Stochastic Receding Horizon Control for Dynamic Option Hedging

Alberto Bemporad

http://www.dii.unisi.it/~bemporad

University of Siena

Department
of Information
Engineering

joint work with **L. Bellucci** and **T. Gabbriellini**, MPS Capital Services

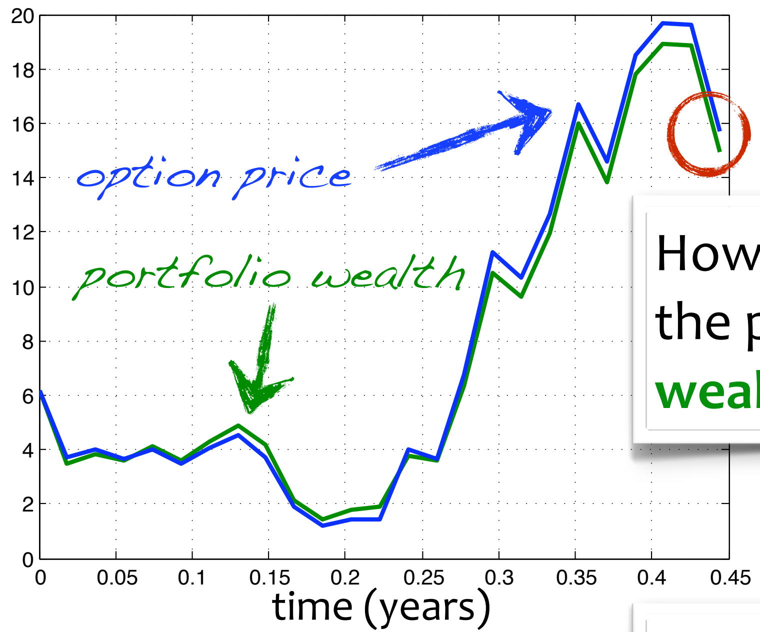
MPS CAPITAL SERVICES

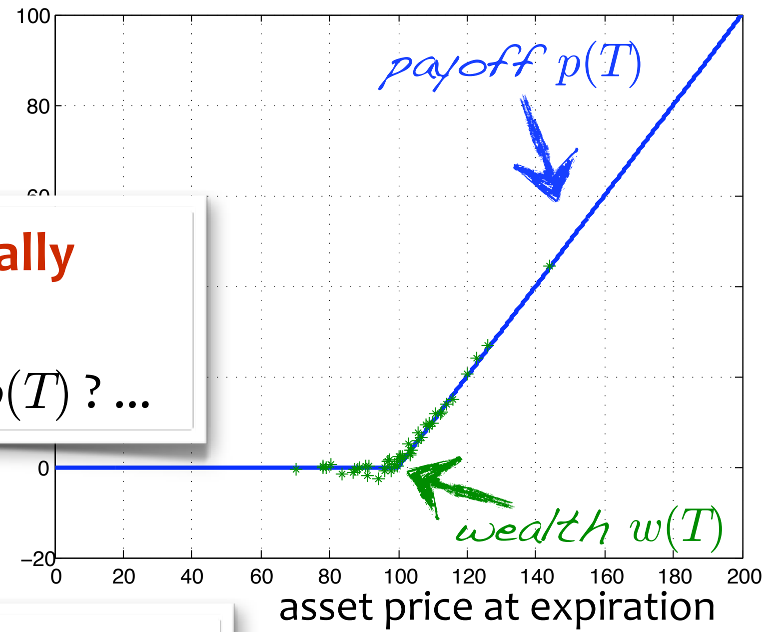
Talk outline

- Dynamic hedging problem for financial options
- Dynamic hedging as a linear stochastic control problem
- Stochastic receding horizon control (SRHC)
- Pricing engine + SRHC = dynamic option hedging tool
- Simulation results

Dynamic hedging problem for financial options

- The financial institution sells a synthetic option to a customer and gets $w(0)$ (€)
- Such money is used to create a portfolio $w(t)$ of underlying assets (e.g. stocks) whose prices at time t are $x_1(t)$, $x_2(t)$, ..., $x_n(t)$
- At the expiration date T, the option is worth the payoff $p(T)$ = wealth (€) to be returned to the customer





How to **adjust dynamically** the portfolio so that **wealth** $w(T) =$ **payoff** $p(T)$? ...

.. for any price realization $x_i(t)$?

Dynamics of traded assets

- Trading instants: $\{0, \Delta t, 2\Delta t, \ldots, t\Delta t, \ldots, (T-1)\Delta t\}$

- Generic asset price dynamics:

price of asset #i at time t

$$\begin{aligned} x_i(t+1) &= f_i(x_i(t), y_i(t), z_i^x(t)) \\ y_i(t+1) &= g_i(y_i(t), z_i^y(t)) \end{aligned}$$

noise

internal variables
(e.g. variance)

- Example: $x_i(t)$=stock price, **log-normal** model (BS, Black-Scholes)

$$dx_i = (\mu dt + \sigma dz_i^x)x_i$$

geometric Brownian motion

$\Longrightarrow$ Ito's lemma

$$x_i(t+1) = e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} z_i^x(t)} x_i(t)$$

$\cancel{y_i(t) = 0, \ z_i^y(t) = 0}$

- Example: $x_i(t)$=stock price, **GARCH(1,1)** model (Heston, Nandi, 2000)

- Example: $x_i(t)$=option price of European call, based on BS model

Note: ***numerical integration*** can be also used to express $x_i(t+1)$, $y_i(t+1)$ as a function of $x_i(t)$, $y_i(t)$.

Portfolio dynamics

- Portfolio wealth at time t:

$$w(t) = u_0(t) + \sum_{i=1}^{n} x_i(t) u_i(t)$$

money in bank account
(risk-free asset)

price of asset #i

number of assets #i

- Assets traded at **discrete-time** intervals under the *self-balancing constraint*:

$$w(t+1) = (1+r)u_0(t) + \sum_{i=1}^{n} x_i(t+1) u_i(t)$$

$$\boxed{w(t+1) \;=\; (1+r)w(t) + \sum_{i=0}^{n} b_i(t) u_i(t)}$$

$$b_i(t) \triangleq x_i(t+1) - (1+r)x_i(t)$$

- **Assumption: transaction costs** can be neglected (this can be relaxed…).

- **Assumption: option pricing engine** is available (more later in this talk …), only focus on dynamic hedging. In particular, $w(0)$ is assigned.

Payoff function

- Let $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ = vector of assets, and $y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$

- Option price $p(t)$: $\quad p(t) = f(x(t), y(t))$

$$p(t) = f(x(0), x(1), \ldots, x(t), y(t))$$ **path-dependent options**

- Payoff $p(T)$: $\quad \boxed{p(T) = f(x(0), x(1), \ldots, x(T))}$

- Examples (n=1):

$$p(T) = \max\{x(T) - K, 0\}$$ **European call**

$$p(T) = \max\left\{0, C + \min_{i \in \{1,\ldots,N_{\mathrm{fix}}\}} \frac{x(t_i) - x(t_{i-1})}{x(t_{i-1})}\right\}$$ **Napoleon cliquet** ($t_i =$ fixing dates)

Talk outline

✓ Dynamic hedging problem for financial options

- Dynamic hedging as a linear stochastic control problem

- Stochastic receding horizon control (SRHC)

- Pricing engine + SRHC = dynamic option hedging tool

- Simulation results

Option hedging = linear stochastic control

- Portfolio dynamics is linear, with multiplicative stochastic noise

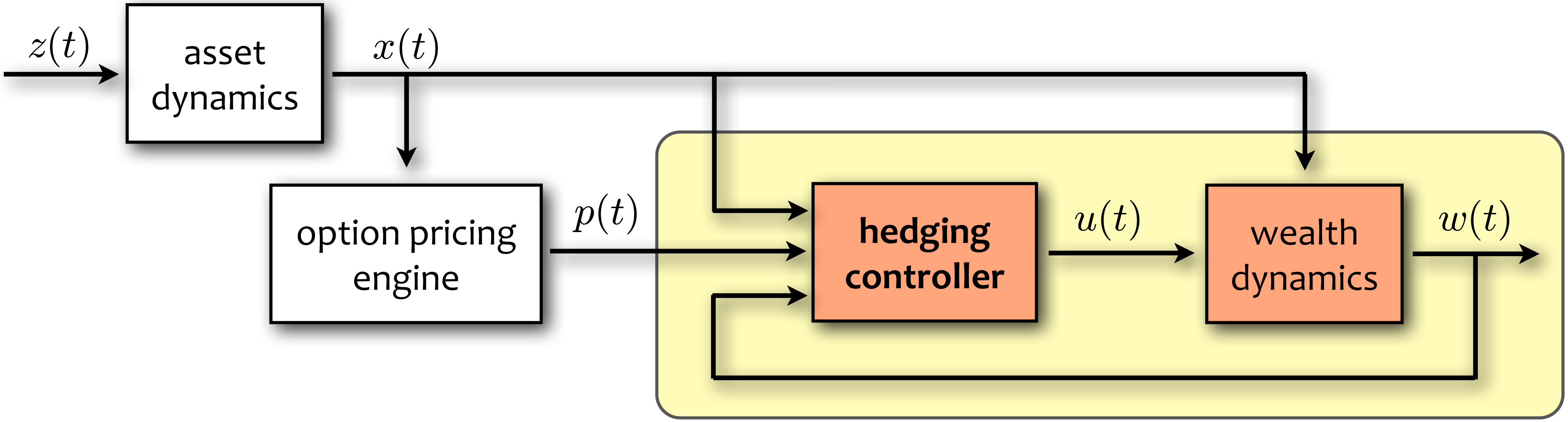
$$w(t+1) = (1+r)w(t) + \sum_{i=0}^{n} b_i(t)u_i(t)$$

input vector

state variable

B matrix affected by stochastic multiplicative noise

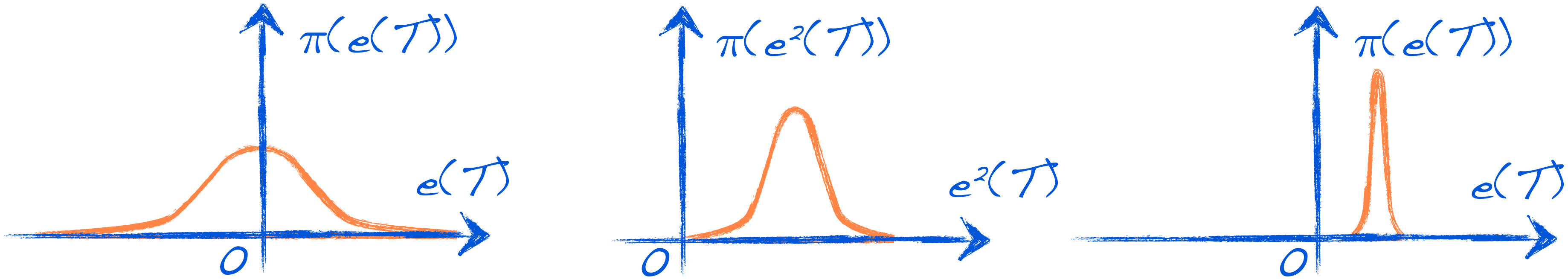
- Block diagram of stochastic control problem:



- **Control objective**: $w(T)$ should be as close as possible to $p(T)$, for any possible realization of the asset prices $x(t)$ (*"tracking w/ disturbance rejection"*)

Control objective

- Define **hedging error** $e(t) \triangleq w(t) - p(t)$ ← *we want e(T) small !*



- Minimize **expected** hedging error: $\min E\left[e(T)\right]^2$

Very risky, variance of e(T) may be large !

- Minimize **expected squared** error: $\min E\left[e(T)^2\right]$

If minimum is 0 then both mean and variance of hedging error are 0

In fact: $E\left[e(T)^2\right] = \underbrace{E\left[(e(T) - E[e(T)])^2\right]}_{\text{variance}} + \alpha \underbrace{E\left[e(T)\right]^2}_{\text{mean}}$ for $\alpha = 1$

- Minimize **variance** of hedging error: $\min E\left[(e(T) - E[e(T)])^2\right]$

How about E[e(T)] ?

Minimum variance control

- Let us minimize the **variance** of hedging error: $\min E\left[(e(T) - E[e(T)])^2\right]$

Theorem *Let $w(0) = p(0)$. Under* **non-arbitrage conditions**, *if a strategy exists such that Var$[w(T) - p(T)] = 0$ (=perfect hedging) then*

$$w(t) - p(t) = 0, \;\; \forall t = 0, 1, \ldots, T$$

and in particular $E[w(T) - p(T)] = 0$, that is the final hedging error is deterministically 0.

(cf. Black-Scholes theory)

Proof: By induction.

Note: $\mathrm{Var}[e(T)]{=}0$ means $e(T){=}w(T)$-$p(T)$ is deterministic

$e(T){>}0$ would imply $w(T){>}p(T)$ ⇒ always **gain** wealth, which is impossible if no arbitrage exists, as $w(0) = (1+r)^{-T} E[p(T)]$

Existing approaches to dynamic option hedging

- **Analytical approaches**: choose $u(t)$ to "reject" exactly the effect of stochastic noise $z(t)$ (Black and Scholes, 1973) (Merton, 1973)

  PROS: lots of insight !

  CONS: limited to simple stock models & payoff functions

- **Multi-stage stochastic programming**: choose $u(t)$ by solving a (large-scale) optimization problem. (Edirisinghe *et al.*, 1993) (Gondzio *et al.*, 2003) (Klaassen, 1998) (Zhao and Ziemba, 1998)

  PROS: pricing & hedging in one shot, check arbitrage conditions

  CONS: rough discretization of probability space & trading dates

- **Stochastic dynamic programming**: (Bertsimas *et al.*, 2001)

  PROS: rather versatile

  CONS: rough discretization of probability space & trading dates

- **Stochastic model predictive control**: solve recursively and on-line an optimization problem over a short prediction horizon (Primbs, 2009) (this talk)

  PROS: very versatile

  CONS: requires on-line optimization

Talk outline

✓Dynamic hedging problem for financial options

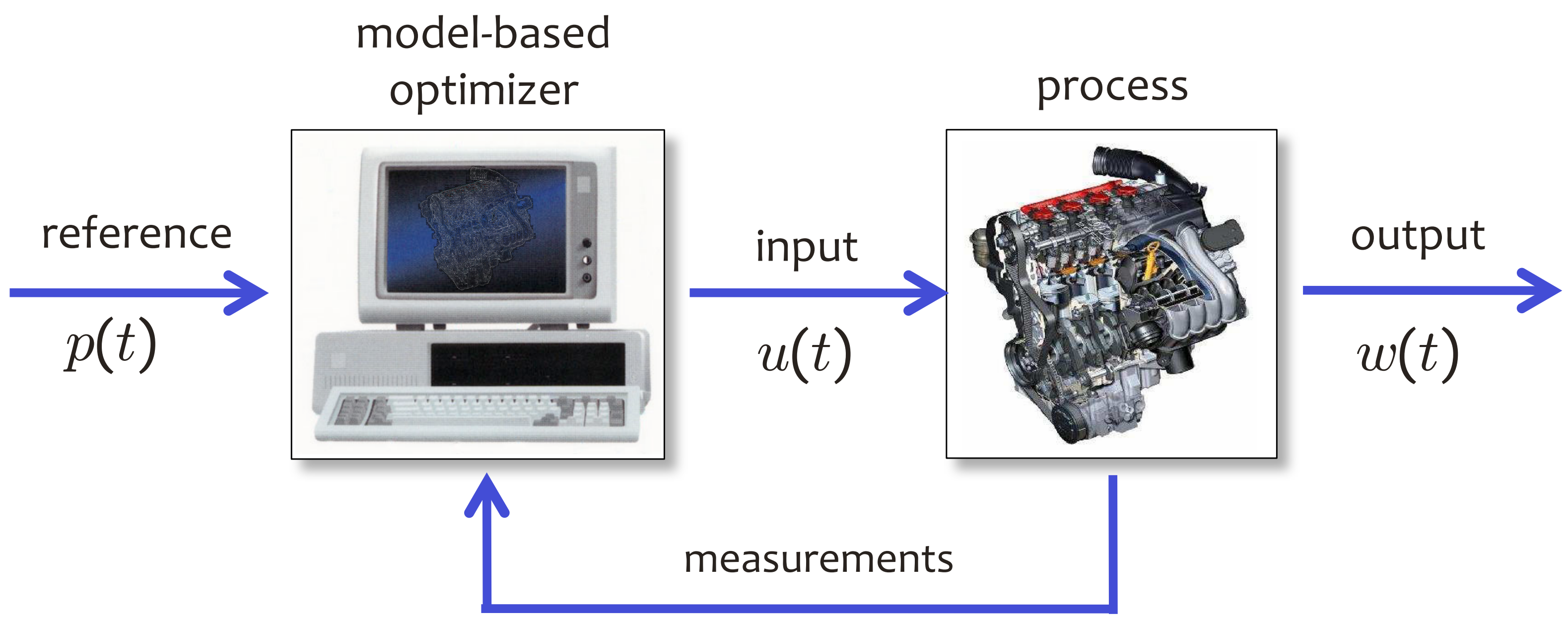
✓Dynamic hedging as a linear stochastic control problem

- Stochastic receding horizon control (SRHC)

- Pricing engine + SRHC = dynamic option hedging tool

- Simulation results

Model Predictive Control (MPC)

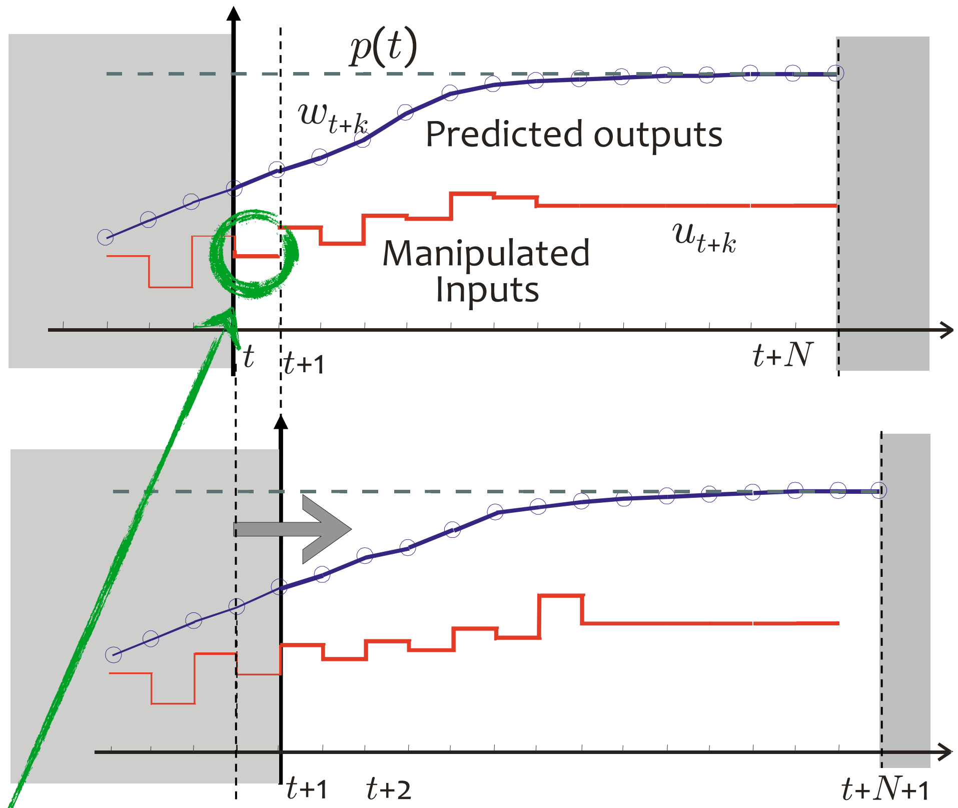


*A **model** of the process is used to **predict** the future evolution of the process in order to optimize the **control** signal*

Receding horizon philosophy

- At time t: solve an **optimal control** problem over a finite future horizon of N steps:

$$\begin{array}{ll} \min\limits_{z} & \sum_{k=0}^{N-1} \|w_{t+k} - p(t)\|^2 + \rho\|u_{t+k}\|^2 \\ \text{s.t.} & w_{t+k+1} = f(w_{t+k}, u_{t+k}) \\ & u_{\min} \leq u_{t+k} \leq u_{\max} \\ & w_{\min} \leq w_{t+k} \leq w_{\max} \\ & w_t = w(t), \;\; k = 0, \ldots, N-1 \end{array}$$



- Only apply the first optimal move $u^*(t)$

- At time $t+1$: Get new measurement $w(t+1)$, repeat the optimization. And so on …

- **Stochastic MPC**: minimize *expected value* of cost function

Stochastic linear MPC

- Assume stochastic model of the process

$$w(t+1) = A(z_1(t))w(t) + B(z_1(t))u(t) + Ez_2(t) \qquad w \in \mathbb{R}^n$$

$z_1(t),\ z_2(t) =$ stochastic noise

- Optimize stochastic performance under (chance) constraints

$$\min E\left[w_N' P w_N + \sum_{k=0}^{N-1} w_k' Q w_k + u_k' R u_k\right]$$

- Ensure mean-square convergence $E[w'(t)w(t)] = 0$

- A few SMPC approaches exist in the control literature

(Schwarme & Nikolaou, 1999)
(Wendt & Wozny, 2000)
(Batina, Stoorvogel, Weiland, 2002)
(van Hessem & Bosgra 2002)
(Munoz de la Pena, Bemporad, Alamo, 2005)
(Couchman, Cannon, Kouvaritakis, 2006)
(Primbs, 2007)
(Oldewurtel, Jones, Morari, 2008)
(Ono, Williams, 2008)
(Bernardini & Bemporad, 2009)

Talk outline

✓Dynamic hedging problem for financial options

✓Dynamic hedging as a linear stochastic control problem

✓Stochastic receding horizon control (SRHC)

- Pricing engine + SRHC = dynamic option hedging tool

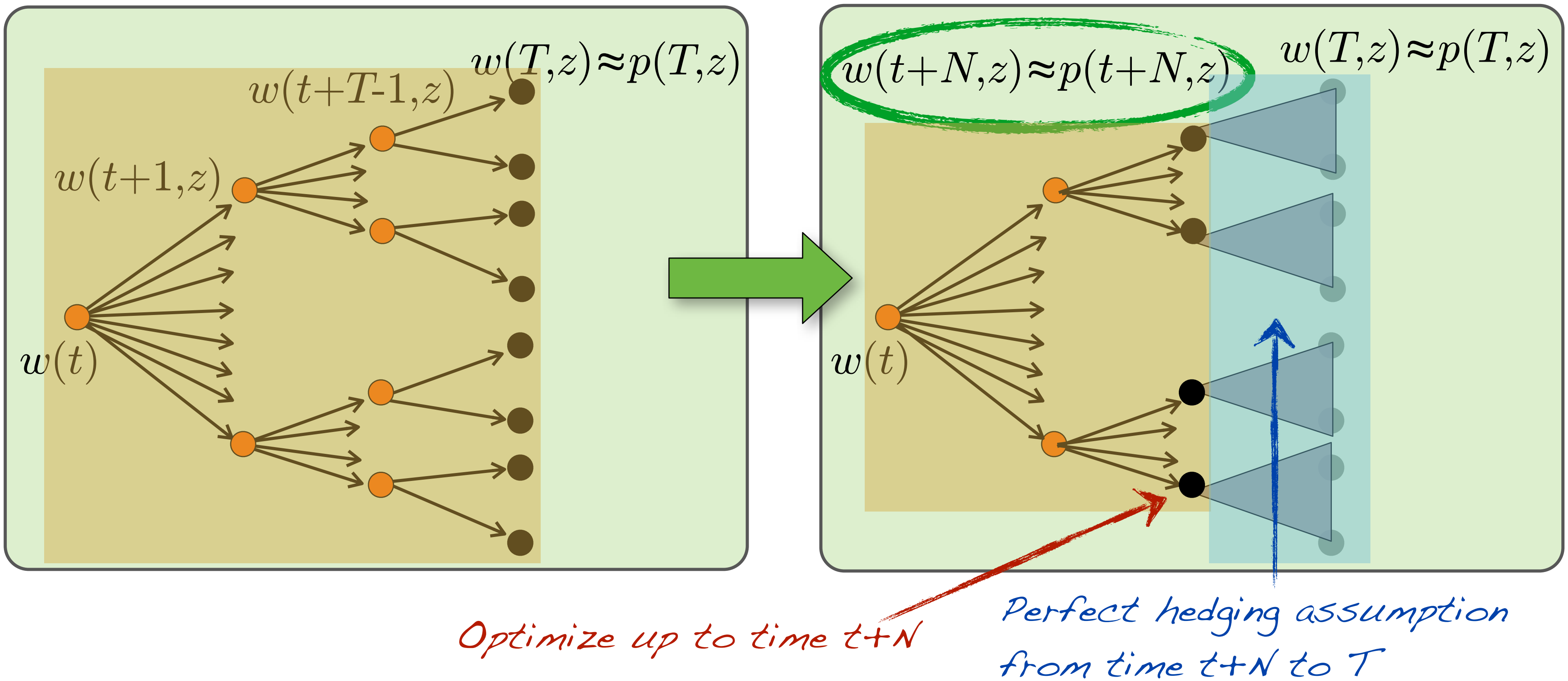
- Simulation results

SMPC for dynamic option hedging

- Stochastic finite-horizon optimal control problem:

$$\begin{aligned} \min_{\{u(k,z)\}} \quad & \mathrm{Var}_z\left[w(T,z) - p(T,z)\right] \\ \text{s.t.} \quad & w(k+1,z) = (1+r)w(k,z) + \sum_{i=0}^{n} b_i(k,z)u_i(k,z), \;\; k = t,\ldots,T-1 \\ & w(t,z) = w(t) \end{aligned}$$

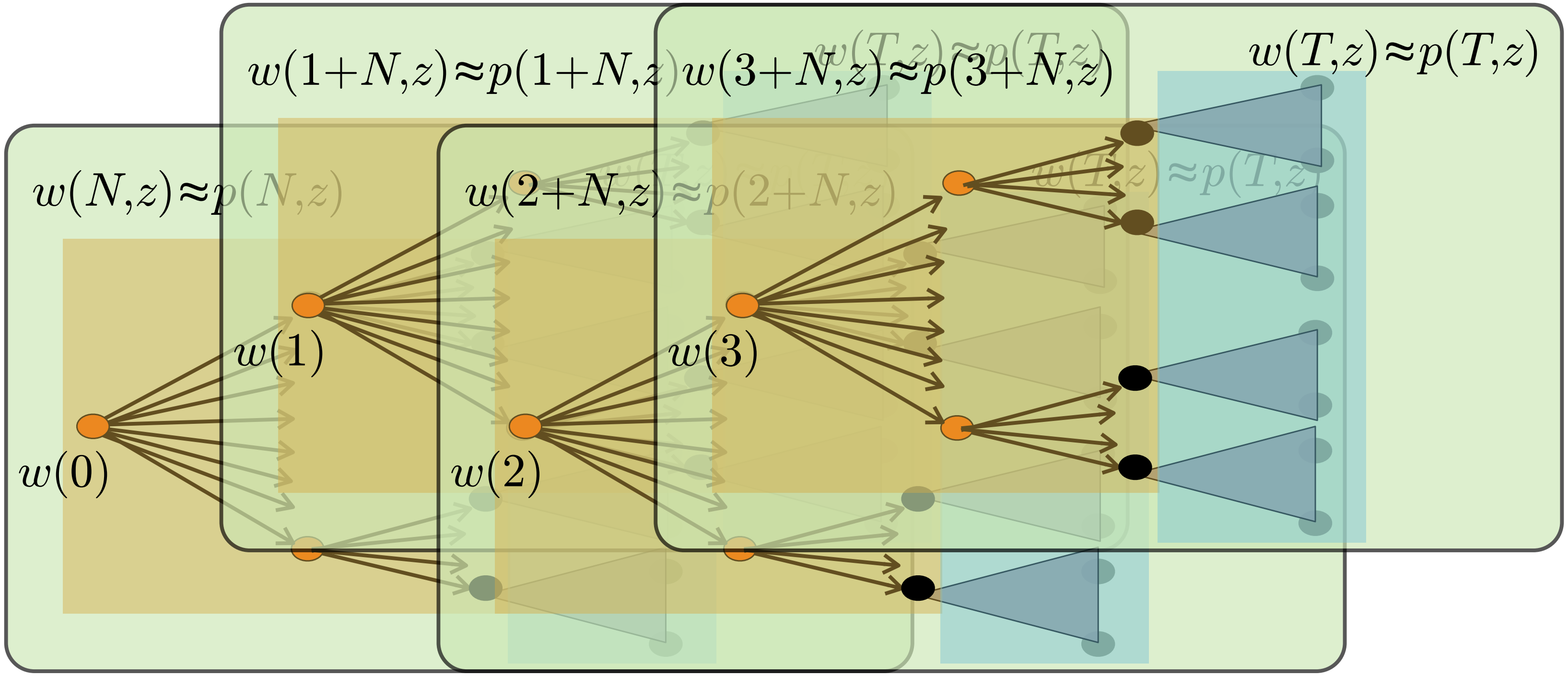
$$z = \{z(t+1),\ldots,z(T)\} \in \mathcal{Z}_t$$



SMPC for dynamic option hedging

- Stochastic finite-horizon optimal control problem (fixed horizon N):

$$\begin{aligned} \min_{\{u(k,z)\}} \quad & \mathrm{Var}_z\left[w(t+N,z)-p(t+N,z)\right] \\ \text{s.t.} \quad & w(k+1,z)=(1+r)w(k,z)+\sum_{i=0}^{n} b_i(k,z)u_i(k,z), \;\; k=t,\ldots,t+N \\ & w(t,z)=w(t) \end{aligned}$$



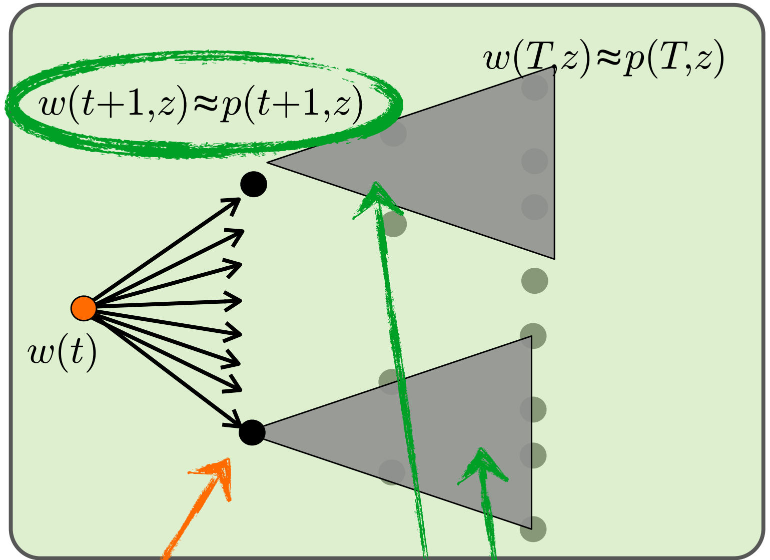
SMPC for dynamic option hedging

- Drawback: the longer the horizon N, the largest the number of scenarios !

- Special case: use N=1 !

$$\min_{u(t)} \quad \text{Var}_z\left[w(t+1,z) - p(t+1,z)\right]$$
$$\text{s.t.} \quad w(t+1,z) = (1+r)w(t) + \sum_{i=0}^{n} b_i(t,z)u_i(t)$$

minimum
variance
control !



✓ **Only one** vector $u(t)$ to optimize (no dependence of u on z !)

✓ No further branching, so we can generate **a lot** of scenarios for z

◉ Need to compute target wealth $p(t+1,z)$ for all z

Perfect hedging assumption
from time t+1 to T

Optimize up to time t+1

SMPC hedging algorithm

- Let t=current hedging date, $w(t)$=wealth of portfolio, $x(t)\in\mathbb{R}^n$ = asset prices

- Use **Monte Carlo simulation** to generate M scenarios of future asset prices

$$x^1(t+1),\ x^2(t+1),\ \ldots,\ x^M(t+1)$$
$$y^1(t+1),\ y^2(t+1),\ \ldots,\ y^M(t+1)$$

- Use a **pricing engine** to generate the corresponding future option prices

$$p^1(t+1),\ p^2(t+1),\ \ldots,\ p^M(t+1)$$

- **Optimize** *sample variance*, get new asset quantities $u(t)\in\mathbb{R}^n$, rebalance portfolio:

$$\min_{u(t)} \sum_{j=1}^{M}\left(w^j(t+1)-p^j(t+1)-\left(\frac{1}{M}\sum_{i=1}^{M} w^j(t+1)-p^j(t+1)\right)\right)^2$$

This is a very simple **least squares** problem with n variables !

(n = number of traded assets)

With transaction costs, the problem can be rewritten as *mixed-integer quadratic program* using a *mixed-logical dynamical* (MLD) reformulation of the portfolio dynamics (Bemporad, Morari, 1999)

Talk outline

✓Dynamic hedging problem for financial options

✓Dynamic hedging as a linear stochastic control problem

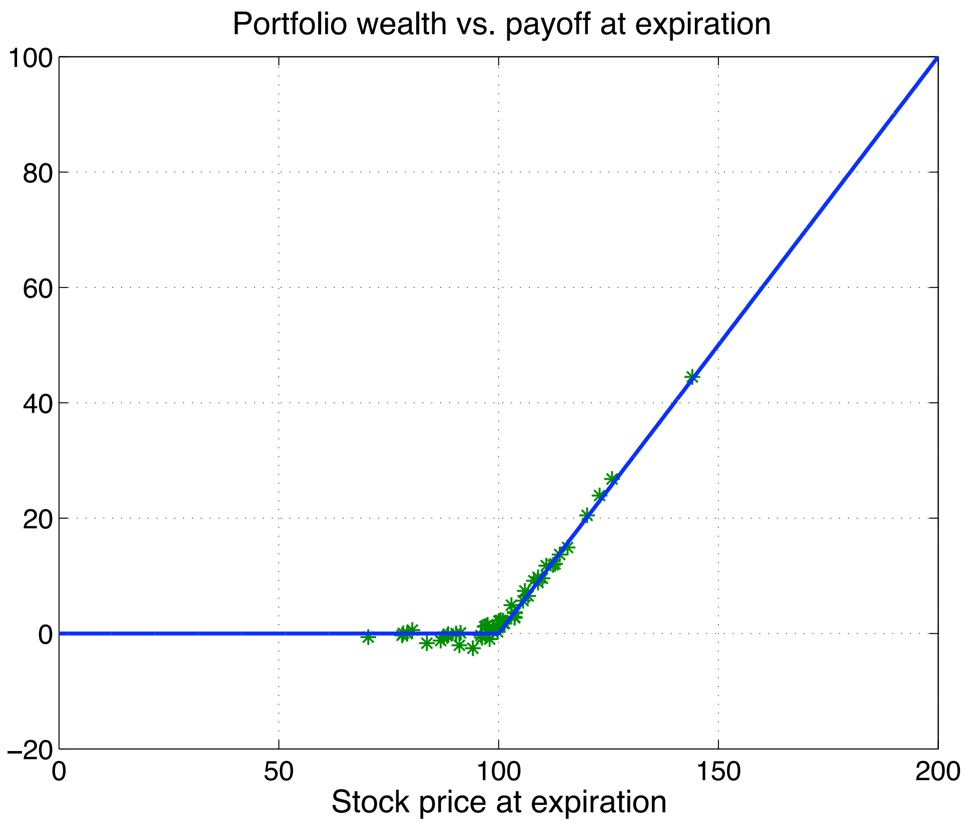
✓Stochastic receding horizon control (SRHC)

✓Pricing engine + SRHC = dynamic option hedging tool

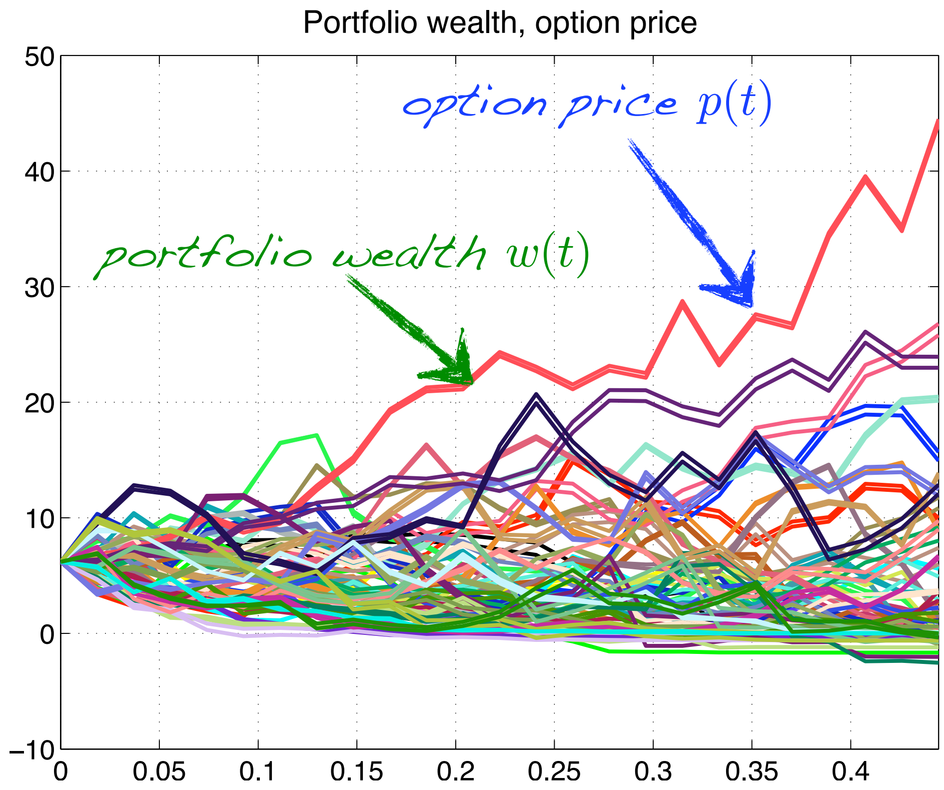
- Simulation results

Is SMPC good for option hedging ?

Example: BS model, European call

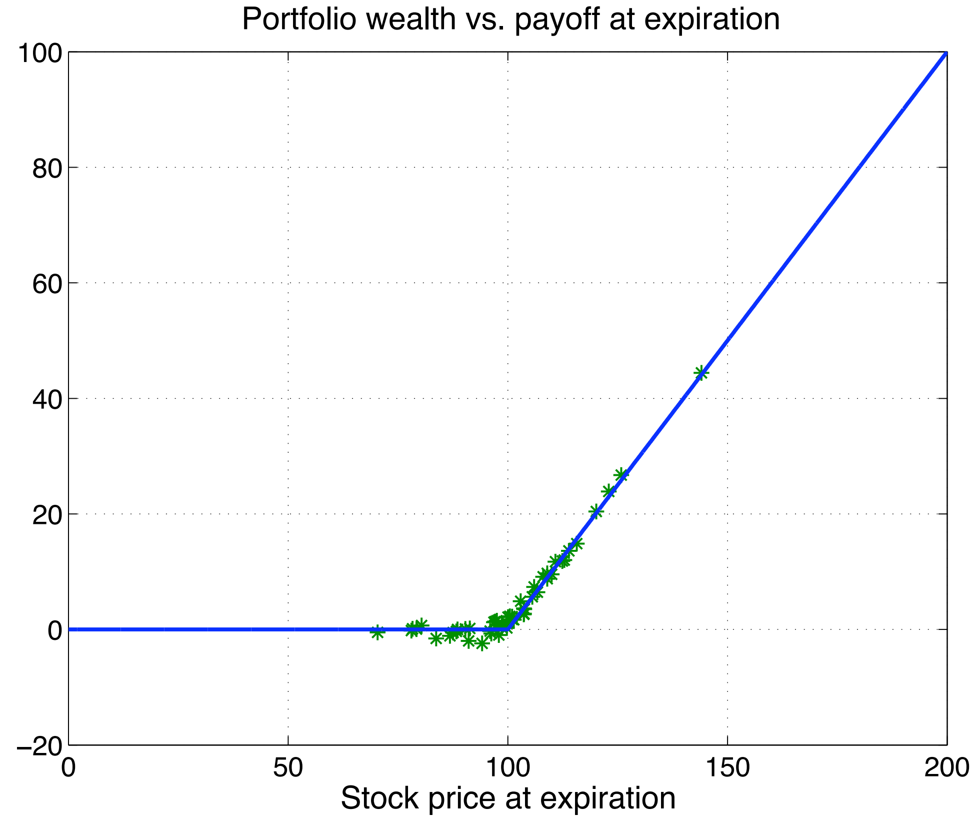


- Black-Scholes model (=log-normal)
- volatility$=0.2$, risk-free$=0.04$
- $T=24$ weeks ($\Delta t=1$ week)
- 50 simulations
- $M=100$ scenarios
- Pricing method: Monte Carlo sim.
- **SMPC**

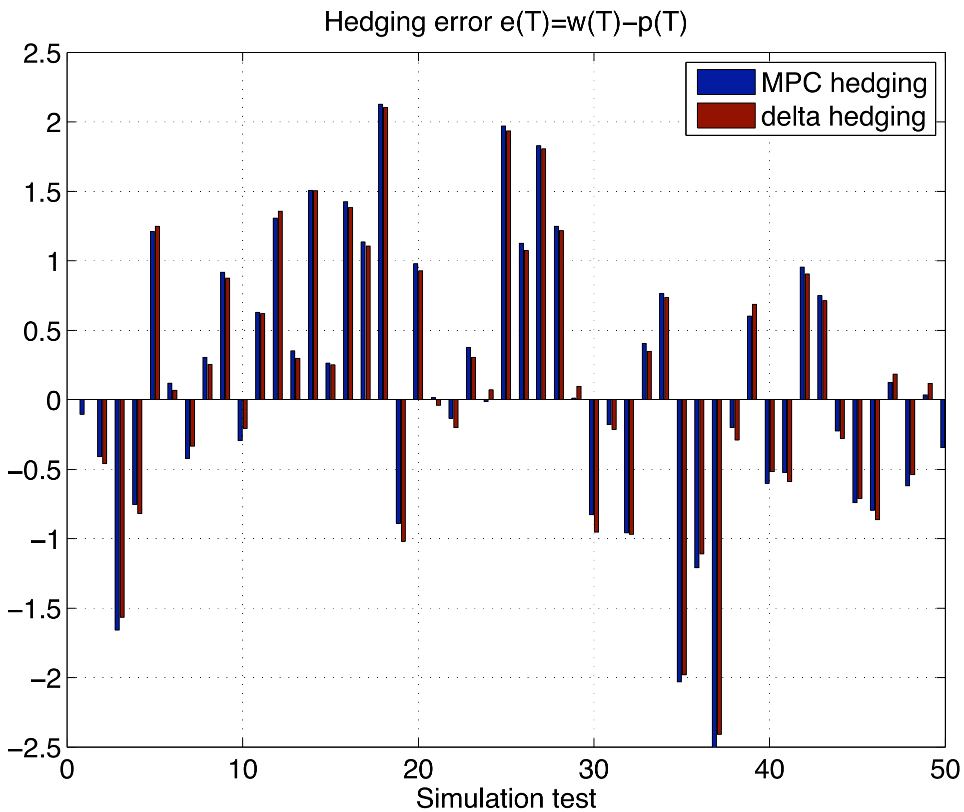


- CPU time $= 7.52$ ms per SMPC step (Matlab R2009 on this mac)

Example: BS model, European call



- Black-Scholes model (=log-normal)
- volatility$=0.2$, risk-free$=0.04$
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- **Delta-hedging**



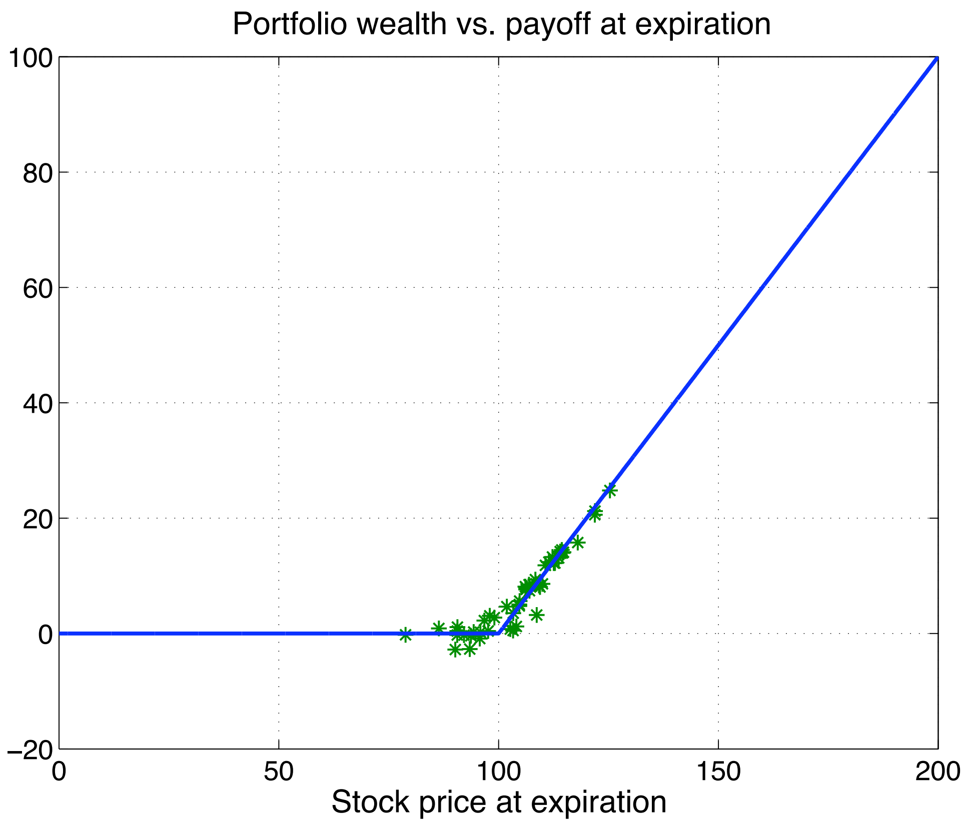
- CPU time $= 0.2$ ms per SMPC step (Matlab R2009 on this mac)

SMPC and delta-hedging are almost indistinguishable

Example: BS model, European call

```
TIME        x(t)        w(t)        p(t)        u0(t)             x(t)*u1(t)    x(t)*dp/dx(t)
-----------------------------------------------------------------------------------------------
t=0.0000: S=100.000, P=  6.196, O=  6.196, P(B)=-52.152, P(S)= 58.348 (BS delta= 57.926)
t=0.0185: S=101.367, P=  6.955, O=  6.865, P(B)=-56.091, P(S)= 63.046 (BS delta= 62.628)
t=0.0370: S= 96.897, P=  4.134, O=  4.261, P(B)=-42.629, P(S)= 46.762 (BS delta= 46.307)
t=0.0556: S= 94.582, P=  2.985, O=  3.108, P(B)=-35.080, P(S)= 38.065 (BS delta= 37.607)
t=0.0741: S= 93.057, P=  2.345, O=  2.415, P(B)=-29.877, P(S)= 32.222 (BS delta= 31.771)
t=0.0926: S= 93.371, P=  2.431, O=  2.395, P(B)=-30.200, P(S)= 32.632 (BS delta= 32.165)
t=0.1111: S= 94.295, P=  2.732, O=  2.591, P(B)=-32.518, P(S)= 35.250 (BS delta= 34.760)
t=0.1296: S= 88.192, P=  0.426, O=  0.859, P(B)=-14.985, P(S)= 15.411 (BS delta= 15.053)
t=0.1481: S= 90.411, P=  0.803, O=  1.199, P(B)=-19.776, P(S)= 20.579 (BS delta= 20.147)
t=0.1667: S= 88.586, P=  0.373, O=  0.754, P(B)=-14.236, P(S)= 14.609 (BS delta= 14.234)
t=0.1852: S= 87.683, P=  0.214, O=  0.544, P(B)=-11.312, P(S)= 11.526 (BS delta= 11.186)
t=0.2037: S= 90.998, P=  0.641, O=  1.000, P(B)=-18.744, P(S)= 19.385 (BS delta= 18.910)
t=0.2222: S= 94.742, P=  1.425, O=  1.867, P(B)=-30.734, P(S)= 32.158 (BS delta= 31.555)
t=0.2407: S= 99.890, P=  3.149, O=  3.945, P(B)=-52.320, P(S)= 55.469 (BS delta= 54.841)
t=0.2593: S=102.720, P=  4.682, O=  5.466, P(B)=-64.736, P(S)= 69.418 (BS delta= 68.857)
t=0.2778: S= 99.723, P=  2.609, O=  3.439, P(B)=-51.468, P(S)= 54.077 (BS delta= 53.379)
t=0.2963: S= 99.591, P=  2.499, O=  3.147, P(B)=-50.513, P(S)= 53.012 (BS delta= 52.268)
t=0.3148: S= 98.178, P=  1.709, O=  2.233, P(B)=-42.460, P(S)= 44.169 (BS delta= 43.336)
t=0.3333: S=100.471, P=  2.709, O=  3.142, P(B)=-55.135, P(S)= 57.845 (BS delta= 57.034)
t=0.3519: S=102.804, P=  4.012, O=  4.363, P(B)=-69.359, P(S)= 73.371 (BS delta= 72.719)
t=0.3704: S= 97.457, P=  0.144, O=  1.202, P(B)=-34.892, P(S)= 35.037 (BS delta= 33.884)
t=0.3889: S= 97.789, P=  0.238, O=  1.030, P(B)=-34.692, P(S)= 34.930 (BS delta= 33.564)
t=0.4074: S= 98.881, P=  0.602, O=  1.089, P(B)=-41.289, P(S)= 41.891 (BS delta= 40.275)
t=0.4259: S= 97.699, P=  0.071, O=  0.308, P(B)=-22.850, P(S)= 22.921 (BS delta= 20.300)
t=0.4444: S= 96.002, P= -0.344, O=  0.000, P(B)= -0.344, P(S)=  0.000 (BS delta=  0.000)
```

Example: Heston model, European call



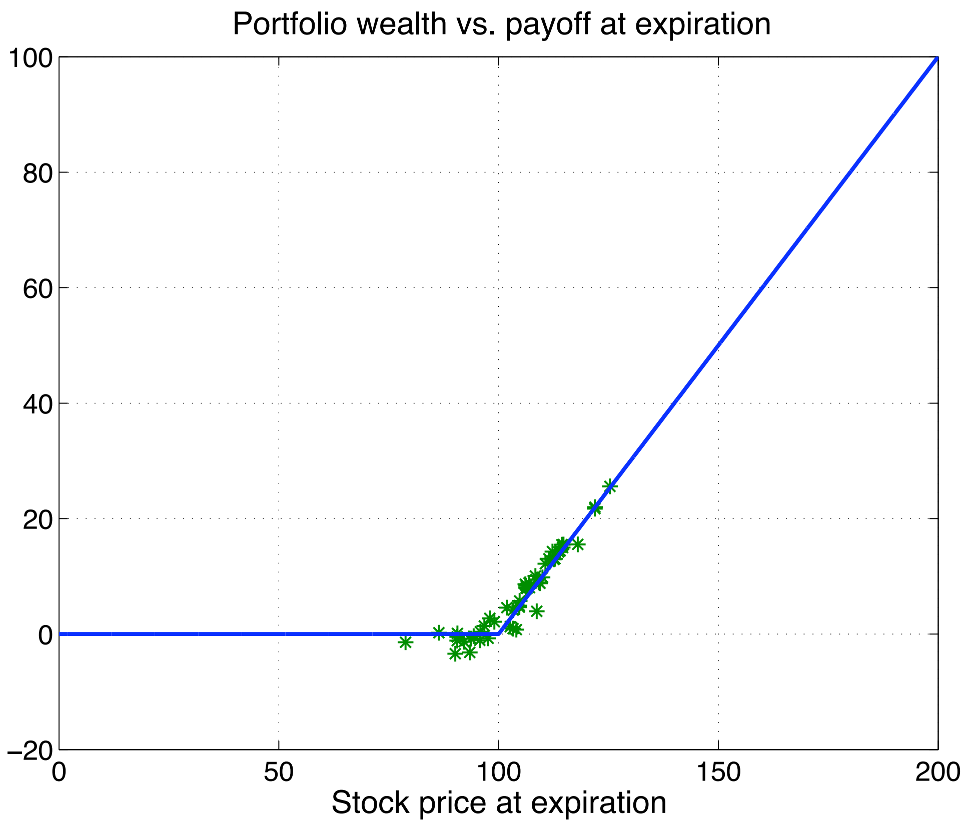
- Heston's model
- T=24 weeks (hedging every week)
- 50 simulations
- M=100 scenarios
- risk-free=0.04
- Pricing method: Monte Carlo sim.
- **SMPC**

- CPU time $= 85.5$ ms per SMPC step (Matlab R2009 on this mac)

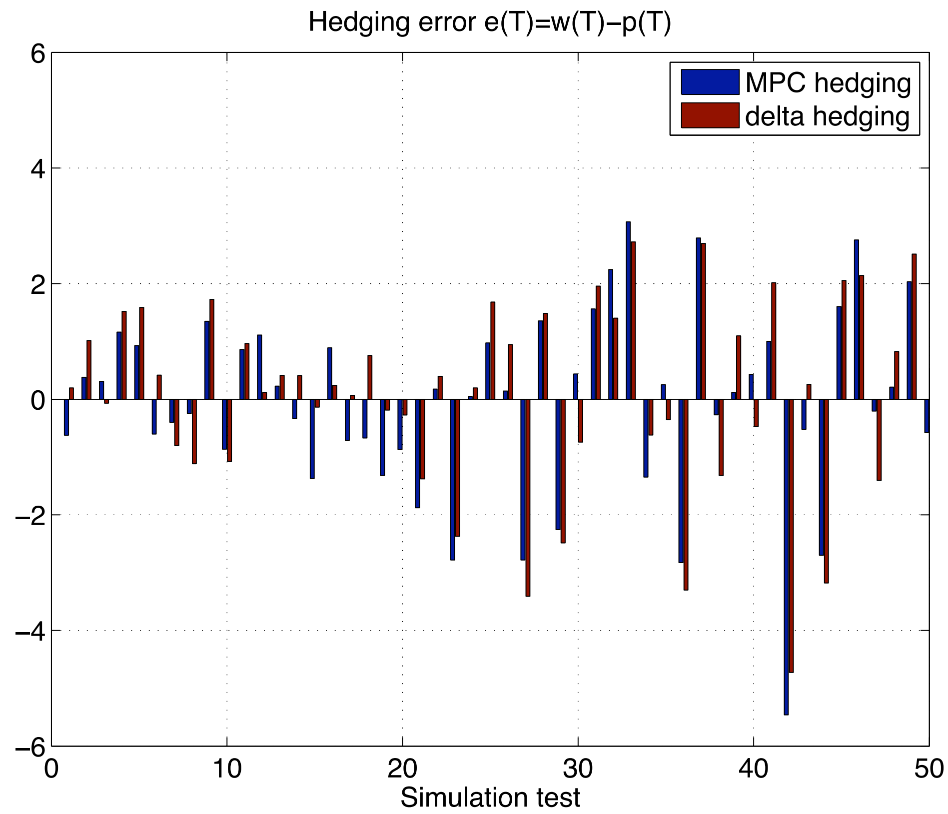
Heston's model

$$\begin{aligned} dx_i(\tau) &= (\mu_i^x d\tau + \sqrt{y_i(\tau)} dz_i^x) x_i(\tau) \\ dy_i(\tau) &= \theta_i(k_i - y_i(\tau)) d\tau + \omega_i \sqrt{y_i(\tau)} dz_i^y \end{aligned}$$

Example: Heston model, European call

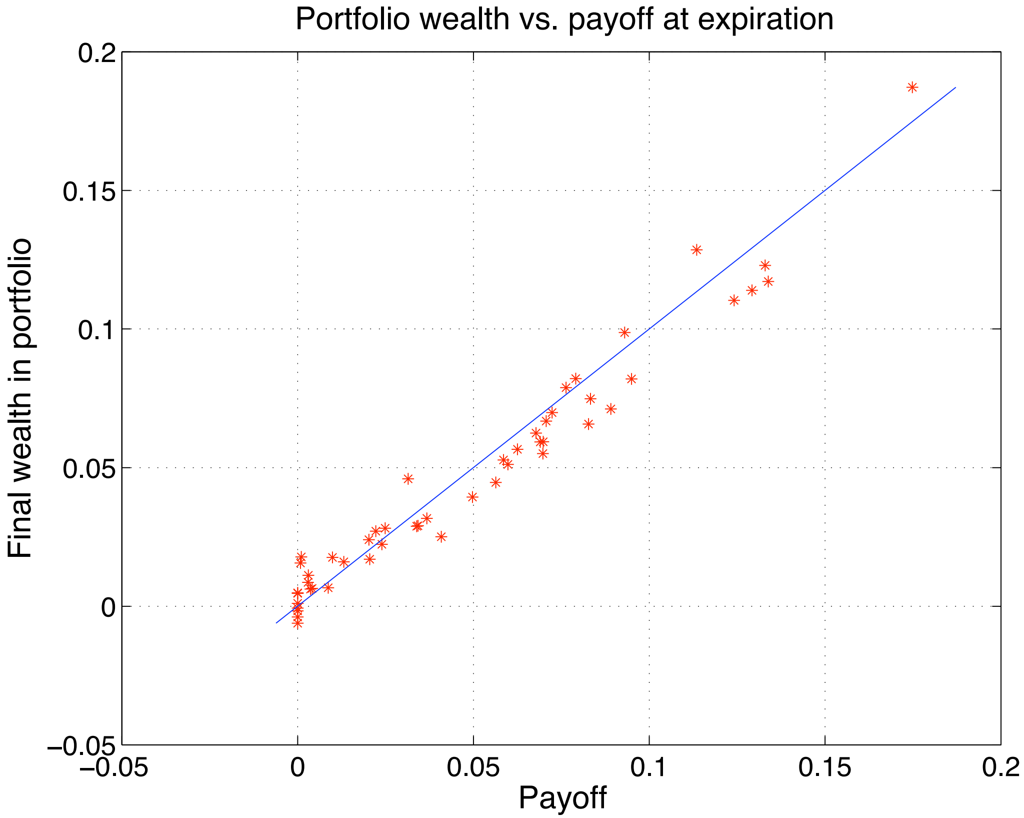


- Heston's model
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- risk-free$=0.04$
- **Delta hedging**



- CPU time $=1.85$ ms per SMPC step (Matlab R2009 on this mac)

Example: BS model, Napoleon cliquet



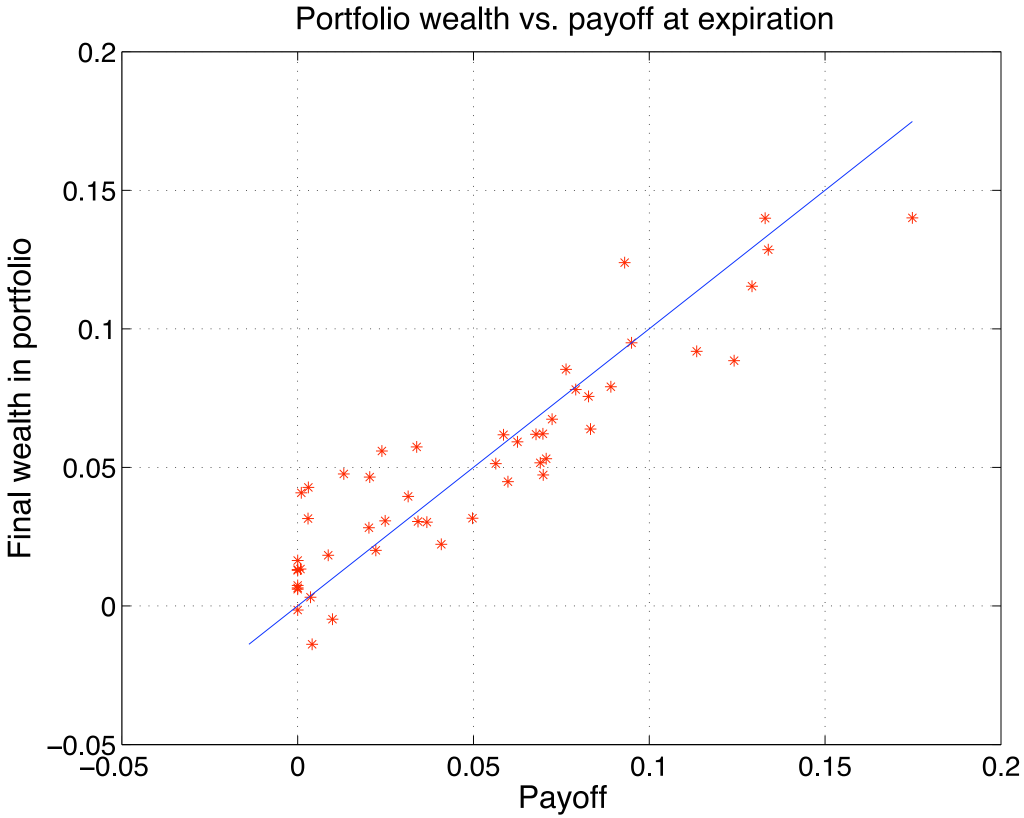
- Black-Scholes model (=log-normal)
- volatility$=0.2$
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- risk-free$=0.04$
- Pricing method: Monte Carlo sim.
- **SMPC: only trade underlying stock**

$$p(T) = \max\left\{0, C + \min_{i\in\{1,\ldots,N_{\text{fix}}\}} \frac{x(t_i) - x(t_{i-1})}{x(t_{i-1})}\right\}$$

$t_i = 0,8,16,24$ weeks

- CPU time $= 1400$ ms per SMPC step (Matlab R2009 on this mac)

Example: BS model, Napoleon cliquet



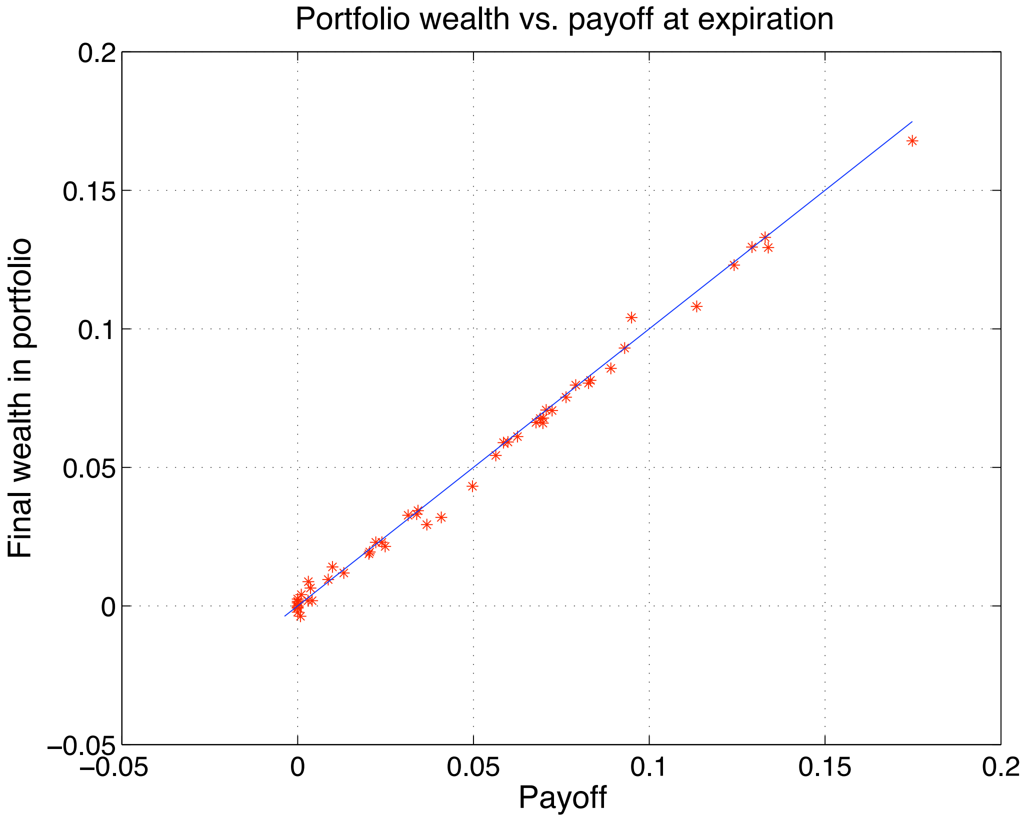
- Black-Scholes model (=log-normal)
- volatility$=0.2$
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- risk-free$=0.04$
- **Delta hedging,**
  only trade underlying stock

$$p(T) = \max\left\{0, C + \min_{i\in\{1,\ldots,N_{\text{fix}}\}} \frac{x(t_i)-x(t_{i-1})}{x(t_{i-1})}\right\}$$

$t_i = 0{,}8{,}16{,}24$ weeks

- CPU time $=2.41$ ms per SMPC step
  (Matlab R2009 on this mac)

Example: BS model, Napoleon cliquet



- Black-Scholes model (=log-normal)
- volatility$=0.2$
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- risk-free$=0.04$
- Pricing method: Monte Carlo sim.
- **SMPC: Trade underlying stock & European call with maturity $t+T$**

$$p(T)=\max\left\{0, C+\min_{i\in\{1,\ldots,N_{\text{fix}}\}}\frac{x(t_i)-x(t_{i-1})}{x(t_{i-1})}\right\}$$

$t_i=0,8,16,24$ weeks

- CPU time $=1625$ ms per SMPC step (Matlab R2009 on this mac)

Approximate option pricing

- **Bottleneck** of the approach for exotic options: price M future option values $p^1(t+1),\ p^2(t+1),\ \ldots,\ p^M(t+1)$
- **Monte Carlo pricing** can be time consuming: say L scenarios to evaluate a single option value ⇒ need to simulate ML paths to build optimization problem (e.g.: M=100, L=10000, ML=10^6)
- Use off-line **function approximation** techniques to estimate $p(t)$ as a function of current asset parameters and other option-related parameters

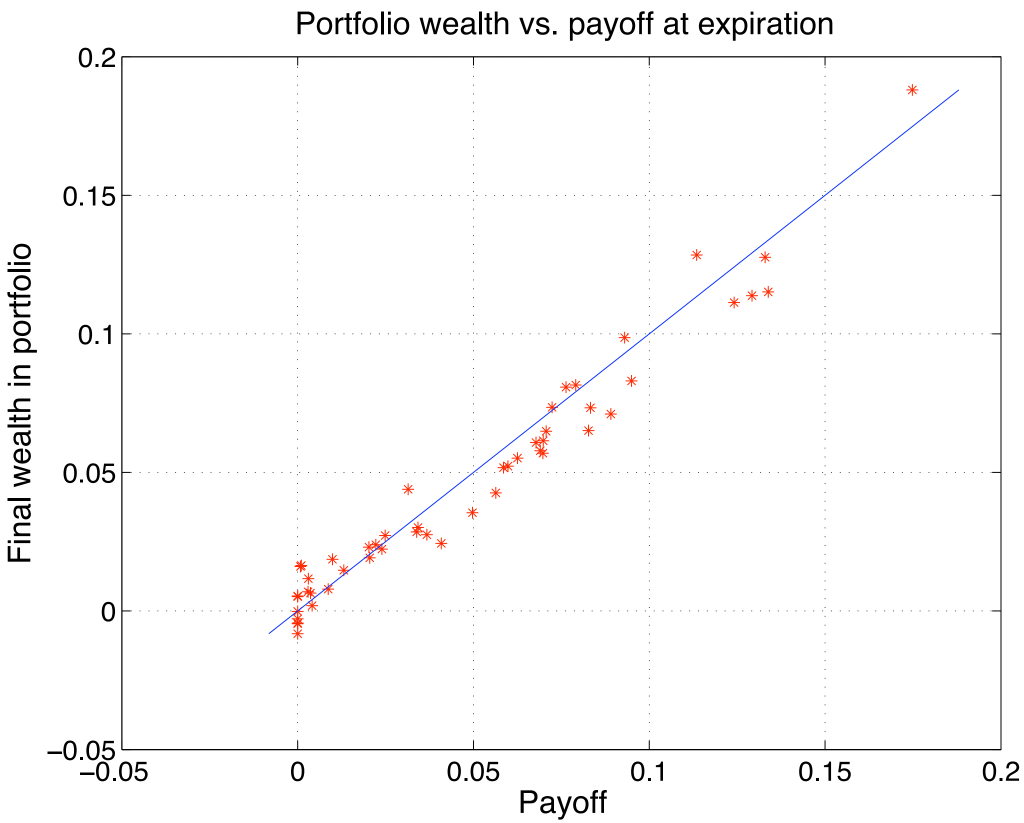
Example: Napoleon cliquet, Heston model

$$p(t) = f(x(t), \sigma(t), x(t_1), \ldots, x(t_{N_{\text{fix}}}))$$

- Most suitable method for estimating pricing function f: **least-squares Monte Carlo** approach based on polynomial approximations

(Longstaff, Schwartz, 2001)

Example: BS model, Napoleon cliquet



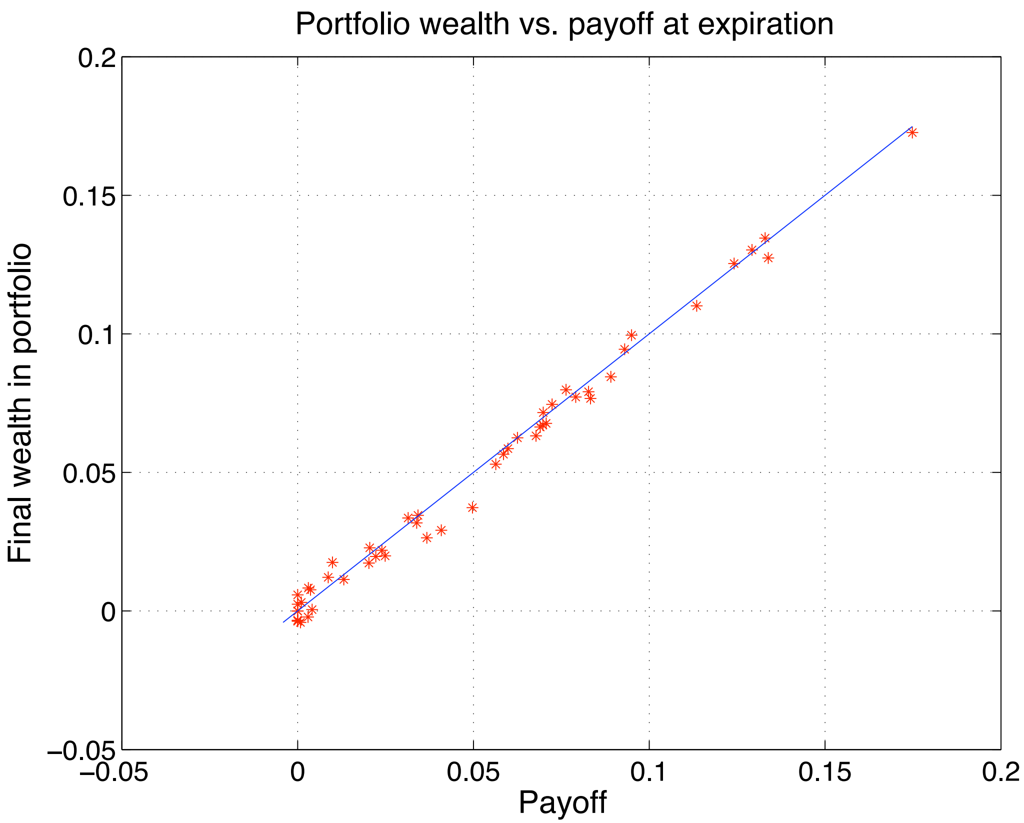
- Black-Scholes model (=log-normal)
- volatility$=0.2$, risk-free$=0.04$
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- Pricing method: LS approximation
- **SMPC: only trade underlying stock**

- ~~CPU time $= 1400$ ms per SMPC step (Matlab R2009 on this mac)~~

- CPU time $=$ 50.5 ms per SMPC step (Matlab R2009 on this mac)
- CPU time $=$ 76.7 s to compute LS approximation (off-line)

Hedging quality very similar !

Example: BS model, Napoleon cliquet



- Black-Scholes model (=log-normal)
- volatility$=0.2$, risk-free$=0.04$
- $T=24$ weeks (hedging every week)
- 50 simulations
- $M=100$ scenarios
- Pricing method: LS approximation
- **SMPC: Trade underlying stock & European call with maturity $t+T$**

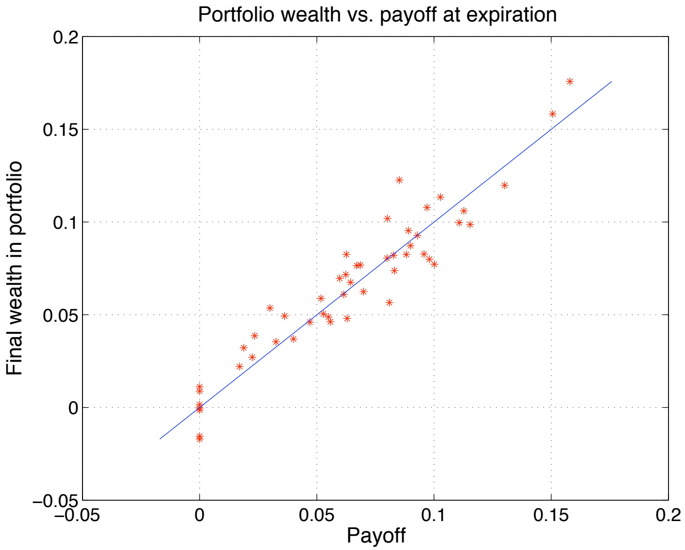
- ~~CPU time $=1625$ ms per SMPC step (Matlab R2009 on this mac)~~

- CPU time $=$ 59.2 ms per SMPC step (Matlab R2009 on this mac)

- CPU time $=$ 76.7 s to compute LS approximation (off-line)

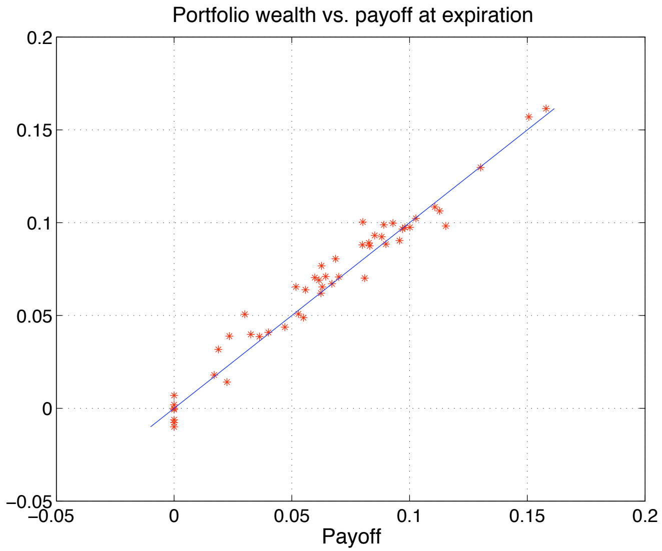
Hedging quality very similar !

Example: Heston model, Napoleon cliquet



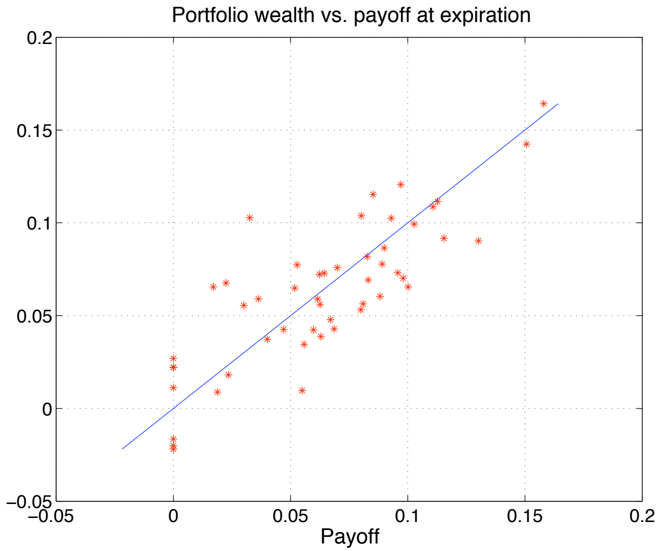
SMPC: only trade underlying stock

CPU time $= 220$ ms per SMPC step



SMPC: Trade underlying stock & European call with maturity $t+T$

CPU time $= 277$ ms per SMPC step



Delta hedging only trade underlying stock

CPU time $= 156$ ms per SMPC step

CPU time $= 156$ s to compute LS approximation (off-line)

Conclusions

- Dynamic option hedging = stochastic control problem
- Propose **Stochastic MPC** as a very **versatile** tool for dynamic option hedging:
  - it's based on Monte Carlo simulation ⇒ can use **arbitrary stock models**
  - can use **any payoff function** for which a numerically efficient pricing engine is available
- Computational demand mostly due to pricing future option values
- **On-line** use of SMPC: suggest trading moves to traders
- **Off-line** use of SMPC: run extensive simulations to quantify the **average hedging error** for a given market model and option type
- On-going work: test SMPC's **robustness** w.r.t. **market model mismatch**