Optimal Control of Investments for Quality of Supply Improvement in Electrical Energy Distribution Networks

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Abstract

This paper considers the problem of deciding multi-period investments for maintenance and upgrade of electrical energy distribution networks. After describing the network as a constrained hybrid dynamical system, optimal control theory is applied to optimize profit under a complex incentive/penalty mechanism imposed by public authorities. The dynamics of the system and the cost function are translated into a mixed integer optimization model, whose solution gives the optimal investment policy over the multi-period horizon. While for a reduced-size test problem the pure mixed-integer approach provides the best optimal control policy, for real-life large scale scenarios a heuristic solution is also introduced. Finally, the uncertainty associated with the dynamical model of the network is taken care of by adopting ideas from stochastic programming.

Key words: Multi-period Investments, Optimal Control, Hybrid Systems, Mixed Integer Optimization, Stochastic Programming

1 Introduction

National regulations were recently applied in several countries for encouraging electrical energy distribution companies to improve the continuity of energy supply. Such regulations include incentives/penalties to energy distribution companies that depend on a few quality indicators. The introduced incentives/penalty mechanisms usually reflect customers' preferences and requirements, and their willingness to pay for quality. Better quality standards are considered for customers that are more sensitive to quality of energy supply. In addition, incentives/penalties are not homogenous on the territory, but depend on the present status of the network, in particular they are larger in areas where the current quality of energy supply is worse. Average standards and yearly rate of improvement standards are more relevant to promoting overall improvement or to maintaining quality and can be used to adjust continuity differentials between regions.

Quality management has become a strategic issue for electricity suppliers. The aforementioned regulations not only impact considerably the economic activities of the supplier, but also provide guidelines to the company management for deciding the multi-annual investment plans for renovating energy distribution lines, in order to maximize the quality of energy supply perceived by customers while satisfying financial and operational constraints.

In this paper we consider the Italian regulation system introduced in 2004 [14]. In Italy, companies that do not achieve the yearly improvement standards must pay a penalty, while companies that exceed the yearly improvement standards receive an incentive payment. Penalties and incentives are defined by complex rules that include dead bands, saturations, penalty cancella-

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tions, etc.

By taking into account national regulations, the state of the network, previous actions and other historical data, the managers of the company currently decide the multi-annual investment plan for maintaining and upgrading the energy distribution lines according to a manual trial-and-error procedure. This activity is time-consuming, does not always bring to optimal choices that exploit the available resources to maximize the resulting quality of energy supply, and in any case always make the management wonder if better plans could have made. The above disadvantages are amplified by the fact that such decisions should be made according to a "rolling horizon" philosophy (or "receding horizon" philosophy, using the terminology of model predictive control [7, 20]), that is, decisions should be re-iterated during the multi-period horizon in order to take into account poorly forecasted or unexpected events.

Planning for investments forms a crucial part of the strategic level decision-making in many other applications. Examples can be found in many different fields, such as heavy process industries, electric utilities or automobile industries [12, 22, 24]. In all these applications the investments require the commitment of substantial capital resources over long periods of time. Algorithms that allow one to plan "optimal" decisions over such a time horizon would automatically provide strategies for maximizing profits. Optimization algorithms based on the solution of linear programs have been successfully applied in economics since the 1940s. However, there are fewer references of mixed integer linear programming, see for example [16, 31]. Furthermore, the uncertainties associated with long-range forecasts make these decision problems very complex. Stochastic programming [6] is one of the most powerful analytical tools to support decision-making under uncertainty. As a scientific field, it exists since many years and has received many contributions (see, for example, `http://stoprog.org`). Stochastic optimization has gathered renewed attention in the last years (see [8, 29] and the references therein), probably due to improvements in computing platforms.

In this paper we propose an automatic method for taking decisions about multi-annual investments that is based on optimal control ideas. After modelling the network as a simple (but large-scale) hybrid dynamical system with integrating dynamics and a piecewise affine output function, and after expressing the nonlinear incentive/penalty function by means of mixed-integer variables, an optimal control problem is solved to optimize the profit of investments. Standard mixed-integer programming solvers provide the best optimal control policy only for reduced-size test problems. In order to cope with the actual large scale scenario of a regional network, a heuristic solution is introduced. Finally, by taking into account the uncertainty associated with the nonlinear function relating investments with benefits, we propose a model to optimize the profit of maintenance and upgrade of a electrical distribution network under uncertainty.

2 Optimal Control Problem

The aim of this section is to set up an optimal control problem to determine the optimal allocation of investments for maintenance and upgrade of electrical energy distribution networks on a multi-period (four years) time basis. In our context, a control action is considered optimal if the profit of the electrical distribution company is maximized (indirectly, this would also imply that the quality of energy supply perceived by customers is maximized).

We treat the electrical distribution network as a (large-scale) discrete-time dynamical system whose sampling time is one year (decisions are taken on a yearly basis), whose states define the quality of energy supply in each individual district, and whose input is the amount of money invested in that district at a given year. The decision of the optimal investments depends on the prediction over a certain number N of future years of the evolution of the quality of the network. The prediction is repeated every year, according to the so-called "receding-horizon" (or "rolling-horizon") principle, over a multi-annual horizon that has been shifted forward by one year. The dynamical model of the network and its current state is needed at a certain year i to evaluate the effect of current and future control actions (investments) in the prediction, and therefore for computing the value of the corresponding performance index.

Contrarily to standard model predictive control problems where the objective is to regulate the state of the system to the origin or to track a given reference profile, here the goal is to optimally controlling the state of the system to a condition (determined a posteriori) where the profit of the investment project is maximized. The objective function is defined by the incentive-penalty mechanism described in Section 2.2.

2.1 Dynamical System

An electrical distribution company must decide the amount of money that must be invested in each district $j \in \mathcal{D}$, where $\mathcal{D} = \{1, 2, \ldots, D\}$ is a finite set of districts, in order to maintain a certain quality of energy supply, which is measured in each district $j \in \mathcal{D}$ by the amount of minutes of power outage per customer per year (Customers Minutes Lost, CML), an indicator of the continuity of supply service.

In order to improve the CML in their districts of competence, distribution companies invest money in maintenance and upgrade of the energy distribution network.

Districts are usually heterogeneous among them, have different sizes and levels of quality, so that each district must have its own investment project. Each project has a cost and provides an expected improvement of quality. In general, CML does not decrease linearly with the invested money, but rather has a more complex nonlinear dependence. In this paper we consider two different kinds of investment projects. The first one are local improvement interventions on a district. We model quality of supply as a piecewise affine function of the invested money (see Figure 1). The second one are big network upgrade projects (such as the construction of a distribution substation) which may affect more than one district and are defined by a fixed cost and a fixed gain. These are yes/no interventions, i.e., either are carried out at that fixed cost producing a fixed gain, or not. In general, each project can be carried out at any year over the optimization horizon, but also time constraints can be addressed in the optimization model.

To evaluate an investment policy along a multi-period time basis, we used a hybrid dynamical model [1,4,19]. The system state is made up of both continuous and binary variables. The continuous variables represent the money invested in local improvement projects, while the binary states indicate if a given upgrade project has been realized. Although the dynamics of the system are very simple, a hybrid model is suited to model both nonlinearities (once these are described as piecewise affine functions), discrete states, and discrete inputs.

For each district $j \in \mathcal{D}$ and each time period $i \in \mathcal{Y}$, where $\mathcal{Y} = \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, we characterize the quality level as a piecewise affine function of the money invested in local improvement projects. The money invested in the local improvement of district $j \in \mathcal{D}$ from the initial time period up to year $i \in \mathcal{Y}$ is denoted by x^c_{ij}. For the investment problem tackled in this paper, the time period goes from 2003 ($i = 0$) to 2007 ($i = N = 4$). This is the period of definition of the Italian normative described in [14].

As mentioned earlier, the investment problem also allows yes/no upgrade investments affecting more than one district. The l-th of such investments, $l \in \mathcal{I} = \{1, \ldots, I\}$, is defined by the cost of the investment and the quality increase in each affected district. For each possible upgrade $l \in \mathcal{I}$, the binary state x^b_{il} indicates if the upgrade has been done by time period $i \in \mathcal{Y}$. The cost of investment l is denoted by C^{up}_l and the quality increase in district j by ΔI_{lj}. Matrix $\Delta I \in \mathbb{R}^{I \times D}$ contains the quality increase factors for all upgrades and districts ($\Delta I_{lj} = 0$ if upgrade investment l does not affect district j).

At each time period $i \in \mathcal{Y}$, one must decide the *continuous* inputs u^c_{ij}, which is the money invested in each district $j \in \mathcal{D}$ for maintenance, and the *binary* input u^b_{il}, which decides whether network upgrade l must be realized. Hence, the equations that determine the dynamics of the system are

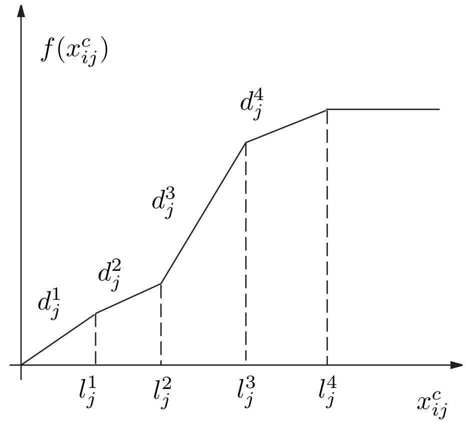

Fig. 1. Quality of electricity supply improvement function.

$$x^c_{i+1,j} = x^c_{ij} + u^c_{ij},\ \forall j \in \mathcal{D}, \tag{1a}$$
$$x^b_{i+1,l} = x^b_{il} \vee u^b_{il},\ \forall l \in \mathcal{I}, \tag{1b}$$

where "$\vee$" denotes the logical or. Hybrid dynamics (1) is a collection of simple continuous linear dynamics and finite state machines.

The CML of district j at time period i is denoted by $\mathcal{C}_{ij}$ and can be described as follows. Consider the piecewise affine function $f : \mathbb{R} \mapsto \mathbb{R}$ defined as

$$f(x^c_{ij}) = \begin{cases} d^1_j x^c_{ij} & \text{if } x^c_{ij} \le l^1_{ij} \\ f(l^1_j) + d^2_j(x^c_{ij} - l^1_j) & \text{if } l^1_j \le x^c_{ij} \le l^2_j \\ f(l^2_j) + d^3_j(x^c_{ij} - l^2_j) & \text{if } l^2_j \le x^c_{ij} \le l^3_j \\ f(l^3_j) + d^4_j(x^c_{ij} - l^3_j) & \text{if } l^3_j \le x^c_{ij} \le l^4_j \\ f(l^4_j) & \text{if } x^c_{ij} \ge l^4_j \end{cases} \tag{2}$$

which is depicted in Figure 1. Then,

$$\mathcal{C}_{ij}(x^c_{ij}) = \mathcal{C}_{0j} - f(x^c_{ij}) - \sum_{l \in \mathcal{I}} x^b_{il} \Delta I_{lj}, \tag{3}$$

where ΔI_{lj} is the quality increase in district j if upgrade l is realized. The minus signs in (3) take into account that CML decreases when the quality of electricity supply increases.

The dynamics of each district j are only coupled by binary upgrade decisions, as graphically shown in Figure 2. A further coupling is due to constraints on the capital that can be invested at each time period i, namely

$$\sum_{j \in \mathcal{D}} u^c_{ij} + \sum_{l \in \mathcal{I}} u^b_{il} C^{\mathrm{up}}_l \le \mathrm{U}_{\max}, \forall i \in \mathcal{Y},$$

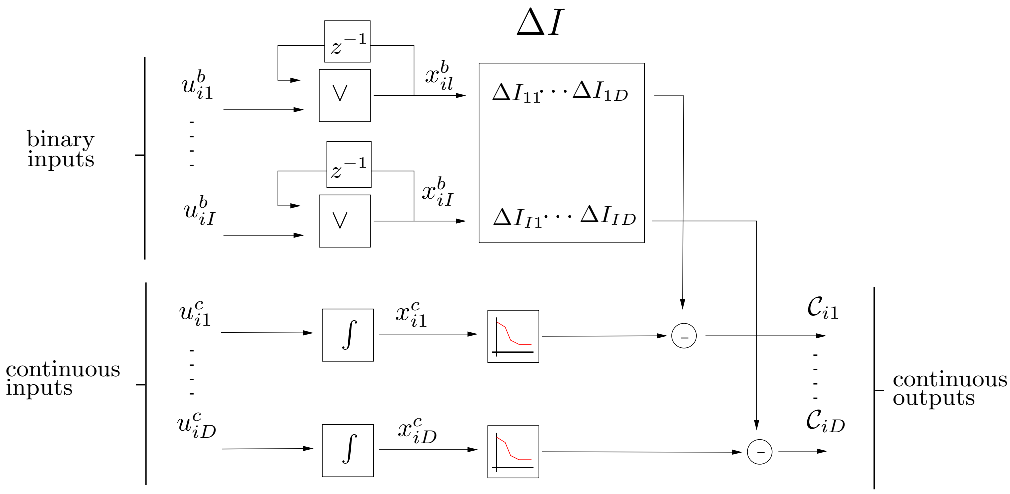

Fig. 2. Hybrid dynamical model relating investments to quality of supplied electrical energy

where $\mathrm{U}_{\max}$ is the maximum amount of money that can be invested each year i. Note that other constraints on the invested money (i.e., on the inputs of the system) and on the quality level (i.e., on the states of the system) may be imposed similarly.

2.2 Cost Function

Distribution agencies must select the allocation of investments depending on complex incentive-penalty mechanisms imposed by national authorities for energy. In this paper we consider the Italian normative described in [14], that we summarize here below.

Each year and for each district the company is given an incentive if the two-year moving average CML of that district is under a given basic standard, or must pay a penalty if it is higher. The objective is to maximize the overall profit, i.e., the difference between the incentive obtained and the invested money or paid penalties, during the time period 2004–2007. The initial state C_{0j} of the network is the CML for year 2003.

The incentive-penalty mechanism is defined by the following set of rules:

- *Incentive-penalty evaluation*: The basic standards for each district j and period i, denoted as S_{ij}, are data computed from the initial quality level $\mathcal{C}_{0j}$ and from the density level ρ_j of the district, where ρ_j can be "High", "Medium" or "Low". Each density level is treated with a different incentive/penalty function.
  The incentive value for a given district j and period i is a piecewise affine function of (i) the two-years moving average of the CML of the district, and of (ii) the basic standard. It is also proportional to the power supplied to the district to both domestic users (P^D_j) and non-domestic users (P^N_j).
  The values of S_{ij}, $\mathcal{C}_{0j}$, ρ_j, P^D_j and P^N_j are known data of the problem, not decision variables. The incentive-penalty evaluation will be described more in detail in Section 3.2.
- *Dead band*: For each district, a dead band is defined around the basic standard level where neither incentives nor penalties are due. For each district j and period i the band is $[S_{ij} - f^-_{ij}, S_{ij} + f^+_{ij}]$, where f^-_{ij} and f^+_{ij} are data provided in [14, Articles 23.1, 23.2 and 23.3].
- *Maximum and minimum incentive-penalty*: Incentives and penalties are saturated: the maximum incentive is J^+_{ij} and the maximum penalty is $-J^-_{ij}$. Such saturation levels depend on the density ρ_j of each district and on the corresponding basic standards S_{ij} and are defined in [14, Articles 23.6 and 23.7].
- *Discount Rate*: We suppose a discount rate $r = 7\%$ to evaluate the Weighted Average Cost of Capital (WACC) with respect to year 2004.
- *Penalty cancellation for 2004-2005*: The law provides a special treatment for the penalties of years 2004 and 2005. If the distribution company incurs in a penalty in either year 2004 or 2005, the debt is not payed immediately. It is instead subdivided in three installments to be payed in the following three years. In addition, if at any time the CML goes under the basic standard, then the remaining installments are canceled.

The above regulations define a nonlinear function relating investments to profit. Such a function is composed by a piecewise affine function and by a set of logical conditions, which can be handled by a suitable mixed-integer optimization model, as detailed in the following section.

3 Optimization Model

In this section we propose a mixed integer optimization model which takes into account both the hybrid dynamics (1)–(3) and piecewise affine/logical relations describing the objective function.

For solving optimal control problems for discrete-time hybrid dynamical systems subject to linear and logical constraints, the Mixed Logical Dynamical (MLD) formalism was introduced in [4]. MLD models allow one to specify the evolution of continuous variables through linear dynamic equations, of discrete variables through propositional logic statements and automata, and the mutual interaction between the two. The key idea of the approach consists of embedding the logic part in the state equations by transforming Boolean variables into 0-1 integers, and by expressing the relations as mixed-integer linear inequalities [4, 23, 25, 33, 34]. The tool HYSDEL (HYbrid System DEscription Language) described [33] allows one to automatically perform such a translation of linear/logical relations into mixed-integer inequalities. A Matlab interface to HYSDEL is provided in the Hybrid Toolbox for Matlab [3].

The hybrid optimal control problem tackled in this paper can be treated using similar techniques. Here the objective is to maximize the total profit cumulated over a set of periods $i \in \mathcal{Y}$ for a given set of districts $j \in \mathcal{D}$. The decision variables are the sums invested each year for improvements and for upgrades: u^c_{ij} is the money invested during period i in district j, u^b_{il} is a binary variable that indicates if upgrade project l is realized during period i. The optimization problem can be schematically stated as follow:

$$\begin{aligned} \max\ & J^{\text{profit}} \\ \text{s.t. } & \text{Constraints defining system dynamics} \\ & \text{Constraints defining } J^{\text{profit}} \\ & \text{Input constraints (money spent)} \\ & \text{State constraints (quality levels).} \end{aligned} \tag{4}$$

In order to express the dynamics of the system and the cost function as (mixed-integer) linear inequalities, several auxiliary variables and linear constraints are needed, as explained below.

3.1 System Dynamics

The following variables are considered in the optimization model: x^c_{ij} is the cumulated money invested in district j during period i and is the continuous state of each district; $\mathcal{C}_{ij}$ is the predicted CML level [min/yr] in district j during period i; $\Delta\mathcal{C}_{ij}$ is the cumulated improvement from the initial CML level in district j during period i. The cumulated investment x^c_{ij} is simply given by the sum of all previous investments

$$x^c_{ij} = \sum_{k=1}^{i} u^c_{kj}. \tag{5}$$

Rather than considering the discrete state x^b_{il} that indicates if a given upgrade l has been realized during year i as an optimization variable, we consider the following binary variables $u^b_{il} \in \{0, 1\}$, where $u^b_{il} = 1$ implies that investment l is done during period i. A given investment can only be done once, so the following constraint is added:

$$\sum_{i \in \mathcal{Y}} u^b_{il} \le 1, \quad \forall\, l \in \mathcal{I}. \tag{6}$$

We assume that a minimum level of investment is made in each district in order to maintain, independently on making upgrade investments, at least the CML level existing at year 2003. For this reason, we consider the difference

$$\Delta\mathcal{C}_{ij} = \mathcal{C}_{0j} - \mathcal{C}_{ij} - \sum_{l \in \mathcal{I}} \sum_{h=1}^{i} u^b_{hl} \Delta I_{jl} = f(x^c_{ij}) \tag{7}$$

as an optimization variable, where $\mathcal{C}_{0j}$ is the CML level at year 2003 and $\Delta I_{jl} \ge 0$ is the CML improvement of district j if upgrade l is realized. By the above assumption, $\Delta\mathcal{C}_{ij} \ge 0$. Function $f(x^c_{ij})$ represents a given local improvement investment project. Each district has a different project which is defined as a piecewise affine function of the cumulated invested money, where each different linear segment of the function represents a different stage of the project. As described in (2), in this paper we consider a continuous piecewise affine function with four different gains (four projects stages) and a saturation. This description well models data provided by the company ENEL. Parameters l^1_{ij} and d^4 model the investment project stage costs and gains, respectively. Figure 1 shows an example of a piecewise affine function with four different gains and a saturation.

In order to represent function $f(x^c_{ij})$ through mixed-integer linear inequalities, the following auxiliary optimization variables are introduced: $\alpha^1_{ij} \in \{0, 1\}$ indicating if $x^c_{ij} \ge l^1_j$; $\alpha^2_{ij} \in \{0, 1\}$ indicating if $x^c_{ij} \ge l^2_j$; $\alpha^3_{ij} \in \{0, 1\}$ indicating if $x^c_{ij} \ge l^3_j$; $s^1_{ij} \ge 0$ denoting the part of the investment under l^1_j; $s^2_{ij} \ge 0$ denoting the part of the investment between l^1_j and l^2_j; $s^3_{ij} \ge 0$ denoting the part of the investment between l^2_j and l^3_j; $s^4_{ij} \ge 0$ denoting the part of the investment between l^3_j and l^4_j. Then, as at optimality certainly holds that $x^c_{ij} \le l^4_j$ because of saturation, the CML level improvement is modeled by the following set of constraints for each district

Seg. h	C_1^h	C_2^h
1	0.240	0.120
2	0.360	0.180
3	0.480	0.240

Table 1
Parameters defining different minutes segments from [14] [k€/(MW(min/yr))].

Density ρ_j	Seg. 1	Seg. 1	Seg. 3	$sg_{1\rho_j}$	$sg_{2\rho_j}$
High	0-25	25-75	+75	25	75
Medium	0-40	40-120	+120	40	120
Low	0-60	60-180	+180	60	180

Table 2
Segment definition ([min/yr]) for different densities as defined in [14]

j and period i:

$$\begin{aligned}
&x_{ij}^c = s_{ij}^1 + s_{ij}^2 + s_{ij}^3 + s_{ij}^4 \\
&\alpha_{ij}^1 l_j^1 \le s_{ij}^1 \le l_j^1 \\
&\alpha_{ij}^2 (l_j^2 - l_j^1) \le s_{ij}^2 \le \alpha_{ij}^1 (l_j^2 - l_j^1) \\
&\alpha_{ij}^3 (l_j^3 - l_j^2) \le s_{ij}^3 \le \alpha_{ij}^2 (l_j^3 - l_j^2) \\
&s_{ij}^4 \le \alpha_{ij}^3 (l_j^4 - l_j^3) \\
&\alpha_{ij}^1 \ge \alpha_{ij}^2 \ge \alpha_{ij}^3 \\
&\Delta\mathcal{C}_{ij} = d_j^1 s_{ij}^1 + d_j^2 s_{ij}^2 + d_j^3 s_{ij}^3 + d_j^4 s_j^4 \\
&s_{ij}^1, s_{ij}^2, s_{ij}^3, s_{ij}^4 \ge 0, \alpha_{ij}^1, \alpha_{ij}^2, \alpha_{ij}^3 \in \{0,1\}.
\end{aligned} \tag{8}$$

3.2 *Profit*

According to [14] the incentive-penalty value for a given district and period depends on the expected CML value, the basic standards and the density of the districts. The basic standard levels S_{ij} of CML reference for a given district j during year i are data of the optimization problem. The expected incentive (or penalty) J_{ij}^t in district j during period i is defined as a piecewise affine function that depends on the two-year moving average CML level $\mathcal{B}_{ij} = \frac{\mathcal{C}_{i-1,j}+\mathcal{C}_{i,j}}{2}$. Without taking into account [14, Article 23] , we have

$$J_{ij}^t = \sum_{h=1}^{3} (P_j^D C_1^h + P_j^N C_2^h)(t_{ij}^h - b_{ij}^h), \tag{9}$$

where P_j^D and P_j^N are data for each district and period, C_1^h and C_2^h, $h = 1,2,3$, are the parameters in [k€/(MW(min/yr))] given in [14, Table 1] and in Table 1 for different segments $[sg_{1\rho_j}, sg_{2\rho_j}]$, $\rho_j \in \{$"High","Medium","Low"$\}$ is the density of the district, reported in Table 2, and where the auxiliary quantities t_{ij}^h represent the number of minutes of S_{ij} over segment h, and b_{ij}^h the number of minutes of $\mathcal{B}_{ij}$ over segment h. While t_{ij}^l are data computable from S_{ij}, b_{ij}^h are instead decision variables. To define them, we need to introduce two binary variables β_{ij}^1, β_{ij}^2 which indicate if $\mathcal{B}_{ij}$ is greater than the limit of each segment

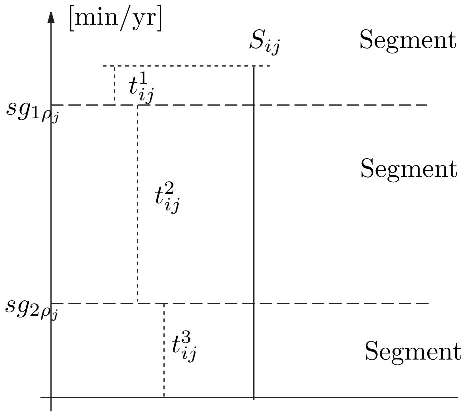

Fig. 3. Auxiliary variable definition

$sg_{1\rho_j}$, $sg_{2\rho_j}$, respectively:

$$[\mathcal{B}_{ij} \in \text{ Segment 1 }] \rightarrow [\beta_{ij}^1 = \beta_{ij}^2 = 0]$$
$$[\mathcal{B}_{ij} \in \text{ Segment 2 }] \rightarrow [\beta_{ij}^1 = 1, \beta_{ij}^2 = 0]$$
$$[\mathcal{B}_{ij} \in \text{ Segment 3 }] \rightarrow [\beta_{ij}^1 = \beta_{ij}^2 = 1] .$$

Accordingly, the cost function J_{ij}^t can be defined by (9) and by the following set of constraints

$$\begin{aligned}
&\mathcal{B}_{ij} = b_{ij}^1 + b_{ij}^2 + b_{ij}^3 \\
&\beta_{ij}^1 sg_{1\rho_j} \le b_{ij}^1 \le sg_{1\rho_j} \\
&\beta_{ij}^2 (sg_{2\rho_j} - sg_{1\rho_j}) \le b_{ij}^2 \le \beta_{ij}^1 (sg_{2\rho_j} - sg_{1\rho_j}) \\
&b_{ij}^3 \le \beta_{ij}^2 \hat{\mathcal{C}} \\
&\beta_{ij}^2 \le \beta_{ij}^1 \\
&b_{ij}^1, b_{ij}^2, b_{ij}^3 \ge 0, \beta_{ij}^1, \beta_{ij}^2 \in \{0,1\}.
\end{aligned} \tag{10}$$

where $\hat{\mathcal{C}}$ is an upper bound on $\mathcal{C}_{ij}$ and ρ_j is the density of the district j.

3.2.1 *Dead Band*

A dead band $[S_{ij} - f_{ij}^-, S_{ij} + f_{ij}^+]$ around the basic standard where neither incentive nor penalty are applied is defined for each district j and year i in [14, Articles 23.1, 23.2 and 23.3] and can be modeled by a set of logical constraints over the value of J_{ij}^t. Introduce two binary variables δ_{ij}^- and δ_{ij}^+ defined as

$$\begin{aligned}
\left[\delta_{ij}^- = 1\right] &\leftrightarrow \left[S_{ij} - \mathcal{C}_{ij} \geq f_{ij}^-\right] \\
\left[\delta_{ij}^+ = 1\right] &\leftrightarrow \left[S_{ij} - \mathcal{C}_{ij} \geq f_{ij}^+\right].
\end{aligned} \tag{11}$$

The profit with dead band J_{ij}^f is given by

$$J_{ij}^f = (1 - \delta_{ij}^- + \delta_{ij}^+) J_{ij}^t. \tag{12}$$

Logic constraints (11) and the nonlinear relation (12) can be modelled by a set of mixed-integer linear constraints using standard procedures [34].

3.2.2 Maximum Incentive/Penalty

The maximum incentives and penalties are values defined in [14, Articles 23.6 and 23.7] that depend on the density ρ_j of the district and on the basic standard S_{ij}. In order to take such saturation constraints into account in the optimization model, a new set of auxiliary variables and constraints is needed.

Let the maximum incentive be J_{ij}^+ and the maximum penalty be $-J_{ij}^-$. The saturated version J_{ij}^s of J_{ij}^f lies in $[J_{ij}^-, J_{ij}^+]$, $\forall j \in \mathcal{D}$ and year $i \in \mathcal{Y}$. Introduce two variables $\gamma_{ij}^- \in \{0,1\}$ and $\gamma_{ij}^+ \in \{0,1\}$, defined as

$$\begin{aligned}
\left[\gamma_{ij}^- = 1\right] &\leftrightarrow \left[J_{ij}^f \geq J_{ij}^-\right] \\
\left[\gamma_{ij}^+ = 1\right] &\leftrightarrow \left[J_{ij}^f \geq J_{ij}^+\right].
\end{aligned} \tag{13}$$

Then, the saturated value function is defined as

$$J_{ij}^s = (1 - \gamma_{ij}^-) J_{ij}^- + (\gamma_{ij}^- - \gamma_{ij}^+) J_{ij}^f + \gamma_{ij}^+ J_{ij}^+. \tag{14}$$

Logic constraints (13) as well as the product $(\gamma_{ij}^- - \gamma_{ij}^+) J_{ij}^f$ of real and binary variables can be linearized by introducing an auxiliary continuous variable and a set of linear constraints, as described in [34].

3.2.3 Profit Definition

As mentioned in Section 2.2, discount rates and penalty cancellations for 2004-2005 must be taken into account. The latter requires the introduction of binary variables ζ_{ij} defining whether the CML level has reached the basic standard or not:

$$[\zeta_{ij} = 1] \leftrightarrow [\mathcal{C}_{ij} - S_{ij} \geq 0]. \tag{15}$$

The overall objective function J^{profit} is defined as

$$\begin{aligned}
&- \sum_{i \in \mathcal{Y}} \sum_{j \in \mathcal{D}} \sigma^{i-1} u_{ij}^c - \sum_{i \in \mathcal{Y}} \sum_{l \in \mathcal{I}} \sigma^{i-1} u_{il}^b C_{il}^{\text{up}} \\
&+ \sigma^2 J_{1j}^s (1 - \zeta_{1j}) + \sigma^3 J_{2j}^s (1 - \zeta_{2j}) + \sigma^4 J_{3j}^s + \sigma^5 J_{4j}^s \\
&+ \tfrac{1}{3} \sigma^2 J_{1j}^s \zeta_{1j} + \tfrac{1}{3} \sigma^3 J_{1j}^s \zeta_{1j} \zeta_{2j} + \tfrac{1}{3} \sigma^4 J_{1j}^s \zeta_{1j} \zeta_{3j} \\
&+ \tfrac{1}{3} \sigma^3 J_{2j}^s \zeta_{2j} + \tfrac{1}{3} \sigma^4 J_{2j}^s \zeta_{2j} \zeta_{3j} + \tfrac{1}{3} \sigma^5 J_{2j}^s \zeta_{2j} \zeta_{4j},
\end{aligned} \tag{16}$$

where $\sigma = \frac{1}{1+r/100}$ and C_l^{up} is the cost for upgrade l. The product between binary variables and expected penalties is again modelled by introducing a set of auxiliary continuous variables and mixed-integer linear constraints as in [34].

3.3 Input and State Constraints

The constraints on the money spent can represent any limits on u_{ij}^c and u_{ij}^b, for example "global" constraints on the overall money spent in different districts, different years, etc. In the case at hand, limited total budget and maximum budget over collections of districts (called "regions") are considered. Also, a constraint on the minimum CML level during the last year is imposed for each district, in order to prevent an excessive worsening of quality (in particular for design reasons the minimum level is equal to the basic standard of 2007). Note that other linear constraints (such as constraints on the maximum difference of quality levels between different districts) could be easily modeled, although they are not considered in our model, which is suited for the Italian normative [14] and include the following constraints:

$$\begin{aligned}
&\sum_{j \in \mathcal{D}} u_{ij}^c + \sum_{l \in \mathcal{I}} u_{il}^b C_l^{\text{up}} \leq \mathrm{U}_{\max}, \forall i \in \mathcal{Y} \\
&\sum_{j \in \mathrm{Reg}_h} u_{ij}^c \leq \mathrm{U}_{\max}^{\mathrm{Reg}_h}, \forall i \in \mathcal{Y}, \forall h \in \mathcal{R} \\
&\mathcal{C}_{4j} \geq S_{4j}, \forall j \in \mathcal{D}.
\end{aligned} \tag{17}$$

where Reg_h is the set of districts of region h, $h \in \mathcal{R} = \{1, \ldots, R\}$, and $\mathrm{U}_{\max}^{\mathrm{Reg}_h}$ is the corresponding yearly limit of investments.

4 Solution Strategies

The optimum control problem providing the desired investment allocation has been recast as the following optimization problem

$$\begin{aligned}
&\max_{\mathbf{u},\mathbf{z}} \; J^{\text{profit}} \\
&\text{s.t.} \;\; (5)\text{–}(17)
\end{aligned} \tag{18}$$

The optimization variables $\mathbf{u}, \mathbf{z}$ are listed in Table 3. These variables have been subdivided into two categories: "actual" decision variables $\mathbf{u} = \{u_{ij}^c, u_{il}^b\}$, that

Name	Description
$\mathbf{u}$	Decision variables (command inputs)
u^b_{il}	Investment decision on upgrade l in time period i [binary]
u^c_{ij}	Money invested in district j on period i [k€]
$\mathbf{z}$	Auxiliary variables
x^c_{ij}	Accumulated money invested in district j on period i [k€]
$\mathcal{C}_{ij}$	CML level in district j on period i [(min/yr)]
$\Delta\mathcal{C}_{ij}$	Accumulated CML level decrease in district j on period i [(min/yr)]
s^l_{ij}	Auxiliary variables for defining piecewise affine function $\Delta\mathcal{C}_{ij}(x^c_{ij})$ [k€]
α_{ij}	Auxiliary binary variable for defining piecewise affine function $\Delta\mathcal{C}_{ij}(x^c_{ij})$ [k€]
$\mathcal{B}_{ij}$	Mean biannual CML level in district j on period i [(min/yr)]
J^t_{ij}	Incentive/penalty [k€]
b^l_{ij}	Auxiliary variables for defining piecewise affine function J^t_{ij} [(min/yr)]
β^l_{ij}	Auxiliary variables for defining piecewise affine function J^t_{ij} [binary]
J^f_{ij}	Incentive/penalty taking into account the dead band [k€]
$\delta^\pm_{ij}$	Auxiliary variables for defining the dead band [binary]
J^s_{ij}	Incentive/penalty taking into account dead band and saturation [k€]
$\gamma^\pm_{ij}$	Auxiliary variables for defining the saturation [binary]
J^{profit}	Profit [k€]
J_{ij}	Auxiliary variable for defining the profit [k€]
ζ_{ij}	Auxiliary variables for defining the profit [binary]

Table 3
MILP formulation: optimization variables

constitute the investment policy over the prediction horizon, and the auxiliary variables $\mathbf{z}$ that are introduced to evaluate the dynamics and the cost function. Note that both $\mathbf{u}$ and $\mathbf{z}$ vectors have binary and real components. Using such a notation, the optimization problem (18) can be written as the following mixed-integer linear program (MILP)

$$\begin{aligned} \max_{\mathbf{u},\mathbf{z}} \ & c^T\mathbf{u} + f^T\mathbf{z} \\ \text{s.t.} \ & A\mathbf{u} \le b \\ & W\mathbf{z} \le h + T\mathbf{u} \\ & \mathbf{u} \in \mathbb{R}^{n_c} \times \{0,1\}^{n_c} \\ & \mathbf{z} \in \mathbb{R}^{m_c} \times \{0,1\}^{m_c}, \end{aligned} \tag{19}$$

Name	Description
$\mathcal{D}$	Set of districts $1,\dots,D$
$\mathcal{Y}$	Set of periods $0,\dots,N$
$\mathcal{I}$	Set of network upgrade investments $1,\dots,I$
$\mathcal{R}$	Set of regions $1,\dots,R$
C^{up}_l	The cost of each investment l [k€]
ΔI_{lj}	CML decrease in district j if upgrade l is realized [(min/yr)]
d^h_j,l^h_j	Parameters of the improvement investment of district j [k€],[(min/yr)/k€]
S_{ij}	Basic standards for each district j and period i [(min/yr)]
t^l_{ij}	Minutes of S_{ij} over segment l [(min/yr)]
$\mathcal{C}_{0j}$	Initial CML level (year 2003) [(min/yr)]
ρ_j	Density of the district ["High", "Medium" or "Low"]
P^D_j	Power of non domestic users for each district j [MWh]
P^N_j	Power of domestic users for each district j [MWh]
f^-_{ij},f^+_{ij}	Size of dead band [(min/yr)]
J^+_{ij},$-J^-_{ij}$	Saturation limits [k€]
r	Discount rate [%]
C^h_1,C^h_2	Parameters in k€/(MW(min/yr))
$sg_{1\rho_j}, sg_{2\rho_j}$	Limits of each segment for a given density ρ_j [(min/yr)]
$\text{U}_{\max}$	Maximum total budget for each year [k€]
Reg_h	Set of districts in region h [$\text{Reg}_h \subseteq 2^{\mathcal{D}}$]
$\text{U}^{\text{Reg}_h}_{\max}$	Maximum budget over region h for each year [k€]

Table 4
Problem parameters

where matrices c, f, A, b, W, h, T depend on the parameters listed in Table 4.

The constraints in (19) are divided into two categories: (i) constraints that affect only the decision variables $\mathbf{u}$, and (ii) the remaining constraints. The first ones are the constraints on budget and upgrade projects. The rest of the constraints model all the auxiliary constraints to define the dynamics of the system and the cost function, as well as state constraints on the CML level. Such a categorization of constraints will be important in Section 5 where uncertainty is taken into account. We will refer to problem (19) as the "deterministic optimization problem", as we are assuming that all its parameters are known. A "stochastic optimization model" that takes into account possible uncertainties in the parameters is presented in Section 5.

4.1 Pure MILP Approach

Problem (19) can be solved by standard MILP solvers, such as GLPK [21] (public domain), Cplex [15] (commercial), or Xpress-MP [10] (commercial), for which Matlab interfaces are available at `http://www.dii.unisi.it/hybrid/tools.html`.

Although the aforementioned MILP solvers are very efficient, MILP is still an $\mathcal{NP}$-hard problem, and its complexity is in general exponential with the number of binary variables. The size of Problem (19) grows linearly with the number N of time periods and the number D of districts. In many cases of practical relevance, the problem may be too large to be solved to optimality. However, most MILP solvers are based on branch and bound algorithms which may give good suboptimal feasible solutions after reaching a predefined time or memory limit. In the study presented in Section 6, the MILP problem cannot be solved to optimality on a standard PC because the solver runs out of memory. In order to handle such complex cases, next section provides a heuristic approach to get good solutions to Problem (19) with limited computations.

4.2 Heuristic Approach

One of the most widely used heuristics are greedy algorithms. The idea is to start from a simple solution, change relevant variables sequentially, each time selecting the variable that achieves the greatest immediate improvement in the objective function. Moreover, once the value of a variable is fixed it is not changed further.

In the case at hand, the algorithm starts from the initial feasible solution where no investment is done, i.e. $u^c_{ij}(0) = 0$ and $u^b_{il}(0) = 0$ for all periods $i \in \mathcal{Y}$, districts $j \in \mathcal{D}$ and upgrades $l \in \mathcal{I}$. The initial profit, $J^{\text{profit}}(0)$, is the predicted profit of this policy. The algorithm keeps track of the money that has not been invested (remaining budget) through variables $\mathrm{U}^{\max}_i(k)$. At iteration k of the greedy algorithm, the value of $\mathrm{U}^{\max}_i(k)$ is the money left for period i. The initial money constraint is equal to the maximum budget, $\mathrm{U}^{\max}_i(0) = \mathrm{U}^{\max}$ for all $i \in \mathcal{Y}$.

At each iteration k, an investment U is added to the current solution and the cost of U is subtracted from the remaining budget. The added investment U is chosen according to an optimality index, which is the ratio between the corresponding obtained profit and the cost of investment U. The investment with the highest optimality index is chosen among the local improvement projects of each district and the different network upgrades. For each district j for which an investment has not been fixed yet, the following MILP problem determines the obtained profit and the corresponding optimal investment:

$$
\begin{aligned}
J^{\text{profit}}_j(k) = \max\ & J^{\text{profit}} \\
\text{s.t.}\ & (5)-(17) \\
& u^c_{ih} = u^c_{ih}(k-1),\ \forall i \in \mathcal{Y},\ \forall h \neq j \in \mathcal{D} \\
& u^b_{il} = u^b_{il}(k-1),\ \forall i \in \mathcal{Y},\ \forall l \in \mathcal{I} \\
& u^c_{ij} \leq \mathrm{U}^{\max}_i(k-1), \forall i \in \mathcal{Y}.
\end{aligned}
\tag{20}
$$

Problem (20) evaluates the profit $J^{\text{profit}}_j(k)$ of the best investment that can be done in district j, with the remaining budget after $k-1$ iterations. The optimality index for a given investment option in district j is then given by

$$
\mathrm{I}_j(k) = \frac{J^{\text{profit}}_j(k) - J^{\text{profit}}(k-1)}{\sum_{i\in\mathcal{Y}} \beta^{i-1} u^c_{ij}}. \tag{21}
$$

For each investment project l that is not fixed, the following MILP problem is solved:

$$
\begin{aligned}
J^{\text{profit}}_l(k) = \max\ & J^{\text{profit}} \\
\text{s.t.}\ & (5)-(17) \\
& u^c_{ij} = u^c_{ih}(k-1),\ \forall i \in \mathcal{Y},\ \forall j \in \mathcal{D} \\
& u^b_{ih} = u^b_{il}(k-1),\ \forall i \in \mathcal{Y},\ \forall h \neq l \in \mathcal{I} \\
& u^b_{il} C^{\text{up}}_l \leq \mathrm{U}^{\max}_i(k-1), \forall i \in \mathcal{Y}.
\end{aligned}
\tag{22}
$$

The optimality index for a given upgrade investment l in the network is

$$
\mathrm{I}_l(k) = \frac{J^{\text{profit}}_l(k) - J^{\text{profit}}(k-1)}{\sum_{i\in\mathcal{Y}} \beta^{i-1} u^b_{il} C^{\text{up}}_l}. \tag{23}
$$

Note that optimization problems (20) and (22) are easy to solve because most of the the decision variables (i.e. the inputs to the system) are fixed. In particular, as the dynamics of each district are independent, only the variables and constraints of that district have to be taken into account to solve the corresponding proboem. In the case of network upgrades, the optimization of each of the districts affected by the investment must be taken into account. At each iteration the investment project with the maximum optimality index is chosen and the current solution and budget is updated as

$$
\mathrm{I}^*(k) = \max_{j\in\mathcal{D}, l\in\mathcal{I}} \{\mathrm{I}_j(k), \mathrm{I}_l(k)\}
$$

If $\mathrm{I}_j(k) = \mathrm{I}^*(k)$ then

$$
\begin{aligned}
&J^{\text{profit}}(k) = J_j^{\text{profit}}(k) \\
&\mathrm{U}_i^{\max}(k) = \mathrm{U}_i^{\max}(k-1) - u_{ij}^c, \quad \forall i \in \mathcal{Y} \\
&u_{ih}^c(k) = u_{ih}^c(k-1), \quad \forall i \in \mathcal{Y}, \forall h \neq j \in \mathcal{D} \\
&u_{ij}^c(k) = u_{ij}^c, \quad \forall i \in \mathcal{Y} \\
&u_{il}^b(k) = u_{ih}^b(k-1), \quad \forall i \in \mathcal{Y}, \forall l \in \mathcal{I}
\end{aligned}
$$

otherwise, if $\mathrm{I}_l(k) = \mathrm{I}^*(k)$,

$$
\begin{aligned}
&J^{\text{profit}}(k) = J_l^{\text{profit}}(k) \\
&\mathrm{U}_i^{\max}(k) = \mathrm{U}_i^{\max}(k-1) - u_{il}^b C I_l, \quad \forall i \in \mathcal{Y} \\
&u_{ij}^c(k) = u_{ih}^c(k-1), \quad \forall i \in \mathcal{Y}, \forall j \in \mathcal{D} \\
&u_{ih}^b(k) = u_{ih}^b(k-1), \quad \forall i \in \mathcal{Y}, \forall h \neq l \in \mathcal{I} \\
&u_{il}^b(k) = u_{il}^b, \quad \forall i \in \mathcal{Y}.
\end{aligned}
$$

The algorithm iterates until the solution does not improve any more, i.e., $J^{\text{profit}}(k) = J^{\text{profit}}(k-1)$. The feasible suboptimal solution is given by $\{u_{ij}^c(k), u_{lj}^b(k)\}$.

The heuristic algorithm presented above only takes into account total budget constraints. To add maximum investment on a given region $h \in \mathcal{R}$ additional variables can be used to carry the remaining budget of the region, as done with variables $\mathrm{U}_i^{\max}(k)$ for the global constraint.

5 Stochastic Optimal Control

A large number of problems in production planning and scheduling, location, transportation, finance, and engineering design require that decisions be made in the presence of uncertainty. Uncertainty, for instance, governs the prices of fuels, the availability of electricity, and the demand for chemicals. In the problem tackled in this paper uncertainty affects the function in (2), which is depicted in Figure 1, between quality of supply and money invested in improvement projects.

Stochastic programming (SP) is a special class of mathematical programming that involves optimization under uncertainty (see [6, 17, 27]). The original applications were agricultural economics, aircraft route planning and production of heating oil back in the 50's. Nowadays SP is becoming a mature theory that is successfully applied in several other application domains, see for instance the survey [28]. For other contributions in control theory of SP techniques the reader is referred to [2, 13, 18]. From the computational viewpoint specific efficient algorithms for stochastic LP and QP are available in the literature (see for example [5, 9, 26, 32]) and commercial solutions to SP were announced recently [11]. In this section a two-stage stochastic integer programming formulation is proposed for the investment problem described in previous sections.

The increase of quality caused by an investment project is very difficult to predict. We suppose a 20% error margin in the predictions: For each district j and upgrade project l we define a pair of independent continuous random variables $\xi_i, \xi_l \in [0.8, 1.2]$. These random variables model the possible error in the forecasted quality increase of each project

$$
\begin{aligned}
\Delta\mathcal{C}_{ij} &= \xi_j f(x_{ij}^c), \\
\Delta I_{lj}(\xi_l) &= \xi_l \Delta I_{lj}.
\end{aligned}
$$

The uncertainty affects the definition of the CML of each district, and therefore the hybrid dynamical model (1)–(3) of the system. As a consequence, constraints (7) and (8) are directly affected by uncertainty. Moreover, all the variables (and so the constraints) defined to evaluate the profit depend on the quality level of each district and period, so they are also affected indirectly by uncertainty.

Stochastic programming optimizes the mean value of the cost function taking into account *causality*. This means that the decision on how to invest the available money must be done "before" knowing the real effect of the investment projects, i.e., before knowing the value of the random variables ξ and so the actual evolution of the CML of the districts. In this problem we propose to use a two-stage model [6]. The investments along the whole prediction horizon are decided as first stage variables that must be decided without any prior knowledge. The recourse variables are the state variables and the auxiliary variables needed to evaluate the expected profit and are decided "a posteriori", once the value of the uncertainty is known.

Using the notation presented in previous sections the stochastic MILP problem is defined as

$$
\begin{aligned}
\max_{\mathbf{u}} \ & c^T\mathbf{u} + E[\max_{\mathbf{z}} f(\Xi)^T\mathbf{z}|\Xi] \\
\text{s.t.} \ & A\mathbf{u} \leq b \\
& W(\Xi)\mathbf{z} \leq h(\Xi) + T(\Xi)\mathbf{u}, \ \forall \Xi \\
& \mathbf{u} \in \mathbb{R}^{n_c} \times \{0,1\}^{n_c} \\
& \mathbf{z} \in \mathbb{R}^{m_c} \times \{0,1\}^{m_c},
\end{aligned} \tag{24}
$$

where $\Xi = \{\xi_i, \xi_l\}$ collects all the random variables associated with uncertainty of the relation between improvement/upgrade investments and CML. Note that the constraints on the budget and the discount rate are deterministic parameters, so matrices c, A and b are fixed because they do not depend on CML, and hence on Ξ.

In general, uncertainty is best modeled as a continuous random variable. However, in complex practical

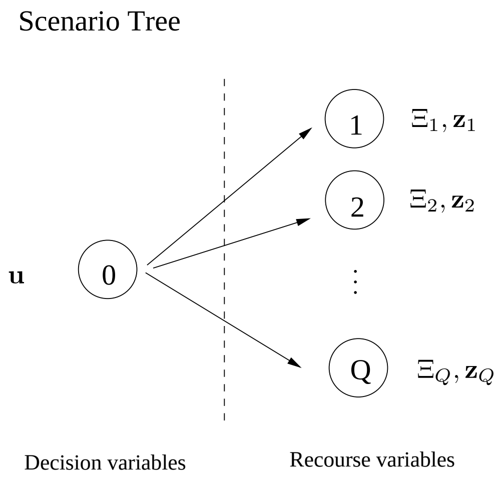

Fig. 4. Tree with Q different scenarios.

problems, continuous probability distributions are often too difficult to handle from a computational viewpoint. Therefore, discrete probability measures have a prominent role in approaches based on stochastic programming. Besides turning integrals into sums, discrete distributions allow equivalent representations of optimization models as block-structured large scaled deterministic optimization mathematical programs.

We resort to sampling each of the independent continuous distributions $\xi_{j,l}$. The number q of possible values of each uncertain variable determines the complexity of the stochastic optimal control problem. For the study tackled in Section 6 we have considered that $\{\xi_i, \xi_l\}$ take values on $\{0.8, 0.9, 1, 1.1, 1.2\}$ with equal probabilities $p = 0.2$, i.e., we consider $q = 5$ possible realizations. In this way, as each variable in independent, the uncertain variable Ξ may take values $\Xi_1, \ldots, \Xi_Q$ with probabilities $p_1, \ldots, p_Q$ respectively, where $Q = q^{D+I}$.

For each fixed value of the uncertainty Ξ_i, referred to as *scenario*, all problem parameters $f(\Xi_i)$, $T(\Xi_i)$, $W(\Xi_i)$, $h(\Xi_i)$ become fixed. By enumerating all possible Q scenarios, a large scale MILP problem can be posed. To each scenario, a set of "recourse" auxiliary variables $\mathbf{z}_i$ are assigned, but the problem optimizes only a single set of decision variables $\mathbf{u}$. In this way, the causality of the decision process is maintained, as illustrated in Figure 4. The (large scale) equivalent MILP problem (24) is

$$\begin{aligned} \max_{\mathbf{u},\mathbf{z}_i} \; & c^T\mathbf{u} + \sum_{i=1}^{Q} p_i f(\Xi_i)^T \mathbf{z}_i \\ \text{s.t.} \; & A\mathbf{u} \le b \\ & W(\Xi_i)\mathbf{z}_i \le h(\Xi_i) + T(\Xi_i)\mathbf{u}, \quad i = 1, \ldots, Q \\ & \mathbf{u} \in \mathbb{R}^{n_c} \times \{0,1\}^{n_c} \\ & \mathbf{z}_i \in \mathbb{R}^{m_c} \times \{0,1\}^{m_c}, \quad i = 1, \ldots, Q. \end{aligned} \tag{25}$$

5.1 Solution Strategies

The complexity of the two-stage stochastic problem (25) heavily depends on the number Q of scenarios. In the optimization model proposed in this paper the uncertainty affects investment projects. If each project is supposed to be independent, and the uncertainty is supposed to take q different values, the number of scenarios of the optimization problem is $Q = q^{D+I}$. Although stochastic programming for linear and quadratic problems is nowadays a mature field, stochastic MILP problems are still in general hard to solve, we refer again to [28] for a complete survey of the state of the art.

Given that even the deterministic MILP approach (19) is too complex to handle instances of the investment allocation problem if interest, we avoid solving (25) using stochastic MILP solution techniques, but rather propose again a heuristic algorithm. By decoupling each investment project using the heuristic approach of Section 4.2, one can deal with each random pair $\{\xi_i, \xi_l\}$ independently. Each local improvement project can be dealt without taking into account the realization of the random variables associated with other districts, as such variables only affect the corresponding CML, and hence evaluating the investment project of district j only requires the enumeration of q scenarios, corresponding to the possible realizations of ξ_j. Districts affected by upgrade projects must take into account also the uncertainty associated with upgrade projects, so the number of scenarios may be larger than q.

Having reduced the number of scenarios in each smaller subproblem associated with a single district, one can use an MILP formulation to solve the stochastic equivalent of the optimization problems described in Section 4.2. In this way, it is possible to obtain a feasible suboptimal investment policy that takes into account the uncertain nature of each investment project. Note that more efficient solvers [29–31] of two-stage stochastic integer problems may be used here in the heuristic algorithm.

6 A Case Study

We apply the optimal control approach developed in the previous sections for solving a real-life investment allocation problem. We consider three different problems [1]:

- P9: System with 9 districts;
- P18: System with 18 districts and 5 network upgrade projects;
- P36: System with 36 districts.

[1] Numerical data were generated by perturbing actual confidential data provided by the Italian electrical company ENEL.

The data defining P18 are provided in Tables 5, 6, 8, and 9. Other parameters that are not provided in these tables (such as dead band values) can be obtained from [14]. Table 6 shows the parameters that define the districts of P18. The districts are divided into $R = 12$ regions. In the optimization problem, maximum investment constraints on some of these regions are also taken into account. Budget constraints are defined in Table 5. The parameters for each local improvement project are given in Table 8. The cost and quality increase parameters of network upgrade investments are given in Table 9. The data of P9 and 36 are not provided for lack of space. Total investment constraints of 500 and 5000 [k€], respectively, are considered.

In the three cases, deviations of up to 20% of estimated CML predictions in all districts are considered as explained in Section 5. The uncertain variables take five possible values ($\{0.8, 0.9, 1, 1.1, 1.2\}$) with equal probability ($p = 0.2$). It is also taken into account that there is a probability $p = 0.2$ that some of the investments fail during the first two years for unexpected reasons.

Table 10 shows the profit and computation time of the different solutions strategies developed in the previous sections for both the deterministic and the stochastic case [2].

For the deterministic problem, the optimal solution has been obtained only for P9. In this case, the MILP solver of CPLEX obtains the optimal investment policy even faster than the greedy algorithm of Section 4.2. For P18 the computer runs out of memory after 2300 s while solving the MILP problem (19) to optimality. The solutions obtained by CPLEX after 200 and 2000 s are shown. For P36, the computer runs out of memory after 1100 s. The reported solution is the one obtained after 1000 s of CPU time. The heuristic algorithm is instead very fast and converges in less that 20 s for P36. Although suboptimal, the obtained solution is considered a valid decision by the company.

The optimization problem associated with the stochastic formulation is in all cases too large to be solved by one large MILP. A suboptimal solution is obtained by using the greedy approach of Sections 4.2, 5.1.

We remark that the decision making problem is quite complex to be solved by hand, and in any case the optimal solutions, obtained using any of the optimization methods mentioned above, are of great use to the investment department, as they give insights on directions to take to pursue the most profitable investments.

[2] The results were obtained in MATLAB 6.3 using CPLEX 9.0 on a AMD Athlon™XP 2800+ with 512 MB of RAM.

U_{max}	$U_{max}^{Reg_2}$	$U_{max}^{Reg_6}$	$U_{max}^{Reg_{11}}$
1000	400	500	400

Table 5
Budget for P18: U_{max} is the total amount of money available each year and $U_{max}^{Reg_h}$ the maximum budget available each year for region h, from 2004 to 2007 [k€]

District	Reg.	CML_{03}	P^D	P^N	ρ_j
3	1	203.4	7.9	6	Low
4	2	40.3	27.6	11.1	High
5	2	91.6	70.9	16.5	Medium
6	2	125.5	15.5	3	Low
8	3	90.3	23.4	8	Medium
9	3	130.4	10	5.8	Low
11	4	80	194.3	60.8	Medium
12	4	139.1	24.3	5.6	Low
17	6	78.9	93.8	32.5	Medium
18	6	113.9	15	5.6	Low
20	7	85.9	53.4	20.1	Medium
23	8	72.1	53.7	18.6	Medium
27	9	86.3	16.2	9.3	Low
29	10	53.8	133.9	42.9	Medium
30	10	65.6	38.3	17.4	Low
31	11	60.6	19.3	6	Medium
32	11	86	17.3	8.4	Low
34	12	67.5	46.1	6.6	Medium

Table 6
District parameters for P18: "Region" denotes the region $h \in \mathcal{R}$ the district belongs to, "CML(2003)" is the initial CML in [min/yr], P^D and P^N [MWh] are the power served to domestic and non-domestic clients, respectively, "Density" is the density of the district

7 Conclusions

In this paper we have proposed a novel application of optimal control of hybrid dynamical systems for solving a management problem in allocation of investments for maintenance of the electrical distribution network in a territory composed by several districts. Because of non-trivial national regulations and of complex relations between invested money and resulting quality of supply, allocating investments for both local improvements and major upgrades is a complex problem, usually solved heuristically after a series of tedious iterations and without any guarantee of having taken the best decision. We have shown that an optimal control setup and optimization tools provide a systematic way of making the best (or at least good) investment allocation. We also have considered a stochastic mixed-integer linear programming formulation for taking into account the uncertainty associated with the relation between investments

District	l^1	l^2	l^3	l^4
3	205.7	845.2	914.1	1000
4	648.6	1000	1000	1000
5	806.1	806.1	883.6	1000
6	385.5	699.3	833.2	1000
8	452.3	809.3	915.3	1000
9	259.1	505.2	609.4	1000
11	496.5	755.9	849.1	1000
12	643.4	860.7	910.8	1000
17	337.2	395.3	717	1000
18	170.6	737.8	1000	1000
20	535.1	653.6	839.2	1000
23	344.1	525.6	686.4	1000
27	117.1	176.5	575.3	1000
29	561.8	887.5	1000	1000
30	214.3	495.5	527.1	1000
31	827	900.9	1000	1000
32	809.1	885.7	885.7	1000
34	471.2	753.7	863.5	1000

Table 7
Parameters of improvement investments for P18: l^h is the h-th stage cost [k€] of the local improvement project, d^h is the gain on the h-th stage [min/k€] (see Figure 1)

District	d^1	d^2	d^3	d^4
3	0.0887	0.0422	0.0470	0.0377
4	0.0068	0.0286	0	0
5	0.0131	0	0.0118	0.0081
6	0.0586	0.0410	0.0350	0.0281
8	0.0391	0.0341	0.0270	0.0287
9	0.0593	0.1138	0.1225	0.0803
11	0.0050	0.0044	0.0070	0.0029
12	0.0417	0.0410	0.0354	0.0341
17	0.0095	0.0234	0.0066	0.0063
18	0.0368	0.0199	0.0183	0
20	0.0104	0.0037	0.0053	0.0040
23	0.0024	0.0024	0.0029	0.0023
27	0.0161	0.0144	0.0082	0.0041
29	0.0023	0.0022	0.0042	0
30	0.0089	0.0064	0.0229	0.0013
31	0.0060	0.0106	0.0096	0
32	0.0115	0.0408	0	0.0077
34	0.0186	0.0168	0.0158	0.0127

Table 8
Parameters of improvement investments for P18: l^h is the h-th stage cost [k€] of the local improvement project, d^h is the gain on the h-th stage [min/k€] (see Figure 1)

Upgrade	13	6	8	2	5
Cost [k€]	703	572	932	662	311
District	ΔI_{lj}	ΔI_{lj}	ΔI_{lj}	ΔI_{lj}	ΔI_{lj}
4			0.2		
5		6	5		
11		1.6			
17		0.9			
18				6.5	
20	1				
23	0.1				
27					4.4
29	0.4				
30	0.5				

Table 9
Parameters of upgrade investments for P18: "Cost" is the cost of the upgrade project, ΔI_{lj} is the quality increase [(min/yr)] for each district. Note that not all the districts are affected by upgrade investments

and benefits, and provided heuristic algorithms for solving the problem suboptimally, that are especially useful when computation time and/or memory constraints are imposed.

The strategies described in this paper are not those that are currently employed by the electrical distribution company "Enel Distribuzione". However, the company is currently considering to apply such strategies for the allocation plan of next year.

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P9	J_{det} [K€]	$E[J_{stc}]$ [K€]	time [s]
MILP	1829	1534	1.7
Greedy	1807	1363	2.3
SP (Greedy)	1649	1603	72.17
P18	J_{det}	$E[J_{stc}]$	time [s]
MILP 200s.	3445	2343	200
MILP 2000s.	3983	2680	2000
Greedy	3795	2530	7.2
SP (Greedy)	3450	3332	125.3
P36	J_{det}	$E[J_{stc}]$	time [s]
MILP 1000s.	13079	12185	1000
Greedy	13021	12285	18.6
SP (Greedy)	12861	12306	563.3

Table 10
Optimization results for P9, P18, P36 and for different solution strategies (MATLAB 6.3 and CPLEX 9.0 on a AMD Athlon(tm) XP 2800+ with 512 MB of RAM): J_{det} is the profit obtained for the deterministic problem, $E[J_{stc}]$ is the expected profit for the stochastic formulation, "time" is the time for computing the solution

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