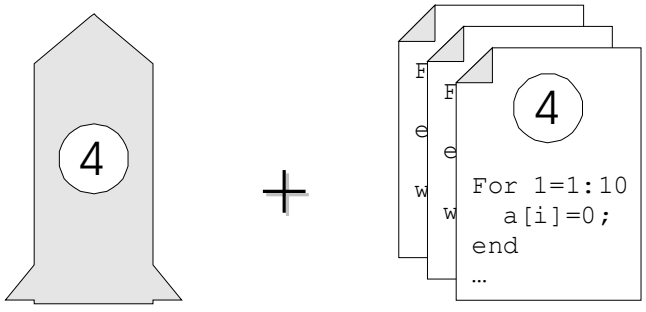
- **CERTIFY** that such combination behaves as desired
  - for ALL initial conditions within a given set
  - for ALL disturbances within a given class

- or **PROVIDE** a counterexample.

Simulation: provides a partial answer (not all possibilities can be tested!)

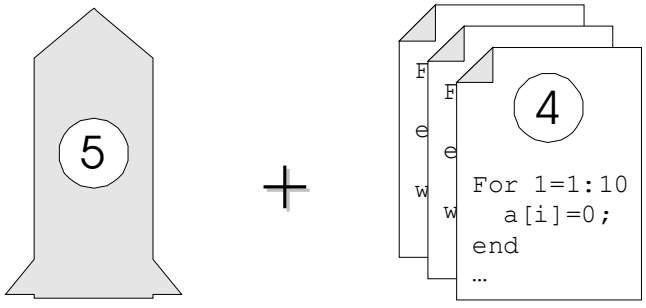
Reachability Analysis: provides the answer.

Ariane 5 - 1996



•Success

Physical beaviour Computer code

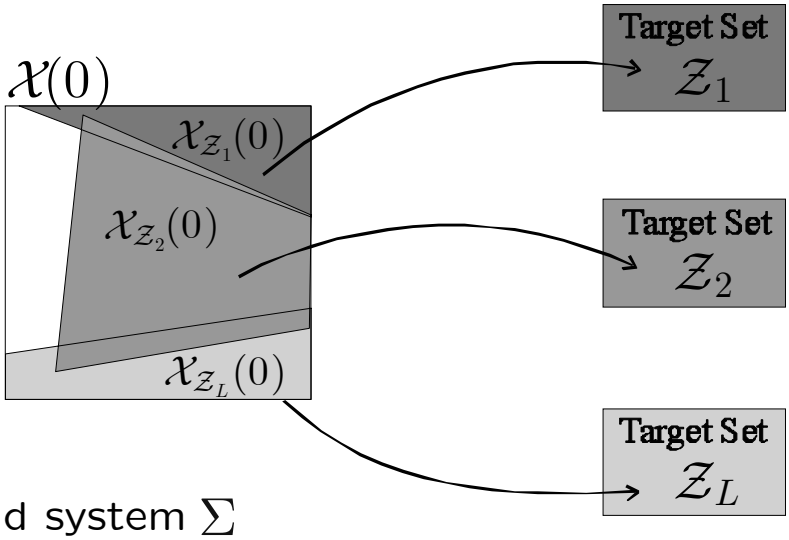


•Failure

Need for verification

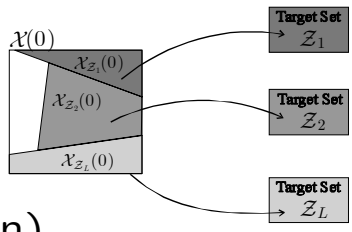
Reachability Analysis/Verification

(Bemporad, Torrisi, Morari, 2000)

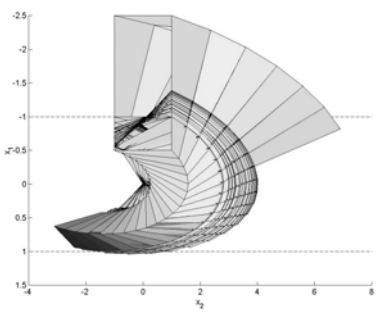


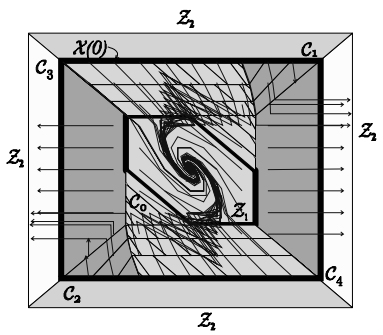
- Given:
  - A hybrid system Σ
  - A set of initial conditions $\mathcal{X}(0)$
  - Target sets $\mathcal{Z}_1,\ \mathcal{Z}_2,\ \ldots,\ \mathcal{Z}_L$ (disjoint)
  - Time horizon $t \leq T_{\max}$
- Problem:
  - Is $\mathcal{Z}_i$ reachable from $\mathcal{X}(0)$ in t steps ?
  - If yes, from which subset $\mathcal{X}_{\mathcal{Z}_i}(0)$ of $\mathcal{X}(0)$?
  - Disturbance/input sequences driving $\mathcal{X}_{\mathcal{Z}_i}(0)$ to $\mathcal{Z}_i$.

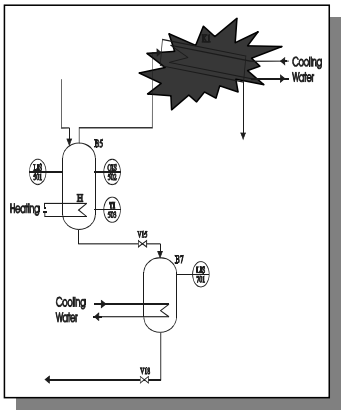
Applications



- Safety ($\mathcal{Z}_1,\ \mathcal{Z}_2,\ \ldots,\ \mathcal{Z}_L$ = unsafe sets)
- Stability ($\mathcal{Z}_1$ = invariant set around the origin)
- Optimal control ($u(0), \ldots, u(T_{\max})$ = optimal strategy, $\mathcal{Z}_1$ = reference set) (Bemporad, Giovanardi, Torrisi, CDC 2000)
- (practical) Liveness ($\mathcal{Z}_1$ = set to be reached within a finite time)







Verification Algorithm via MILP

- Simple solution: Solve $\forall T>0$ the mixed-integer linear feasibility test

$$\begin{cases} x(0) \in \mathcal{X}(0) \\ x(T) \in \mathcal{Z}_i \\ u(t) \in \mathcal{U}, 0 \le t \le T \\ x(t+1) = Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \\ E_2 \delta(t) + E_3 z(t) \le E_1 u(t) + E_4 x(t) + E_5 \end{cases}$$

with respect to $x(0), \{u(t), \delta(t), z(t)\}_{t=0}^{T}$

- Only practical for small problems ! because number of free integer variables $\delta(0), \delta(1), \ldots, \delta(T)$ grows with T
- **Efficient Solution**: Exploit the special structure of the problem.

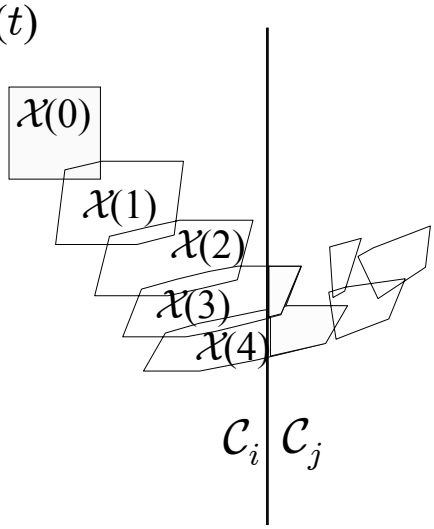
(Bemporad, Torrisi, Morari, 2000)
(Torrisi, Bemporad, Giovanardi, 2003)

Reach-Set Evolution

$T_{\max} < \infty$, discrete-time $\Rightarrow$ Decidable but $\mathcal{NP}$-hard

Reachability analysis algorithm:

- Compute the polyhedral reach set $\mathcal{X}(t)$
- Detect switching
- Describe new intersections $\mathcal{X}(t) \cap \mathcal{C}_j$
- Stopping criteria for a single exploration
- Organize the search



(Bemporad, Torrisi, Morari, 2000)

Reach Set Computation

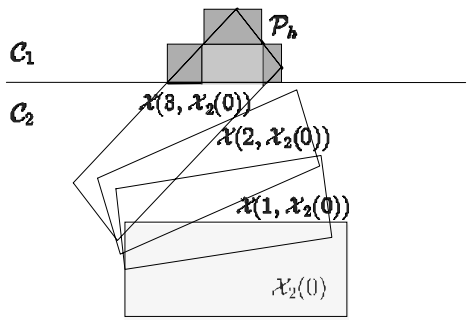
Reach set implicitly defined by linear inequalities

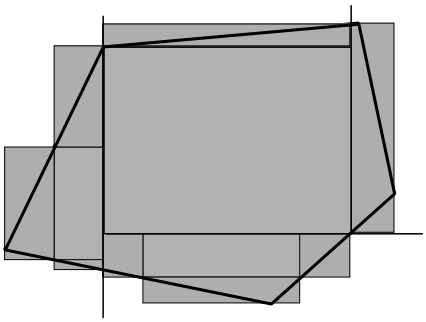
$$\begin{cases} x \in \mathcal{X}(0) \\ K_i^x \left(A_i^k x + \sum_{k=0}^{t-1} A_i^j [B_i u(t-1-k) + f_i] \right) + \\ \quad + K_i^u u(t) \leq H, \;\; k = 1, \ldots, t \\ u_{\min} \leq u(k) \leq u_{\max}, \;\; k = 0, \ldots, t-1 \end{cases}$$

where $\mathcal{X}_i = \{(x,u) : \;\; K_i^x x + K_i^u u \leq H\}$ is the current region

- Simple to compute
- Number of constraints grows linearly with time
- Explicit form also possible via projection methods (e.g. CDD, Fukuda 1997)

Approximation of Intersections

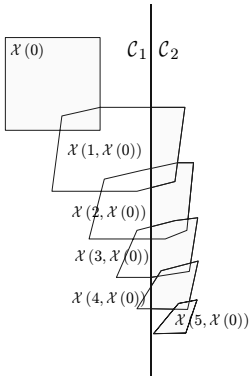




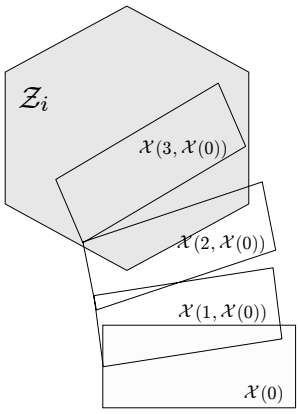
- ✓ Simple to compute via Linear Programming (LP)
- ✓ Can approximate with arbitrary precision
- ✓ Trade off between conservativeness and complexity
- ✓ Both inner and outer approximations in one shot
- ✓ Approximate computation of projections

Stopping Criteria

- Reach set has left the current region

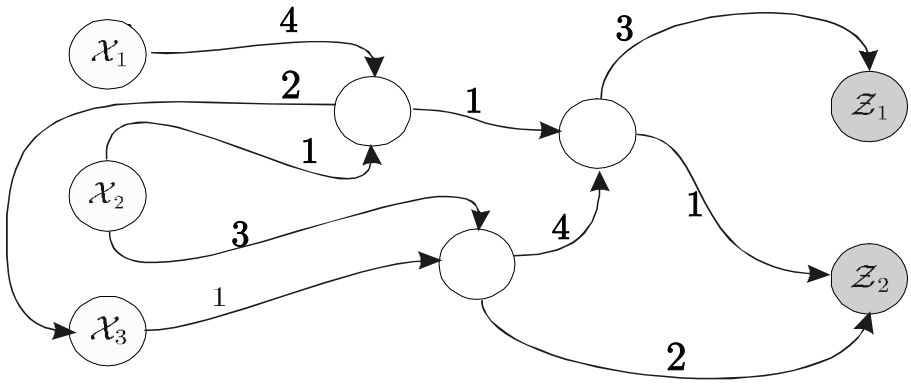


- Reach set is all inside a target $\mathcal{Z}_i$



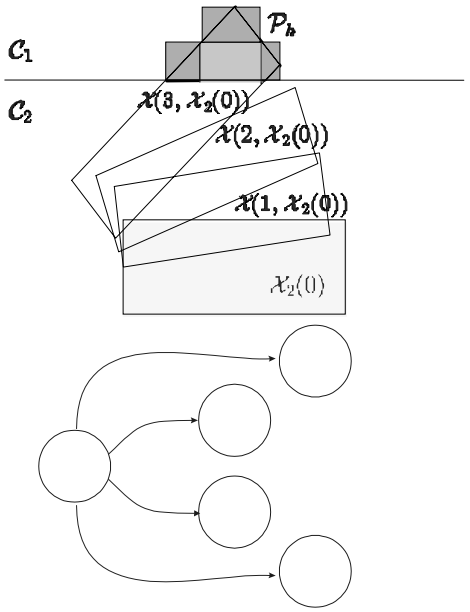
- $t > T_{\max}$ (to guarantee termination)

Graph of Evolution

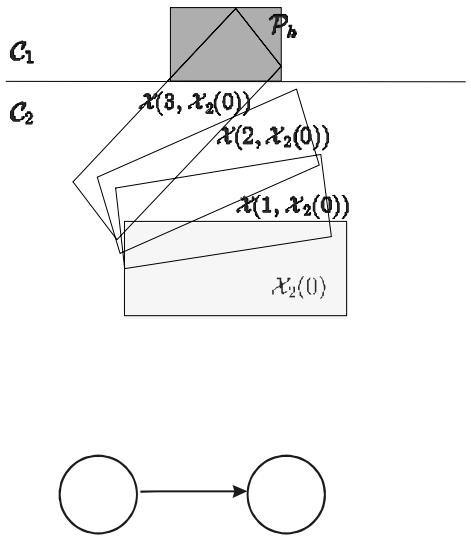


- The graph is initialized with all the initial subsets $\mathcal{X}_i=\mathcal{X}(0)\cap\mathcal{C}_i$ and all the target sets
- A node is added for each initial region of a new exploration
- If a set can reach another set, an oriented arc is drawn
- The time needed to reach the set is a weight associated with the arc

Conservativeness vs Graph Complexity



- Less conservative
- Graph more complicate

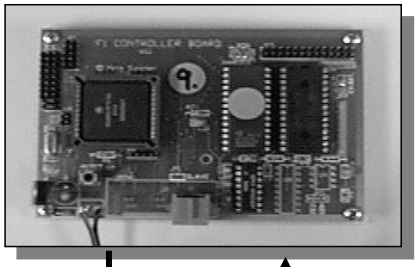


- Graph less complicate
- More conservative

LP determines if a switching sequence is feasible

Verification Example: Cruise Control System

Cruise Control System



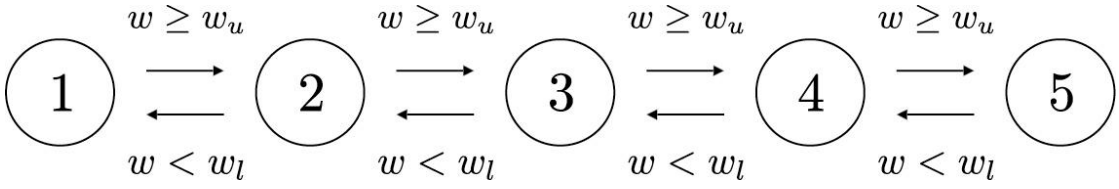
GOAL:

Verify if a given switching controller satisfies certain specifications

(Torrisi, Bemporad, 2001)

Cruise Control System

Gear selector:



Speed controller:

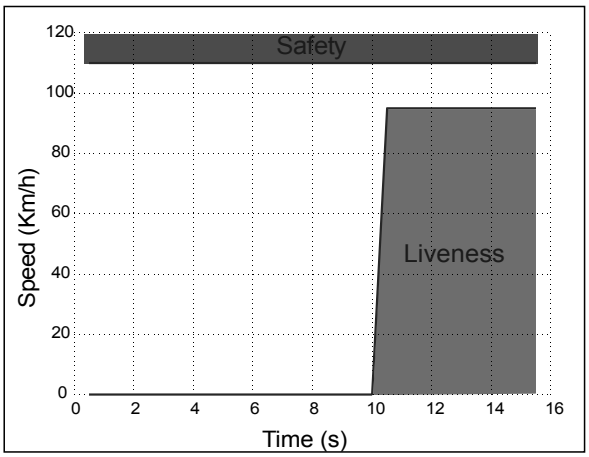
$$e(t+1) = e(t) + T_s(v_r(t) - v(t)) \quad (+ \text{ saturation})$$

$$u_e(t) = \begin{cases} k_e(v_e(t) - v(t)) + i_e e(t) & \text{if } v(t) < v_r + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_b(t) = \begin{cases} k_b(v_e(t) - v(t)) & \text{if } v(t) \geq v_r + 1 \\ 0 & \text{otherwise} \end{cases}$$

Verification Problem

Question: Will the cruise control reach the desired speed reference within 10 s without exceeding the speed limit?



Safety $\mathcal{Z}_1 = \{v : v > v_r + r_{\text{toll}}\}$

Liveness $\mathcal{Z}_2 = \{v, t : v < v_r - 2r_{\text{toll}}, t > 10/T_s\}$

$r_{\text{toll}} = 5$ km/h

Hysdel Model (HYbrid Systems DEscription Language)

```
SYSTEM car {
INTERFACE {
    STATE { REAL speed, err, vr; BOOL gear1, gear2, gear3, gear4, gear5; }
    PARAMETER {…}}
IMPLEMENTATION {
    AUX {
        REAL Fe1, Fe2, Fe3, Fe4, Fe5, w1, w2, w3, w4, w5, DCe1, DCe2, DCe3, DCe4,zut, zub, ierr, torque, F_brake;
        BOOL dPWL1, dPWL2, dPWL3, dPWL4, sd, su, verr, sat_torque, sat_F_brake, no_sat;}
    LOGIC { no_sat = ~(sat_torque | sat_F_brake) & verr; }
    AD {dPWL1 = wPWL1 - (w1 + w2 + w3 + w4 + w5) <= 0; dPWL2 = wPWL2 - (w1 + w2 + w3 + w4 + w5) <= 0;
        dPWL3 = wPWL3 - (w1 + w2 + w3 + w4 + w5) <= 0; dPWL4 = wPWL4 - (w1 + w2 + w3 + w4 + w5) <= 0;

        sd = (w1 + w2 + w3 + w4 + w5) - wl <= 0; su = wu - (w1 + w2 + w3 + w4 + w5) <= 0; verr = speed - vr - 2 <= 0;
        sat_torque = - zut + (DCe1 + DCe2 + DCe3 + DCe4) + 1 <= 0; sat_F_brake = - zub + max_brake_force <= 0;          }
    DA {Fe1 = {IF gear1 THEN torque / speed_factor * Rgear1}; Fe2 = {IF gear2 THEN torque / speed_factor * Rgear2};
        Fe3 = {IF gear3 THEN torque / speed_factor * Rgear3}; Fe4 = {IF gear4 THEN torque / speed_factor * Rgear4};
        Fe5 = {IF gear5 THEN torque / speed_factor * Rgear5};

        w1 = {IF gear1 THEN speed / speed_factor * Rgear1}; w2 = {IF gear2 THEN speed / speed_factor * Rgear2};
        w3 = {IF gear3 THEN speed / speed_factor * Rgear3}; w4 = {IF gear4 THEN speed / speed_factor * Rgear4};
        w5 = {IF gear5 THEN speed / speed_factor * Rgear5};

        DCe1 = {IF dPWL1 THEN (aPWL2) + (bPWL2) * (w1 + w2 + w3 + w4 + w5) ELSE (aPWL1) + (bPWL1) * (w1 + w2 + w3 + w4 + w5)};
        DCe2 = {IF dPWL2 THEN (aPWL3 - aPWL2) + (bPWL3 - bPWL2) * (w1 + w2 + w3 + w4 + w5)};
        DCe3 = {IF dPWL3 THEN (aPWL4 - aPWL3) + (bPWL4 - bPWL3) * (w1 + w2 + w3 + w4 + w5)};
        DCe4 = {IF dPWL4 THEN (aPWL5 - aPWL4) + (bPWL5 - bPWL4) * (w1 + w2 + w3 + w4 + w5)};

        zut = {IF verr THEN kt * (vr - speed) + it * err}; zub = {IF ~verr THEN - kb * (vr - speed) - ib * err};
        torque ={IF sat_torque THEN (DCe1 + DCe2 + DCe3 + DCe4) + 1 ELSE zut}; F_brake = {IF sat_F_brake THEN max_brake_force ELSE zub};
        ierr = {IF no_sat THEN err + Ts * (vr - speed)};}
    CONTINUOUS {
        speed = speed + Ts / mass * (Fe1 + Fe2 + Fe3 + Fe4 + Fe5 - F_brake - beta_friction * speed); err = ierr; vr = vr;}
    AUTOMATA {
        gear1 =                (gear2 & sd) | (gear1 &       ~su); gear2 = (gear1 & su) | (gear3 & sd) | (gear2 & ~sd & ~su);
        gear3 = (gear2 & su) | (gear4 & sd) | (gear3 & ~sd & ~su); gear4 = (gear3 & su) | (gear5 & sd) | (gear4 & ~sd & ~su);
        gear5 = (gear4 & su)                | (gear5 & ~sd );}
    MUST {
        -w1 <= -wemin; w1 <= wemax; -w2 <= -wemin; w2 <= wemax; -w3 <= -wemin; w3 <= wemax; -w4 <= -wemin; w4 <= wemax; -w5 <= -wemin;
        w5 <= wemax; -F_brake <= 0; F_brake <= max_brake_force; torque - (DCe1 + DCe2 + DCe3 + DCe4) - 1 <= 0;
        -((REAL gear1) + (REAL gear2) + (REAL gear3) + (REAL gear4) + (REAL gear5)) <= -0.9999;
        (REAL gear1) + (REAL gear2) + (REAL gear3) + (REAL gear4) + (REAL gear5)  <= 1.0001;
        dPWL4 -> dPWL3; dPWL4 -> dPWL2; dPWL4 -> dPWL1; dPWL3 -> dPWL2; dPWL3 -> dPWL1; dPWL2 -> dPWL1;}}
```

Hybrid Model

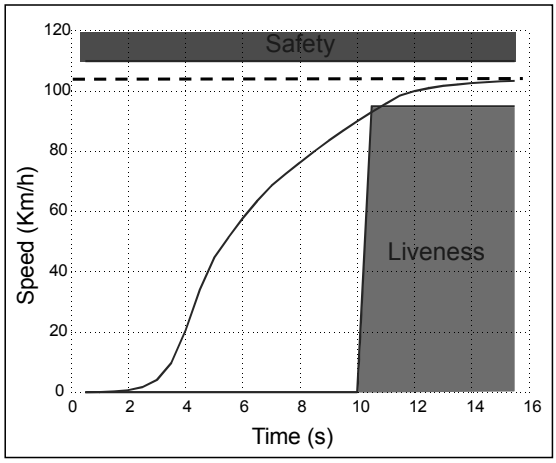
- MLD model

$$\begin{aligned} x'(k) &= Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) + B_5 \\ y(k) &= Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) + D_5 \\ E_2 \delta(k) + E_3 z(k) &\leq E_1 u(k) + E_4 x(k) + E_5 \end{aligned}$$

- 3 continuous states: v, v_r, e (speed, reference and tracking error)
- 5 binary states: g_1, g_2, g_3, g_4, g_5 (gears)
- 19 auxiliary continuous vars: (5 traction force, 5 engine speed, 5 reset/saturation, 4 PWL max engine torque)
- 15 auxiliary binary vars: (4 PWL max torque breakpoints, 4 saturations 5 logic updates, 2 gear switching conditions)
- 173 mixed-integer inequalities

Verification Results

- For all $v_r \in [30, 70]\ km/h$ the controller satisfies both liveness & safety properties
- CPU time: ~2.5h (Matlab 5.3, PC650MHz)



- For $v_r \in [30, 120]\ km/h$ the verification algorithm finds the first counterexample after ~7m

Conclusions

- **Hybrid systems** as a framework for new applications, where both logic and continuous dynamics are relevant
- **Mixed Logical Dynamical (MLD)** systems as discrete-time, *computation-oriented* models for hybrid systems

```
/* HEAT EXCHANGE model  - (C) 2001 by A. Bemporad, Zurich, March 13, 2001 */

SYSTEM furnaces {

INTERFACE {
        STATE {
                REAL t1,t2,u0; }
        INPUT {
                BOOL heat1,heat2; }
        PARAMETER {
                REAL Ts=0.08; /* sampling time, seconds */
                REAL Bd1=0.07688365361336; /* discretization of B matrix, heat 1 active */
                REAL Bd2=0.07392810551689; /* discretization of B matrix, heat 2 active */
                REAL A11=.9231; /* discretization of A matrix */
                REAL A22=.8521; /* discretization of A matrix */

                REAL u0max=10; /* upperbound on u0 */
                REAL e = 1e-6; /* precision for strict inequalities */  }
}

IMPLEMENTATION {
                AUX {REAL z1,z2; }

                DA {    z1 = (IF heat1 THEN Bd1*u0 [Bd1*u0max,0,e]);
                        z2 = (IF heat2 THEN Bd2*u0 [Bd2*u0max,0,e]); }

                CONTINUOUS { t1 = A11*t1+z1;
                             t2 = A22*t2+z2;
                             u0 = u0; }

                MUST   {  -heat1 | -heat2;  /* heat1 and heat2 cannot be both active */ }
}
}
```

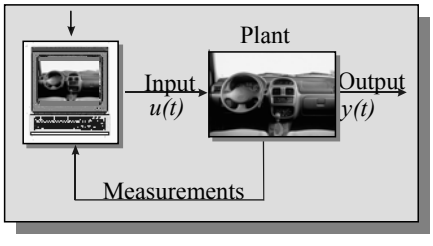
HYSDEL

$$
\begin{aligned}
x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \\
y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \\
E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5
\end{aligned}
$$

Conclusions

- **Hybrid systems** as a framework for new applications, where both logic and continuous dynamics are relevant
- **Mixed Logical Dynamical (MLD)** systems as discrete-time, *computation-oriented* models for hybrid systems

- **Supervisory MPC controllers** and **State Estimation/Fault Detection** schemes can be synthesized via on-line mixed-integer programming (MILP/MIQP)
- **Piecewise Linear Optimal Controllers** can be synthesized via off-line multiparametric programming for fast-sampling applications



$$
u(x) = \begin{cases} F_1x + G_1 & \text{if} \quad H_1x \leq K_1 \\ \vdots & \vdots \\ F_Nx + G_N & \text{if} \quad H_Nx \leq K_N \end{cases}
$$

Conclusions

- **Hybrid systems** as a framework for new applications, where both logic and continuous dynamics are relevant
- **Mixed Logical Dynamical (MLD)** systems as discrete-time, *computation-oriented* models for hybrid systems
- **Supervisory MPC controllers** and **State Estimation/Fault Detection** schemes can be synthesized via on-line mixed-integer programming (MILP/MIQP)
- **Piecewise Linear Optimal Controllers** can be synthesized via off-line multiparametric programming for fast-sampling applications
- **Safety Analysis** properties can be formally verified

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