# Suboptimal Explicit RHC via Approximate Multiparametric Quadratic Programming

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**Abstract**

Algorithms for solving multiparametric quadratic programming (mp-QP) were recently proposed in [1, 2] for computing explicit Receding Horizon Control (RHC) laws for linear systems subject to linear constraints on input and state variables. The reason for this interest is that the solution to mp-QP is a piecewise affine function of the state vector and thus it is easily implementable on-line. The main drawback of solving mp-QP *exactly* is that whenever the number of linear constraints involved in the optimization problem increases, the number of polyhedral cells in the piecewise affine partition of the parameter space may increase exponentially. In this paper we address the problem of finding *approximate* solutions to mp-QP, where the degree of approximation is arbitrary and allows to trade off between optimality and a smaller number of cells in the piecewise affine solution. We provide analytic formulas for bounding the errors on the optimal value and the optimizer, and for guaranteeing that the resulting suboptimal RHC law provides closed-loop stability and constraint fulfillment.

**Keywords**

Receding horizon control, model predictive control, multiparametric programming, convex quadratic programming, error bounds, piecewise linear control.

## 1 Introduction

In industrial applications the design of feedback controllers must often cope with the presence of constraints over actuators and other process components. Such constraints must be adequately handled by the control design. Receding Horizon Control (RHC) (also referred to as Model Predictive Control —MPC—, especially in industry), has become the accepted standard for complex constrained multivariable control problems in the process

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industries [3]. Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. At the next time step the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy. The solution relies on a linear dynamic model, respects all input and output constraints, and optimizes a quadratic performance index.

Although RHC has long been recognized as the winning alternative for constrained systems, due to the considerable on-line computation effort its applicability has been limited to relatively slow systems.

For RHC based on linear prediction models and a quadratic performance index, in [1] the authors proposed a new approach to move off-line all the computations necessary for the implementation of RHC while preserving all its other characteristics. The approach consists of solving off-line the optimization problem associated with RHC for all the expected measurement values by using *multiparametric quadratic programming* (mp-QP) solvers. The resulting feedback controller inherits all the stability and performance properties of linear RHC, and is piecewise affine. For this reason, the on-line computation associated with *explicit* RHC controllers reduces to a function evaluation of a piecewise affine mapping. The approach is therefore extremely promising, as it broadens the scope of applicability of linear RHC to small-size/fast-sampling applications. Alternative approaches for obtaining explicit RHC solutions were investigated in [4, 5].

The problem of reducing on-line computation, although addressed by explicit RHC techniques, is not yet solved. In fact, whenever the number of constraints involved in the optimization problem increases, the number of linear gains associated with the piecewise affine control algorithm may increase exponentially, which still makes the on-line implementation of the piecewise affine controller prohibitive on low-cost control hardware.

The technique proposed in [4] attempts to reduce complexity by reducing *a priori* the allowed combinations of active constraints, based on engineering insight on the control problem.

In this paper we propose a new algorithm for reducing the complexity of the explicit RHC controller, by computing *suboptimal* solutions to the multiparametric quadratic problem. The idea is based on relaxing the first order Karush-Kuhn-Tucker (KKT) optimality conditions (except primal feasibility, so that the computed move is feasible) by some arbitrary degree $\epsilon$, which serves as a design knob for tuning the complexity of the controller. We show that for $\epsilon \to \infty$ the complexity of the controller is reduced to an affine control law (highly suboptimal), while for $\epsilon \to 0$ the controller converges to the explicit RHC controller [1] (fully optimal with respect to the chosen performance index). We analyze a general relaxation scheme where all KKT conditions (except primal feasibility) may be relaxed, and a particular one where only dual feasibility is relaxed. For the general perturbation scheme, we show how to compute *a posteriori* the maximum error between the optimizer and the suboptimizer. For the particular perturbation scheme, we also provide a criterion for choosing $\epsilon$ so that the distances between the optimal and the suboptimal value, and between the exact and the approximate solution are bounded *a priori*, and so that the resulting suboptimal RHC law provides closed-loop stability and constraint fulfillment.

## 2 Receding Horizon Control

We start by briefly reviewing basic facts on RHC and mp-QP (see [1] for details). Consider the discrete-time linear time invariant system

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u(t) \tag{1}$$

where $x \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^{n_u}$ are the state and input vectors respectively, and the pair $(\mathcal{A}, \mathcal{B})$ is stabilizable. Consider the problem of regulating the state $x(t)$ to the origin while fulfilling the constraints

$$D_1x(t) + D_2u(t) \leq d \tag{2}$$

at all time instants $t \geq 0$, where $d$ has strictly positive components. Assume that a full measurement of the state $x(t)$ is available at the current time $t$. Then, the optimization problem

$$\begin{array}{ll} \min\limits_{U} & x_T'Px_T + \sum\limits_{k=0}^{T-1} [x_k'Qx_k + u_k'Ru_k] \\ \text{subj. to} & D_1x_k + D_2u_k \leq d,\ k = 0, \dots, T-1 \\ & x_T \in \Omega \\ & x_{k+1} = \mathcal{A}x_k + \mathcal{B}u_k,\ k = 0, \dots, T-1 \\ & x_0 = x(t) \end{array} \tag{3}$$

is solved with respect to the column vector $U \triangleq [u_0' \ \dots \ u_{T-1}']' \in \mathbb{R}^r$, $r \triangleq n_uT$, at each time $t$, where $x_k$ denotes the predicted state vector at time $t+k$, obtained by applying the input sequence $u_0, \dots, u_{k-1}$ to model (1) starting from the state $x(t)$. In (3), we assume that $Q$ and $R$ are symmetric and positive definite matrices, $P$ is symmetric and nonnegative definite, and the set of terminal states $\Omega$ is polyhedral and contains the origin.

The MPC control law is based on the following idea: At time $t$ compute the optimizer $U^*(x(t)) = [(u_0^*)' \ \dots \ (u_{T-1}^*)']'$ to problem (3), apply

$$u(t) = u_0^* = I^1U^*(x(t)), \qquad I^1 \triangleq [I_{n_u}\ 0\ \dots\ 0] \tag{4}$$

as input to system (1), and repeat the optimization (3) at the next time step $t+1$, based on the new measured (or estimated) state $x(t+1)$. By substituting $x_k = \mathcal{A}^kx(t) + \sum_{j=0}^{k-1} \mathcal{A}^j\mathcal{B}u_{k-1-j}$ in (3), this can be written as

$$\begin{array}{ll} \min\limits_{U} & \frac{1}{2}U'HU + x'(t)C'U + \frac{1}{2}x'(t)Yx(t) \\ \text{subj. to} & AU \leq b + Fx(t), \end{array} \tag{5}$$

where $H = H'$ is positive definite, and $H$, $C$, $Y$, $A$, $b$, $F$ are easily obtained from (3), $H \in \mathbb{R}^{r\times r}$, $C \in \mathbb{R}^{r\times n}$, $Y \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{q\times r}$, $b \in \mathbb{R}^q$, $F \in \mathbb{R}^{q\times n}$. The optimization problem (5) is a quadratic program (QP) which depends on the current state $x(t)$, and therefore is a *multiparametric quadratic* program (mp-QP).

# 3 Multi-Parametric Quadratic Programming

Consider the optimization problem

$$(QP_x) \qquad \begin{array}{ll} \min\limits_{U} & \frac{1}{2}U'HU + x'C'U + \frac{1}{2}x'Yx \\ \text{subj. to} & AU \leq b + Fx, \end{array} \tag{6}$$

where $U \in \mathbb{R}^r$ is the vector of decision variables, $x \in \mathbb{R}^n$ is the parameter vector (for ease of notation, we removed the explicit reference to time), and all data are defined as in Section 2. As only the optimizer $U^*$ is needed, the term involving $Y$ is usually removed from (6). Here we retain such term since it is essential to the arguments of Subsection 5.2. We say that a parameter vector $x$ is *feasible* if the corresponding problem $(QP_x)$ admits a solution, i.e., there exists a vector $U$ satisfying the constraints of $(QP_x)$; we denote by $X_f \subseteq \mathbb{R}^n$ the set of feasible parameter vectors. As a consequence of the definition, $X_f$ is the orthogonal projection onto the $x$-space of the polyhedron $\{(U, x) : AU - Fx \leq b\}$, i.e., $X_f = \{x : \exists U \text{ s.t. } AU - Fx \leq b\}$. Thus $X_f$ is a convex polyhedron. Let $\phi^* : X_f \mapsto \mathbb{R}$ denote the *value function*, which associates with every $x \in X_f$ the optimal value of $(QP_x)$. As $H$ is positive definite, for every $x \in X_f$ the corresponding quadratic program has a unique optimal solution.

Multiparametric quadratic programming (mp-QP) amounts to determining the optimal solution $U^*$, and the value function $\phi^*$ as explicit functions of $x$, for all $x \in X_f$.

Let the rows of $A$ be indexed by $M \triangleq \{1, 2, \ldots, q\}$. For any $N \subseteq M$ we denote by $A_N$ the submatrix of $A$ consisting of rows indexed by $N$. Analogously, if $s \in \mathbb{R}^q$ then we denote by $s_N$ the subvector of $s$ consisting of entries indexed by $N$. Finally, we recall that a constraint of $(QP_x)$ is *active* at a vector $U$ if it is satisfied as an equality by $U$.

**Definition 1** *Let $U^*(x)$ be the optimal solution of $(QP_x)$. The* optimal partition *associated with $x$ is the partition $(B(x), N(x))$ of $M$ where $N(x)$ is the index set of active constraints at $U^*(x)$ and $B(x) = M \setminus N(x)$.*

**Definition 2** *Let $(B, N) = (B(x), N(x))$ for some $x \in X_f$. We call* critical region *associated with $(B, N)$ the set of parameters $CR^* \triangleq \{x \in X_f : N(x) = N\}$.*

The following result can be found in [1, Theorem 2]. In view of the following development we restate here its proof.

**Theorem 1** *Let $H$ be positive definite. Let $(B, N)$ be an optimal partition, and let $CR^*$ be the associated critical region. Assume that the rows of $A_N$ are linearly independent. Then, the optimizer $U^*$ and the associated vector of Lagrange multipliers $\lambda^*$ are the following, uniquely defined, affine functions of $x$ over $CR^*$: $U^*(x) = Z_U x + \zeta_U$, $\lambda_N^*(x) = Z_\lambda x + \zeta_\lambda$, $\lambda_B^*(x) = 0$, where $Z_\lambda = -(A_N H^{-1} A_N')^{-1}(F_N + A_N H^{-1} C)$, $\zeta_\lambda = -(A_N H^{-1} A_N')^{-1} b_N$, $Z_U = -H^{-1} A_N' Z_\lambda - H^{-1} C$, $\zeta_U = -H^{-1} A_N' \zeta_\lambda$.*

*Proof:* Once an optimal partition $(B, N)$ is fixed, the first-order KKT conditions for problem $(QP_x)$ may be written as follows [6, p. 504]:

$$HU + Cx + A'\lambda = 0, \tag{7a}$$
$$A_B U + s_B = b_B + F_B x, \ \ s_B \geq 0, \tag{7b}$$
$$A_N U + s_N = b_N + F_N x, \ \ s_N = 0, \tag{7c}$$
$$\lambda_B = 0, \tag{7d}$$
$$\lambda_N \geq 0. \tag{7e}$$

where $\lambda \in \mathbb{R}^q$ is the vector of Lagrange multipliers, and $s_B$, $s_N$ are a partition of the vector of primal slack variables $s \in \mathbb{R}^q$. We solve (7a) for $U$,

$$U = -H^{-1}(A_N' \lambda_N + Cx) \tag{8}$$

and substitute the result into (7c), getting $-A_N H^{-1}(A_N' \lambda_N + Cx) - b_N - F_N x = 0$. Assuming that $A_N$ is full row rank, $(A_N H^{-1} A_N')^{-1}$ exists and therefore we obtain

$$\lambda_N = -(A_N H^{-1} A_N')^{-1}(b_N + (F_N + A_N H^{-1} C)x). \tag{9}$$

Thus $\lambda$ is an affine function of $x$. We can substitute $\lambda_N$ from (9) into (8) to obtain

$$U = H^{-1} A_N' (A_N H^{-1} A_N')^{-1}(b_N + (F_N + A_N H^{-1} C)x) - H^{-1} C x \tag{10}$$

and note that $U$ is also an affine function of $x$. Relations (9) and (10) lead to the assertion.
□

Theorem 1 characterizes the solution only locally in the neighborhood of a specific $x_0$, as it does not provide the construction of the set $CR^*$ where this characterization remains valid. On the other hand, this region can be characterized immediately. By construction, conditions (7a), (7c) and (7d) are satisfied as identities by $U^*(x)$ and $\lambda^*(x)$. By substituting in (7b) and (7e) the expressions of $U^*(x)$ and $\lambda^*(x)$ we get

$$(A_B Z_U - F_B)x \leq b_B - A_B\zeta_U, \tag{11a}$$

$$-Z_\lambda x \leq \zeta_\lambda. \tag{11b}$$

Representation (11) may be improved by removing possible redundant inequalities. Obviously, $CR^*$ is a polyhedron in the $x$-space, and represents the largest set of $x \in X_f$ such that the combination of active constraints at the minimizer corresponds to the chosen index set $N$.

### 3.1 Degeneracy

So far, we have assumed that the rows of $A_N$ are linearly independent. It can happen, however, that by solving $(QP_x)$ one determines a set of active constraints for which this assumption is violated. For instance, this happens when more than $r$ constraints are active at the optimizer $U^*(x) \in \mathbb{R}^r$, i.e., in a case of *primal degeneracy*. In this case the vector of Lagrange multipliers $\lambda^*$ might not be uniquely defined, as the dual problem of $(QP_x)$ is not strictly convex (instead, *dual degeneracy* cannot occur because we assumed $H$ positive definite, which implies that the minimizer is always unique). In [1], the authors suggest a simple way to handle degeneracy by extracting from $A_N$ an arbitrary subset of $rank(A_N)$ linearly independent rows, and proceed with the corresponding reduced set of active constraints.

### 3.2 Continuity and Convexity Properties

The result stated below makes use of the following definition.

**Definition 3** *A function $z : X \mapsto \mathbb{R}^m$, where $X \subseteq \mathbb{R}^n$ is a polyhedral set, is* piecewise affine *(resp.* piecewise quadratic*) if the following hold: (1) it is possible to partition $X$ into finitely many convex polyhedral regions $CR_i$, $i = 1, \ldots, p$; (2) inside $CR_i$, $z$ is an affine (resp. quadratic) function, for all $i = 1, \ldots, p$.*

Continuity of the value function $\phi^*$ and the solution $U^*$, can be shown as simple corollaries of the linearity result of Theorem 1. This fact, together with the convexity of the set of feasible parameters $X_f$, and of the value function $\phi^*$, is proved in the next theorem [1, Theorem 4].

**Theorem 2** *Consider the multiparametric quadratic program $(QP_x)$ and let $H$ be positive definite. Then the optimizer $U^* : X_f \mapsto \mathbb{R}^r$ is continuous and piecewise affine, and the value function $\phi^* : X_f \mapsto \mathbb{R}$ is continuous, convex and piecewise quadratic.*

We note that the same continuity and convexity results can be obtained as special cases of the general nonlinear results in [7, Chapter 2].

## 4 Approximate mp-QP

Let the parameter vector $x \in X_f$ be arbitrarily chosen[1], and let $(B, N)$ be the corresponding optimal partition. In order to obtain a suboptimal solution to $(QP_x)$, we relax the

[1]A vector $x \in X_f$ can be computed by finding a pair $(U, x)$ satisfying $AU - Fx \leq b$, e.g., via linear programming.

KKT conditions (7) as

$$-\epsilon_1 \le HU + Cx + A'\lambda \le \epsilon_1, \tag{12a}$$

$$A_B U + s_B = b_B + F_B x, \qquad s_B \ge 0, \tag{12b}$$

$$A_N U + s_N = b_N + F_N x, \qquad 0 \le s_N \le \epsilon_2, \tag{12c}$$

$$-\epsilon_4 \le \lambda_B \le \epsilon_4, \qquad \lambda_N \ge -\epsilon_3. \tag{12d}$$

where $\epsilon_1 \in \mathbb{R}^r$, $\epsilon_2, \epsilon_3 \in \mathbb{R}^{|N|}$, $\epsilon_4 \in \mathbb{R}^{|B|}$ are the relaxation vectors that determine the degree of approximation, $\epsilon_k \ge 0$ (componentwise) for $k = 1, \dots, 4$. The relaxed KKT conditions (12) define a polyhedron in the $(U, x, \lambda, s)$-space. The *approximate critical region* is defined as the projection onto the $x$-space of such a polyhedron, and it is denoted by $CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ or, in short, by $CR_\epsilon$.

Assume for the moment that $CR_\epsilon$ has been computed (this issue will be discussed in Section 4.1). Then, the rest of the space $X_f \setminus CR_\epsilon$ has to be explored and new critical regions generated. An effective approach for partitioning $X_f \setminus CR_\epsilon$ by polyhedral sets is based on the following theorem (cf. [1, Theorem 3]).

**Theorem 3** *Let $X \subseteq \mathbb{R}^n$ be a polyhedron, and let $CR_\epsilon = \{\theta \in X : Gx \le g\}$ be a nonempty polyhedral subset of $X$, where $G \in \mathbb{R}^{h \times n}$. Also let*

$$R_i = \left\{ x \in X : \begin{array}{l} G_{\{i\}}x > g_{\{i\}} \\ G_{\{j\}}x \le g_{\{j\}}, \forall j < i \end{array} \right\} i = 1, \dots, h,$$

*where $G_{\{i\}}$ denotes the $i$-th row of $G$ and $g_{\{i\}}$ denotes the $i$-th entry of $g$. Then (i) $X = \left(\cup_{i=1}^h R_i\right) \cup CR_\epsilon$; (ii) $CR_\epsilon \cap R_i = \emptyset$ for all $i$ and $R_i \cap R_j = \emptyset$ for all $i \ne j$; i.e., $\{CR_\epsilon, R_1, \dots, R_h\}$ is a partition of $X$.*

After partitioning the rest of the space, we proceed recursively: we choose for each region $R_i$ a new vector $x_0$, compute the approximate critical region $CR_\epsilon$, compute the rest of the space $R_i \setminus CR_\epsilon$, and so on. Clearly, in order to minimize the number of regions $R_i$ generated at each recursion, before applying Theorem 3 it is convenient to remove all redundant inequalities from the representation of $CR_\epsilon$.

### 4.1 Orthogonal Projections

Before proceeding further, it is useful to rewrite the approximate KKT conditions (12a) in the form

$$HU + Cx + A'_N\lambda_N + A'_B\lambda_B + \nu = 0, \quad -\epsilon_1 \le \nu \le \epsilon_1, \tag{13}$$

where $\nu \in \mathbb{R}^r$ represents the violation of the first KKT condition (7a). From (13) we obtain $U = -H^{-1}(A'_N\lambda_N + A'_B\lambda_B + Cx + \nu)$ and thus, by substitution into (12c) and under the assumption that $A_N H^{-1} A'_N$ is invertible,

$$\lambda_N = E_\nu \nu + E_s s_N + E_\lambda \lambda_B + Z_\lambda x + \zeta_\lambda, \tag{14}$$

where $E_s \triangleq (A_N H^{-1} A'_N)^{-1}$, $E_\nu \triangleq -E_s A_N H^{-1}$ and $E_\lambda \triangleq E_\nu A'_B$.

The approximated critical region $CR_\epsilon$ is now the projection onto the $x$-space of the polyhedron in the $(\nu, s_N, \lambda_B, x)$-space described by the inequalities

$$-\epsilon_1 \le \nu \le \epsilon_1, \tag{15a}$$

$$\begin{aligned} &- A_B H^{-1}(A'_N(E_\nu \nu + E_s s_N + E_\lambda \lambda_B + Z_\lambda x + \zeta_\lambda) \\ &\quad + A'_B\lambda_B + Cx + \nu) \le b_B + F_B x, \end{aligned} \tag{15b}$$

$$0 \le s_N \le \epsilon_2, \tag{15c}$$

$$-\epsilon_4 \le \lambda_B \le \epsilon_4, \; E_\nu \nu + E_s s_N + E_\lambda \lambda_B + Z_\lambda x + \zeta_\lambda \ge -\epsilon_3. \tag{15d}$$

Rather than projecting with respect to the whole set of variables $\nu$, $s_N$, $\lambda_B$, we can restrict the amount of relaxations, and accordingly distinguish among the following three cases:

A. Case $\epsilon_2 = 0$. This special case implies $s_N = 0$, and therefore amounts to fix the index set $N$ of constraints which are active at the optimizer of $(QP_x)$. The projection is performed only with respect to $\nu$, $\lambda_B$.

B. Case $\epsilon_2 = 0$, $\epsilon_4 = 0$. This special case implies $\lambda_B = 0$, $s_N = 0$, and corresponds to avoid the relaxation of the second KKT condition (7b). Equivalently, it implies that the given optimal partition $(B, N)$ is maintained. The simplification of the projection procedure is obvious: we only need to project with respect to $\nu$.

C. Case $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$. In this final special case, we only relax the nonnegativity condition on the Lagrange multipliers corresponding to non-active constraints of $(QP_x)$. Hence we need no projection, as similarly to (11) for the exact case, the approximated critical region reduces to

$$(A_B Z_U - F_B)x \leq b_B - A_B \zeta_U, \tag{16a}$$

$$- Z_\lambda x \leq \epsilon_3 + \zeta_\lambda. \tag{16b}$$

## 4.2 Properties of Approximated Critical Regions

Since the primal feasibility of the optimizer is never relaxed, the approximate critical region is always contained in $X_f$. It is however of interest to study its behavior as a function of the amount of relaxation.

**Lemma 1** *Let $(B, N) = (B(x), N(x))$ for some $x \in X_f$, and let $CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ be the associated approximate critical region. Then the following statements hold:*

*i) if $\epsilon_k \leq \epsilon'_k$, $\forall k = 1, \ldots, 4$, then $CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \subseteq CR(\epsilon'_1, \epsilon'_2, \epsilon'_3, \epsilon'_4)$;*

*ii) $\bigcap_{\delta_1,\delta_2,\delta_3,\delta_4 \geq 0} CR(\delta_1, \delta_2, \delta_3, \delta_4) = CR^*$, where $\delta_1 \in \mathbb{R}^r$, $\delta_2, \delta_3 \in \mathbb{R}^{|N|}$, $\delta_4 \in \mathbb{R}^{|B|}$, and $CR^*$ the exact critical region represented by (11);*

*iii) $\forall \epsilon_3, \epsilon_4 \geq 0$, $\bigcup_{\delta_1 \geq 0} CR(\delta_1, 0, \epsilon_3, \epsilon_4) = X_N$, where $X_N$ is the projection onto the $x$-space of $\{(U, x) : A_B U - F_B x \leq b_B, A_N U - F_N x = b_N\}$;*

*iv) $\forall \epsilon_3, \epsilon_4 \geq 0$, $\bigcup_{\delta_1,\delta_2 \geq 0} CR(\delta_1, \delta_2, \epsilon_3, \epsilon_4) = X_f$.*

*Proof:* In order to prove the lemma, we refer to the relaxed KKT conditions (12). Statements (*i*) and (*ii*) follow immediately from (12). To prove statement (*iii*), we show that (a) $\bigcup_{\delta_1 \geq 0} CR(\delta_1, 0, \epsilon_3, \epsilon_4) \subseteq X_N$, and (b) $\bigcup_{\delta_1 \geq 0} CR(\delta_1, 0, \epsilon_3, \epsilon_4) \supseteq X_N$. Take a vector $\bar{x} \in \bigcup_{\delta_1 \geq 0} CR(\delta_1, 0, \epsilon_3, \epsilon_4)$. Then $\bar{x} \in CR(\bar{\delta}_1, 0, \epsilon_3, \epsilon_4)$ for some $\bar{\delta}_1 \geq 0$, and hence there exists a vector $\bar{U}$ such that $(\bar{U}, \bar{x})$ satisfies (12b), (12c), which in turn implies that $\bar{x} \in X_N$, and proves (a). Vice versa, take $\bar{x} \in X_N$. Then there exists a vector $\bar{U}$ such that $(\bar{U}, \bar{x})$ satisfies (12b), (12c). Eq. (12d) is satisfied for instance by taking $\bar{\lambda} = 0$, while (12a) is satisfied, e.g., for $\epsilon_1 = \bar{\delta}_1 = \left|H\bar{U} + C\bar{x}\right|$ (where $|\cdot|$ is intended componentwise), which in turn implies that $\bar{x} \in CR(\bar{\delta}_1, 0, \epsilon_3, \epsilon_4)$, and therefore proves (b). The proof of statement (*iv*) is similar to the proof of (*iii*) and thus omitted. □

### 4.3 Approximate Optimizer

So far, we have described a suboptimal method for partitioning the parameter set $X_f$, but contrarily to the exact case described in Theorem 1, we have not specified yet an approximate optimizer, which will be denoted by $\widehat{U}(x)$. Similarly to the exact case, we wish to have $\widehat{U}(x)$ to be a piecewise affine function of $x$ (defined over the partition into approximate critical regions given by the recursive method mentioned above), and such that $\widehat{U}(x)$ is primal feasible for all $x \in CR_\epsilon$, for each approximate critical region $CR_\epsilon$. Moreover, we wish that $\widehat{U}(x)$ is as close as possible to the exact solution $U^*(x)$.

For case C, it turns out that, for any given index set $N$, a good choice is to set $\widehat{U}(x)$ as in (10), because it provides primal feasibility for all $x \in CR_\epsilon$, and optimality for $x \in CR^* \subseteq CR_\epsilon$ (i.e., $\widehat{U}(x) = U^*(x)$, $\forall x \in CR^*$).

For cases A and B, primal feasibility should be instead explicitly enforced. To this end, the following lemma may be useful.

**Lemma 2** *Let $V = \{V_1, \dots, V_h\}$ be a set of vectors of $\mathbb{R}^n$ such that $CR_\epsilon \subseteq \mathrm{conv}(V)$. Let $\widehat{U}(x)$ be an affine function of $x$. Then, $A\widehat{U}(V_i) \leq b + FV_i$ for all $V_i \in V$ implies $A\widehat{U}(x) \leq b + Fx$ for all $x \in CR_\epsilon$.*

*Proof:* Straightforward, by convexity. □

A natural choice for $V$ is the set of vertices of $CR_\epsilon$. Although good packages exist for determining the set of vertices of $CR_\epsilon$ (see [8]), for high dimensional $x$-spaces this might be computationally too expensive. In alternative, the set $V$ can be obtained by determining a union of hyper-rectangles which outer approximates $CR_\epsilon$ [9]. After a set $V$ fulfilling Lemma 2 is chosen, we compute the affine suboptimal solution $\widehat{U}(x) = \widehat{Z}x + \widehat{\zeta}$, where $\widehat{Z}$ and $\widehat{\zeta}$ are obtained by solving the following constrained quadratic least squares problem

$$\min_{Z \in \mathbb{R}^{r \times n},\ \zeta \in \mathbb{R}^r} \sum_{i=1}^{h} \|\mathcal{W}[U^*(V_i) - (ZV_i + \zeta)]\|^2 \tag{17a}$$

$$\text{subj. to } A(ZV_i + \zeta) \leq b + FV_i, \quad i = 1, 2, \dots, h, \tag{17b}$$

which provides the best fit to the optimal solutions $U^*(V_i)$ under the constraint of primal feasibility over $\mathrm{conv}(V) \supseteq CR_\epsilon$, where $\mathcal{W}$ is a weighting matrix. When the approximate mp-QP is used to solve an RHC problem, a sensible choice for $\mathcal{W}$ is $\mathcal{W} = \left[\begin{smallmatrix} I_{n_u} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, as only the first $n_u$ components of the solution are used to build the suboptimal explicit RHC law. Moreover, for the approximate region which contains the origin (i.e., the region corresponding to the empty combination of active constraints), in (17) we impose $\zeta = 0$, so that it is possible to achieve asymptotic convergence to the origin.

**Remark 1** Consider the original RHC problem of Section 2. If the constraints (2) have the popular form $u_{\min} \leq u(t) \leq u_{\max}$ (hard constraints), $y_{\min} - \sigma \leq y(t) \leq y_{\max} + \sigma$ (soft constraints) with $u_{\min} \leq 0 \leq u_{\max}$, $y_{\min} \leq 0 \leq y_{\max}$, and $\sigma \geq 0$ is an additional slack optimization variable, then problem (17) is always solvable. Indeed, at least the input sequence $U = 0$ is feasible for all $x \in \mathrm{conv}(V)$ for some sufficiently large $\sigma$. In general, unless some other particular hypotheses on $A$, $b$, $F$ are assumed, problem (17) may be infeasible, especially for large $\epsilon$. In this case, a possibility is to iteratively reduce (e.g., halve) the entries of $\epsilon$ until feasibility of (17) is reached. □

**Remark 2** Contrarily to the exact case, the overall piecewise affine function may not be continuous. Note that the approximate critical regions we obtain are always closed sets, whereas by Theorem 3 we should apply our method to a partition of $X_f$ formed by sets $R_i$ which are defined by both strict and non-strict inequalities. Actually, we propose to apply the search for an approximate critical region in the closure of such sets $R_i$. The resulting approximate descriptions of the optimizer is then redundant, in the sense that it may be defined more than once for some $x \in X_f$. For such vectors $x$, we arbitrarily define $\widehat{U}(x)$ as one of the possible values.

### 4.4 Approximate Value Function

Because of the property of primal feasibility given by (17b) (Case A,B) or (16a) (Case C), the following proposition follows immediately.

**Proposition 1** *Let* $\widehat{\phi}(x) \triangleq \frac{1}{2}\widehat{U}(x)'H\widehat{U}(x) + x'C'\widehat{U}(x) + \frac{1}{2}x'Yx$ *be the approximate value function, and* $\phi^*$ *the (exact) value function of problem* ($QP_x$)*. Then,* $\widehat{\phi}(x) \geq \phi^*(x)$ *for all* $x \in X_f$*, i.e.,* $\widehat{\phi}(x)$ *is an upper-bound for* $\phi^*(x)$*.*

In Lemma 3 we will give a bound on the gap between $\widehat{\phi}(x)$ and $\phi^*(x)$, valid for Case C.

### 4.5 Suboptimality Figures

Once the suboptimal solution to the mp-QP problem has been determined, it is interesting to compute (*a posteriori*) the degree of suboptimality of the resulting approximate explicit RHC controller with respect to the original RHC problem. In other words, the difference between the first $n_u$ components of $\widehat{U}(x)$ and $U^*(x)$. To this end, we define the *absolute error* $e_{\text{abs}} \triangleq \max_{x\in X_f\cap\overline{X}} \|I^1(\widehat{U}(x) - U^*(x))\|_\infty$ and the *relative error* $e_{\text{rel}} \triangleq \max_{x\in X_f\cap\overline{X}} \left\{\|I^1(\widehat{U}(x) - U^*(x))\|_\infty / \|x\|_1\right\}$, where $\overline{X} \subset \mathbb{R}^n$ is a bounded polyhedron, containing the state vector of interest. Typically, $\overline{X}$ is a box: $\overline{X} = \{x : \underline{x} \leq x \leq \overline{x}\}$. The following proposition shows that such errors can be computed numerically.

**Proposition 2** *Let the exact optimizer* $U^*(x)$ *and the approximate optimizer* $\widehat{U}(x)$ *be given. The absolute error* $e_{\text{abs}}$ *can be computed by solving* $2n_u$ *Mixed Integer Linear Programs (MILPs). Furthermore, the relative error* $e_{\text{rel}}$ *can be computed by solving* $2n_u$ *mono-parametric Mixed Integer Linear Programs (p-MILPs) and one maximization of a piecewise hyperbolic scalar function.*

*Proof:* See Appendix A for a constructive proof. □

### 4.6 A Priori Error Bounds

Analytic forms for expressing the error between the optimizer and a feasible vector can be found in [10, 11] for linear complementarity problems. Although in principle these results may be applied to our mp-QP context, they rely on the existence of constants whose determination is not constructively given. Therefore, in this paper we follow a different route and develop a direct approach to analyze the error between the optimal and the suboptimal solution.

Consider the special case $\epsilon_1 = 0$, $\epsilon_2 = 0$, and $\epsilon_4 = 0$ (Case C). Inside $CR_\epsilon$, defined by (16), we take as approximate optimizer $\widehat{U}(x) \triangleq Z_U x + \zeta_U$, and we take as approximate vector of Lagrange multipliers $\widehat{\lambda}_N(x) \triangleq Z_\lambda x + \zeta_\lambda$, $\widehat{\lambda}_B(x) \triangleq 0$, where $Z_U$, $\zeta_U$, $Z_\lambda$, and $\zeta_\lambda$ are defined as in Theorem 1. As already mentioned, this choice provides primal feasibility for all $x \in CR_\epsilon$, and optimality for all $x \in CR^* \subseteq CR_\epsilon$. Accordingly, we take as approximate value function $\widehat{\phi}(x) \triangleq \frac{1}{2}\widehat{U}'(x)H\widehat{U}(x) + x'C'\widehat{U}(x) + \frac{1}{2}x'Yx$. Our goal is to impose *a priori* a bound $\rho$ on the absolute error: $\max_{x\in X_f} \|I^1(\widehat{U}(x) - U^*(x))\|_\infty \leq \rho$.

**Lemma 3** *Let* $\epsilon_1 = 0$*,* $\epsilon_2 = 0$*, and* $\epsilon_4 = 0$*. Then, for all* $x \in CR_\epsilon$*,*

$$\widehat{\phi}(x) - \phi^*(x) \leq \frac{1}{2}\epsilon_3' A_N H^{-1} A_N' \epsilon_3. \tag{18}$$

*Proof:* Since $U = -H^{-1}(Cx + A'\lambda)$, Dorn's dual of problem ($QP_x$) may be written as [6, pp. 232-233]

$$(QD_x): \quad \max_\lambda \left\{-\frac{1}{2}\lambda' A H^{-1} A'\lambda - [b + (F + AH^{-1}C)x]'\lambda - \frac{1}{2}x'C'H^{-1}Cx : \lambda \geq 0\right\} + \frac{1}{2}x'Yx.$$

By convexity, $(QP_x)$ and $(QD_x)$ have the same optimum $\phi^*(x)$. Since $\widehat{\lambda}_N(x)+\epsilon_3 \geq 0$, $\widehat{\lambda}_B(x)=0$ is feasible for $(QD_x)$ for all $x \in CR_\epsilon$, and by noting that $b_N+(F_N+A_NH^{-1}C)x = -A_NH^{-1}A_N'\widehat{\lambda}_N(x)$ we have:

$$\begin{aligned}
\phi^*(x)-\frac{1}{2}x'Yx \;&\geq\; -\frac{1}{2}(\widehat{\lambda}_N(x)+\epsilon_3)'A_NH^{-1}A_N'(\widehat{\lambda}_N(x)+\epsilon_3)\\
&\quad -[b_N+(F_N+A_NH^{-1}C)x]'(\widehat{\lambda}_N(x)+\epsilon_3)-\frac{1}{2}x'C'H^{-1}Cx\\
&= \frac{1}{2}\widehat{\lambda}_N(x)'A_NH^{-1}A_N'\widehat{\lambda}_N(x)-\frac{1}{2}\epsilon_3'A_NH^{-1}A_N'\epsilon_3-\frac{1}{2}x'C'H^{-1}Cx.
\end{aligned}$$

Furthermore, by noting that $\widehat{U}(x) = -H^{-1}(A_N'\widehat{\lambda}_N(x)+Cx)$ we get:

$$\begin{aligned}
\widehat{\phi}(x)-\frac{1}{2}x'Yx \;&=\; \frac{1}{2}\widehat{U}'(x)H\widehat{U}(x)+x'C'\widehat{U}(x)\\
&= \frac{1}{2}\widehat{\lambda}_N(x)'A_NH^{-1}A_N'\widehat{\lambda}_N(x)-\frac{1}{2}x'C'H^{-1}Cx.
\end{aligned}$$

Inequality (18) clearly follows. □

**Lemma 4** *Let $\epsilon_1=0$, $\epsilon_2=0$, $\epsilon_4=0$, and let $\Delta U(x) \triangleq \widehat{U}(x)-U^*(x)$. Then, for all $x \in CR_\epsilon$,*

$$\Delta U'(x)H\Delta U(x) \leq \epsilon_3'A_NH^{-1}A_N'\epsilon_3. \tag{19}$$

*Proof:* We have $\widehat{\phi}(x)-\phi^*(x) = \frac{1}{2}\widehat{U}'(x)H\widehat{U}(x)+x'C'\widehat{U}(x)-\frac{1}{2}U^{*\prime}(x)HU^*(x)+x'C'U^*(x)$, and so

$$\hat{\phi}(x)-\phi^*(x) = -\frac{1}{2}\Delta U'(x)H\Delta U(x)+\Delta U'(x)(H\widehat{U}(x)+Cx). \tag{20}$$

Define the function

$$f(t) \triangleq \frac{1}{2}(\widehat{U}(x)-t\Delta U(x))'H(\widehat{U}(x)-t\Delta U(x))+x'C'(\widehat{U}(x)-t\Delta U(x))+\frac{1}{2}x'Yx.$$

Note that $f(t)$ is the objective value of $(QP_x)$ associated with $\widehat{U}(x)-t\Delta U(x)$, which is feasible for $(QP_x)$ for all $t\in[0,1]$, as $\widehat{U}(x)$ and $U^*(x)$ are both feasible. Since $f(1)=\phi^*(x)$, then $f(t)$ must be decreasing on a left neighbor of $t=1$. Hence, $f'(t) = \Delta U'(x)H\Delta U(x)t - \Delta U'(x)(H\widehat{U}(x)+Cx) \leq 0$ if $t=1$, and so

$$\Delta U'(x)(H\widehat{U}(x)+Cx) \geq \Delta U'(x)H\Delta U(x). \tag{21}$$

From (20) we then obtain

$$\widehat{\phi}(x)-\phi^*(x) \geq \frac{1}{2}\Delta U'(x)H\Delta U(x) \tag{22}$$

which, in addition to (18), implies the thesis. □

**Lemma 5** *Let $z \in \mathbb{R}^r$, and consider the following optimization problem*

$$V^* = \begin{array}{ll} \max_z & \|I^1z\|_\infty \\ \text{subj. to} & z'Hz \leq \alpha \end{array} \tag{23}$$

*where $H$ is positive definite. Then, $V^* = \max_\alpha \left\{\sqrt{\alpha[H^{-1}]_{ii}} : i=1\dots n_u\right\}$, where $[\cdot]_{ij}$ denotes the $(i,j)$-th entry of $[\cdot]$.*

*Proof:* Consider the optimization problem $\max_z\{c'z : z'Hz \leq \alpha\}$. Since at the optimizer $z^*$ the quadratic constraint must be active, the optimization problem is equivalent to $\max_z\{c'z : z'Hz = \alpha\}$. For the latter problem, denoting by $\beta$ the Lagrange multiplier associated with the quadratic constraint, the necessary optimality conditions are $c+2Hz\beta = 0$, $z'Hz - \alpha = 0$, from which we obtain $z = -(H^{-1}c)/(2\beta)$, $\beta = \pm(\sqrt{c'H^{-1}c})/(2\sqrt{\alpha})$, and finally the KKT points $z = \mp(H^{-1}c\sqrt{\alpha})/(\sqrt{c'H^{-1}c})$. The maximum is therefore $\sqrt{\alpha c'H^{-1}c}$. By letting $c = \pm I^1_{\{i\}}$, $i = 1, \ldots, n_u$, where $I^1_{\{i\}}$ denotes the $i$-th row of $I^1$, we prove the lemma. □

**Theorem 4** *Let $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$, and assume that for each optimal partition $(B, N)$ the corresponding approximated critical region $CR_\epsilon$ is generated by setting $\epsilon_3 = \epsilon(N)\underline{1}$, where $\underline{1} \triangleq [1\ 1\ \ldots\ 1]'$, and*

$$\epsilon(N) \leq \frac{\rho}{\sqrt{\underline{1}'A_N H^{-1} A_N' \underline{1}}} \cdot \min_{i=1\ldots n_u} \frac{1}{\sqrt{[H^{-1}]_{ii}}}. \tag{24}$$

*Then* $\max_{x\in X_f} \|I^1(\widehat{U}(x) - U^*(x))\|_\infty \leq \rho$.

*Proof:* As a consequence of Lemma 4, $z = \widehat{U}(x) - U^*(x)$ satisfies the ellipsoidal constraint (19) for all $x \in CR_\epsilon$. By setting $\alpha = \epsilon^2(N)\underline{1}'A_N H^{-1} A_N' \underline{1}$, Lemma 5 guarantees that $\|I^1(\widehat{U}(x) - U^*(x))\|_\infty \leq \max_{i=1,\ldots,n_u}\left\{\sqrt{\epsilon^2(N)\underline{1}'A_N H^{-1} A_N' \underline{1}[H^{-1}]_{ii}}\right\} \leq \rho$ for all $x \in CR_\epsilon$, and for all approximated critical regions $CR_\epsilon$. □

# 5 Suboptimal RHC

This section discusses the two main issues regarding RHC policies, namely the *feasibility* of the optimization problem (3) at each time step $t \geq 0$, and the *stability* of the resulting closed-loop system.

## 5.1 Feasibility

As stressed in the previous section, primal feasibility is maintained in the approximate mp-QP solution. Note that when the RHC setup of Section 2 is augmented by additional constraints aimed at guaranteeing feasibility at each time step $t$ [12], such constraints will be also fulfilled by the suboptimal RHC solution. For instance, if constraints which enforce the predicted terminal state $x_T$ to lie in a polyhedral invariant set [13] are included, feasibility at each time step $t$ is guaranteed. This point is clarified below in the proof of Theorem 6, where we indeed show that $X_f$ is an invariant set. In conclusion, the feasibility of the RHC problem at each time $t$ does not depend on optimality.

## 5.2 Stability

The suboptimal controller proposed in this paper does not inherit directly intrinsic nominal stability properties of its "optimal" RHC counterpart based on the exact minimization of (3).

As the closed-loop suboptimal RHC system, composed by a linear plant in feedback with the suboptimal explicit RHC controller, is a piecewise affine system, *a posteriori* stability criteria based on piecewise (or common) quadratic Lyapunov functions [14–16] or reachability analysis [15] can be applied to analyze if a certain suboptimal RHC controllers is stabilizing (this will be exemplified in Section 6).

On the other hand, we are interested in synthesizing suboptimal RHC controllers that, by construction and independently on the particular value of the tuning parameters, are stabilizing, or in other words, in providing *a priori* stability guarantees.

**Lemma 6** *Let $d > 0$ in (2) (i.e., the interior of the polyhedron given by (2) contains the origin). Then, the critical region $CR_\emptyset$ corresponding to the empty combination of active constraints is a full dimensional subset of $\mathbb{R}^n$.*

*Proof:* We prove that there exists a scalar $\alpha > 0$ such that $\Phi_\alpha \triangleq \{x \in \mathbb{R}^n : \|x\| \leq \alpha\} \subset CR_\emptyset$. This is equivalent to show that for each state $x_0 \in \Phi_\alpha$ the unconstrained optimal control sequence $U = -H^{-1}Cx_0$ satisfies the constraints in (3), and is therefore optimal also for the constrained problem (3). To this end, it is enough to find a small enough positive scalar $\alpha$ such that $x_0$ and $U = [u_0' \ \dots \ u_{T-1}']'$ satisfy the constraints in (3). Let $\Theta \triangleq -H^{-1}C$, and $u_k = \Theta_{\{k\}}x_0$ denote the $k$-th control move, $k = 0, \dots, T-1$. Since $x_k = \mathcal{A}^k x_0 + \sum_{j=0}^{k-1} \mathcal{A}^j \mathcal{B} u_{k-1-j}$, if $\alpha > 0$ is sufficiently small we obtain

$$D_1 x_k + D_2 u_k = \left( D_1 \mathcal{A}^k + D_1 \sum_{j=0}^{k-1} \mathcal{A}^j \mathcal{B} \Theta_{\{k-1-j\}} + D_2 \Theta_{\{k\}} \right) x_0 \leq d$$

for all $k = 0, \dots, T-1$, and for all $x_0 \in \Phi_\alpha$. □

Before proceeding further, we recall the following from [17].

**Definition 4** *Consider the linear autonomous system $x(t+1) = \bar{A}x(t)$ and the polyhedron $\mathcal{P} \triangleq \{x : Bx \leq c\}$. The set $\Omega \triangleq \{x : B\bar{A}^t x \leq c, \ \forall t \geq 0\}$ is called the* maximum output admissible set *(MOAS) contained in $\mathcal{P}$.*

**Theorem 5** *Let $\bar{A}$ be a strictly Hurwitz matrix (all eigenvalues contained in the interior of the unit disk), let $\mathcal{P}$ be bounded, and let $0 \in \text{int}(\mathcal{P})$, where $\text{int}(\mathcal{P})$ denotes the interior of $P$. Then, the MOAS $\Omega$ contained in $\mathcal{P}$ is determined by a finite number of facet inequalities.*

*Proof:* See [17, Theorem 4.1]. □

We now concentrate on the following special class of suboptimal RHC laws, that will be referred to as *sRHC*:

(a) $P$ is the solution of the Riccati equation $P = (\mathcal{A} + \mathcal{B}\mathcal{K})' P (\mathcal{A} + \mathcal{B}\mathcal{K}) + Q + \mathcal{K}' R \mathcal{K}$, where $\mathcal{K} \triangleq -(R + \mathcal{B}' P \mathcal{B})^{-1} \mathcal{B}' P \mathcal{A}$;

(b) the set $\mathcal{P} \triangleq \{x : (D_1 + D_2\mathcal{K})x \leq d\}$ is bounded and contains the origin in its interior[2];

(c) the terminal set $\Omega$ is the MOAS contained in $\mathcal{P}$;

(d) only dual feasibility is relaxed ($\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$), and $\epsilon_3 \triangleq \epsilon \underline{1}$, $\epsilon \geq 0$;

(e) the first critical region generated by the suboptimal multiparametric solver is $CR_\emptyset \triangleq \{x : Wx \leq w\}$, $w \in \mathbb{R}^{n_w}$, associated with the void combination of active constraints $(B_1, N_1) \triangleq (M, \emptyset)$.

Note that assumption (e) implies that the critical region associated with $(M, \emptyset)$ is not approximated.

**Definition 5** *A function $f : \mathcal{X} \mapsto \mathbb{R}$ is said to be* positive definite *if $f(x) > 0$ for all $x \in \mathcal{X}$ and $f(x) = 0$ if and only if $x = 0$. Function $f$ is said to be* negative definite *if $-f$ is positive definite.*

[2]This hypothesis is satisfied for instance when the constraints (2) have the popular form $u_{\min} \leq u(t) \leq u_{\max}$, $x_{\min} \leq x(t) \leq x_{\max}$, with $u_{\min} < 0 < u_{\max}$, $x_{\min} < 0 < x_{\max}$.

**Theorem 6** *Consider the suboptimal RHC controller* sRHC *defined above, and let* $\gamma$ *be the maximum positive number for which the ellipsoid* $\mathcal{E} \triangleq \{x : x'Qx \le \gamma\}$ *is contained in* $CR_\emptyset$. *Let* $(B_h, N_h)$, $h = 2, \ldots, \widehat{\ell}$, *be the optimal partitions of the approximate solution to the mp-QP problem (6), and let* $CR_{\epsilon(N_h)}$ *denote the associated approximate critical regions. If* $\epsilon(N_h)$ *is chosen satisfying*

$$\epsilon(N_h) \le \sqrt{\frac{2\gamma}{\underline{1}' A_{N_h} H^{-1} A'_{N_h} \underline{1}}} \tag{25}$$

*for all* $h = 2, \ldots, \widehat{\ell}$, *then* sRHC *asymptotically stabilizes system (1) while fulfilling the constraints (2) at each time* $t \ge 0$, *for all* $x(0) \in X_f$.

*Proof:* In order to prove the theorem, we show that the exact value function $\phi^*$ is a Lyapunov function for system (1) in closed-loop with the suboptimal controller *sRHC*. Let $\widehat{U} \triangleq [\widehat{u}'_0 \ \ldots \widehat{u}'_{T-1}]'$ be the suboptimizer at time $t$, $t \ge 0$. At time $t+1$, consider the vector of inputs $\tilde{U} \triangleq [\widehat{u}'_1 \ \ldots \widehat{u}'_{T-1} \ (\mathcal{K}\widehat{x}_T)']'$, where $\widehat{x}_k \triangleq \mathcal{A}^k x(t) + \sum_{j=0}^{k-1} \mathcal{A}^j \mathcal{B} \widehat{u}_{k-1-j}$, $k = 0, \ldots, T$. By the definition of $\Omega$, the condition $\widehat{x}_T \in \Omega$ implies that $\widehat{x}_{T+1} \triangleq (\mathcal{A} + \mathcal{B}\mathcal{K})\widehat{x}_T \in \Omega$, which together with the feasibility of $\widehat{U}$ at time $t$ implies the feasibility of $\tilde{U}$ at time $t+1$, which also proves that $x(t+1) \in X_f$. Then,

$$\begin{aligned}
&\phi^*(x(t+1)) - \phi^*(x(t)) \le \tfrac{1}{2}\tilde{U}'H\tilde{U} + x'(t+1)C'\tilde{U} + \tfrac{1}{2}x'(t+1)Yx(t+1) - \phi^*(x(t)) \\
&\quad = \widehat{x}'_{T+1}P\widehat{x}_{T+1} + \textstyle\sum_{k=1}^{T} (\widehat{x}'_k Q \widehat{x}_k + \widehat{u}'_k R \widehat{u}_k) - \phi^*(x(t)) \\
&\quad = \widehat{x}'_T(\mathcal{A} + \mathcal{B}\mathcal{K})'P(\mathcal{A} + \mathcal{B}\mathcal{K})\widehat{x}_T + \textstyle\sum_{k=0}^{T-1} (\widehat{x}'_k Q \widehat{x}_k + \widehat{u}'_k R \widehat{u}_k) + \widehat{x}'_T Q \widehat{x}_T + (\mathcal{K}\widehat{x}_T)'R(\mathcal{K}\widehat{x}_T) \\
&\qquad - \widehat{x}'_0 Q \widehat{x}_0 - \widehat{u}'_0 R \widehat{u}_0 - \phi^*(x(t)) \\
&\quad = \widehat{x}'_T[(\mathcal{A} + \mathcal{B}\mathcal{K})'P(\mathcal{A} + \mathcal{B}\mathcal{K}) + Q + \mathcal{K}'R\mathcal{K}]\widehat{x}_T + \textstyle\sum_{k=0}^{T-1} (\widehat{x}'_k Q \widehat{x}_k + \widehat{u}'_k R \widehat{u}_k) \\
&\qquad - x(t)'Qx(t) - u(t)'Ru(t) - \phi^*(x(t)) \\
&\quad = \widehat{x}'_T P \widehat{x}_T + \textstyle\sum_{k=0}^{T-1} (\widehat{x}'_k Q \widehat{x}_k + \widehat{u}'_k R \widehat{u}_k) - x(t)'Qx(t) - u(t)'Ru(t) - \phi^*(x(t)) \\
&\quad = \widehat{\phi}(x(t)) - x(t)'Qx(t) - u(t)'Ru(t) - \phi^*(x(t)).
\end{aligned}$$

For all $x(t) \in CR_\emptyset$, we have $\widehat{\phi}(x(t)) = \phi^*(x(t))$, and therefore $\phi^*(x(t+1)) - \phi^*(x(t)) < 0$ for all $x(t) \in CR_\emptyset \setminus \{0\}$. Consider now $x(t) \in CR_{\epsilon(N_h)}$, $h = 2, \ldots, \widehat{\ell}$. Since $x(t)'Qx(t) > \gamma$ for all $x(t) \notin CR_\emptyset$, if $\epsilon(N_h)$ satisfies (25) then

$$\begin{aligned}
&\phi^*(x(t+1)) - \phi^*(x(t)) \le \widehat{\phi}(x(t)) - \phi^*(x(t)) - x(t)'Qx(t) - u(t)'Ru(t) \le \\
&\quad \tfrac{1}{2}\epsilon(N_h)^2 \underline{1}' A_{N_h} H^{-1} A'_{N_h} \underline{1} - x(t)'Qx(t) - u(t)'Ru(t) < \gamma - \gamma - u(t)'Ru(t) \le 0.
\end{aligned} \tag{26}$$

By letting $\Delta\phi^*(x) \triangleq \phi^*(\mathcal{A}x + \mathcal{B}I^1\widehat{U}(x)) - \phi^*(x)$, Eq. (26) proves that $\Delta\phi^*$ is a negative definite function. Since $\phi^*(x) \ge x'Qx$ and $Q$ is positive definite, it follows that $\phi^*$ is positive definite, and radially unbounded ($\phi^*(x) \to \infty$ for $\|x\| \to \infty$). We can therefore apply LaSalle's invariance principle for discrete-time systems [18, Theorem 4.2] on the level sets of $\phi^*$ to conclude that the origin is asymptotically stable with domain of attraction $X_f$. □

In conclusion, whenever a new optimal partition $(B_h, N_h)$ is generated by the recursive algorithm, Lemma 3, Theorem 4, and Theorem 6 provide constructive criteria for choosing the relaxation $\epsilon(N_h)$, so that error bounds on the value function and the optimizer and stability can be guaranteed a priori.

## 5.3 Complexity

The suboptimal RHC control law is $\widehat{u}_0(x) = I_1\widehat{U}(x)$. As approximate critical regions where the first $n_u$ components of $\widehat{U}(x)$ are the same and whose union is a convex set can be joined during a post-processing phase [19], similarly to the exact explicit solution of RHC [1], $\widehat{u}_0(x)$ has the following piecewise affine form

$$\widehat{u}_0(x) = \bar{F}^i x + \bar{g}^i \quad \text{if} \quad \bar{H}^i x \le \bar{k}^i, \ i = 1, \dots, \widehat{\ell}_{\text{rhc}}, \tag{27}$$

where the polyhedral sets $\{x : \bar{H}^i x \le \bar{k}^i\}$, $i = 1, \dots, \widehat{\ell}_{\text{rhc}}$ partition $X_f$, and clearly $\widehat{\ell}_{\text{rhc}} \le \widehat{\ell}$.

While *off-line* complexity of the suboptimal RHC algorithm can be investigated similarly to [1], *on-line* complexity of (27) is more interesting, especially from an application point of view. The simplest way to implement the piecewise affine feedback law (27) is to store the polyhedral cells $\{x : \bar{H}^i x \le \bar{k}^i\}$, perform an on-line linear search through them to locate the one which contains the current state $x(t)$, lookup the corresponding $\bar{F}^i$, $\bar{g}^i$, and evaluate $\bar{F}^i x(t) + \bar{g}^i$. This search procedure can be easily parallelized, or more efficiently organized according to a balanced search tree, a research topic currently under investigation.

# 6 Examples

**Example 6.1**

The second order non-minimum phase system with transfer function $2(s-1)/(s^2+2s+5)$ is sampled with $T_s = 0.1$ s to obtain the discrete time state-space representation

$$\left\{ \begin{array}{rcl} x(t+1) & = & \begin{bmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{bmatrix} x(t) + \begin{bmatrix} 0.1271 \\ 0.0132 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1.4142 & -0.7071 \end{bmatrix} x(t). \end{array} \right. \tag{28}$$

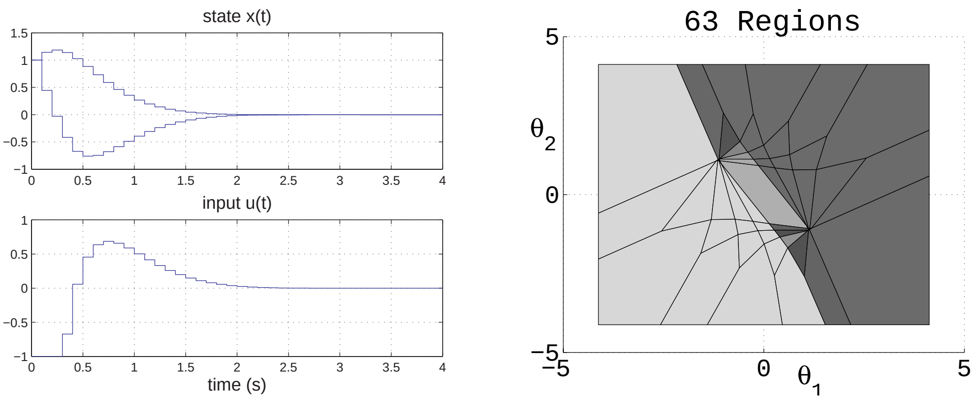
The task is to regulate the system to the origin while fulfilling the input constraint $-1 \le u(t) \le 1$. To this aim, we design an RHC controller based on the optimization problem (3) with $T = 6$, $R = 0.1$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $D_1 = 0$, $D_2 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, $d = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\Omega = \mathbb{R}^2$, and $P$ solves the Lyapunov equation $P = \mathcal{A}' P \mathcal{A} + Q$. Note that this choice of $P$ corresponds to setting $u_k = 0$ for $k \ge 6$ and to minimize $\sum_{k=0}^{\infty} x_k' x_k + 0.1 u_k^2$. The mp-QP problem associated with the RHC law has the form $(QP_x)$ with

$$H = \begin{bmatrix} 0.3799 & 0.1573 & 0.1321 & 0.1055 & 0.0790 & 0.0537 \\ 0.1573 & 0.3799 & 0.1573 & 0.1321 & 0.1055 & 0.0790 \\ 0.1321 & 0.1573 & 0.3799 & 0.1573 & 0.1321 & 0.1055 \\ 0.1055 & 0.1321 & 0.1573 & 0.3799 & 0.1573 & 0.1321 \\ 0.0790 & 0.1055 & 0.1321 & 0.1573 & 0.3799 & 0.1573 \\ 0.0537 & 0.0790 & 0.1055 & 0.1321 & 0.1573 & 0.3799 \end{bmatrix}, \ C = \begin{bmatrix} 1.1951 & 0.4086 \\ 1.0259 & 0.1305 \\ 0.8410 & -0.1030 \\ 0.6517 & -0.2896 \\ 0.4673 & -0.4293 \\ 0.2952 & -0.5243 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The explicit RHC controller was computed using the exact mp-QP Algorithm of [1], and the corresponding polyhedral partition of the state-space is depicted in Figure 1.

In order to reduce the number of regions, we apply the approximate mp-QP algorithm. By setting $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$, and choosing a constant $\epsilon_3$, we get the solutions shown in Figure 2 (for simplicity, from now on we let all the components of $\epsilon_k$ to be equal, and denote by $\epsilon_k$ the single component). Each approximate mp-QP solution was computed in less than 15 s of cpu on a Pentium III 650 MHz running Matlab 5.3. Note that despite




(a) Closed-loop RHC

(b) Polyhedral partition of the state-space: exact solution

Figure 1: Explicit RHC controller: exact solution

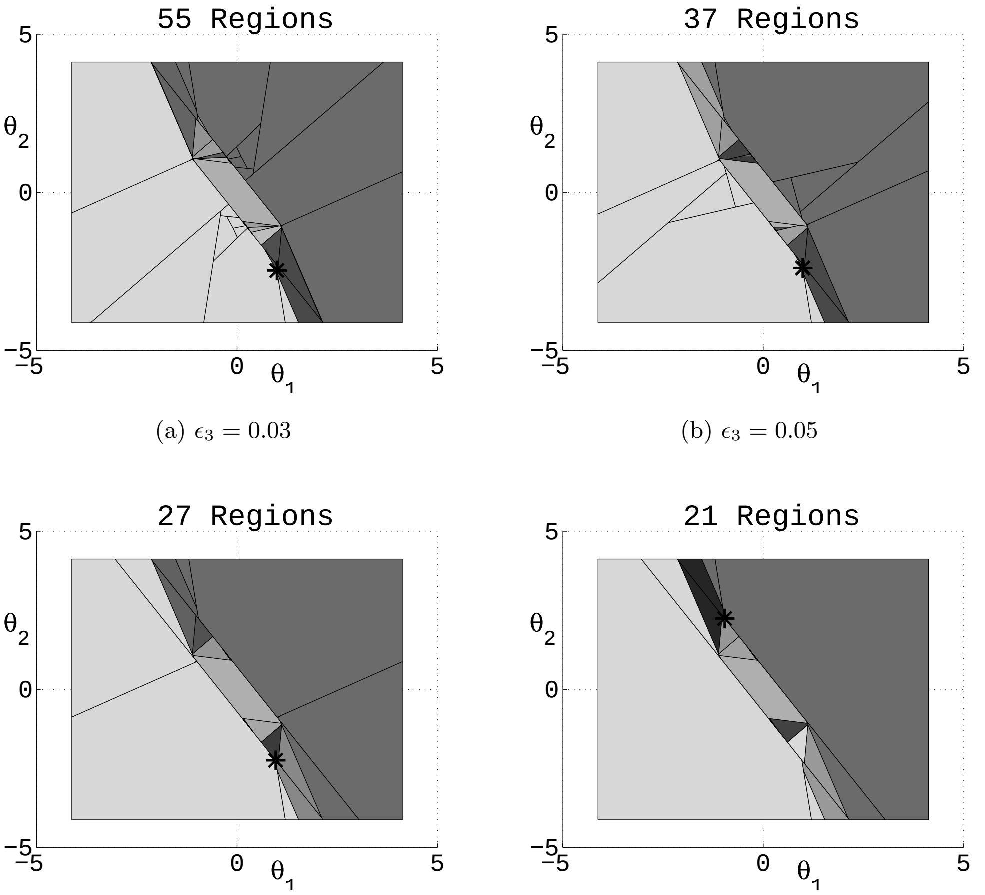
| $\epsilon_3$ | $L$ | $e_{\text{abs}}$ |
|---|---|---|
| 0.03 | $\begin{bmatrix} 8.0052 & 3.8681 \\ 3.8681 & 17.3940 \end{bmatrix}$ | 0.09530 |
| 0.05 | $\begin{bmatrix} 9.3106 & 4.3013 \\ 4.3013 & 17.9287 \end{bmatrix}$ | 0.15884 |
| 0.15 | $\begin{bmatrix} 10.2754 & 4.7303 \\ 4.7303 & 19.0681 \end{bmatrix}$ | 0.28158 |
| 0.2 | $\begin{bmatrix} 11.0924 & 5.1412 \\ 5.1412 & 20.2768 \end{bmatrix}$ | 0.28158 |

Table 1: Relaxation tolerance, common quadratic Lyapunov function, and maximum absolute error for the solutions shown in Figure 2

the relaxation of dual feasibility ($\epsilon_3 > 0$), the region containing the origin does not change with respect to the exact solution. This is justified by the fact that, being $N = \emptyset$, the constraints defining the critical region are all of the form $AU^*(x) \leq b + Fx$, and therefore are not affected by the relaxation. Closed-loop RHC trajectories are indistinguishable from those depicted in Figure 1(a), obtained using the exact explicit RHC law. For all the suboptimal RHC laws, the closed-loop system is quadratically stable, as it admits the common quadratic Lyapunov function $U'LU$ [16], where $L$ is reported in Table 1. The maximum absolute error $e_{\text{abs}}$, also reported in Table 1, was computed according to Proposition 2 running CPLEX 7.0 [20] on the same machine (about 10 s of cpu time per computation), and the state where such an error is achieved is marked by an asterisk in Figure 2.

By choosing $\epsilon_3$ adaptively in accordance with Theorem 4, we obtain the results shown in Table 2. It is apparent that the a posteriori error bound $e_{\text{abs}}$ is always smaller than the pre-specified a priori error bound $\rho$. This is not surprising, as the choice for $\epsilon_3$ suggested by Theorem 4 is based on the conservative over-estimate (19). Moreover, for $x \in CR_\epsilon$, the piecewise affine function $\widehat{U}(x) - U^*(x)$ does not span the whole ellipsoidal set described by the constraint in (23), so that further conservativeness is introduced. The fact that the intrinsic polyhedral structure of the partition may not allow to reach the a priori error bound $\rho$ is further testified by the fact that as $\rho$ increases, $e_{\text{abs}}$ saturates at 0.28158 (cf. Table 1).

We next vary all $\epsilon_i$. To maintain the solution $\widehat{U}(x)$ exact in the region where no constraint is active, we do not relax the KKT conditions for such a region, and set $\epsilon_i =$



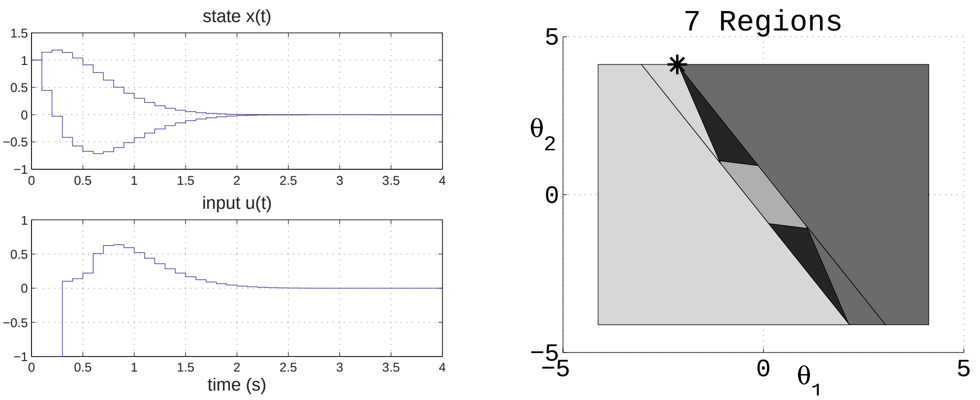

(a) $\epsilon_3 = 0.03$ (b) $\epsilon_3 = 0.05$

(c) $\epsilon_3 = 0.15$ (d) $\epsilon_3 = 0.2$

Figure 2: Approximate mp-QP solutions for different values of $\epsilon_3$, and $\epsilon_1 = \epsilon_2 = \epsilon_4 = 0$

10000, $i = 1, \ldots, 4$. The suboptimal RHC control law

$$u = \begin{cases} \begin{bmatrix} -2.2356 & -1.4078 \end{bmatrix} x & \text{if } \begin{bmatrix} -2.2356 & -1.4078 \\ 2.2356 & 1.4078 \\ -1.3618 & -0.4618 \\ 1.3618 & 0.4618 \\ 0.1751 & 1.1118 \\ -0.1751 & -1.1118 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#1)} \\ 1 & \text{if } \begin{bmatrix} 2.2356 & 1.4077 \\ 0 & -0.2425 \\ -0.2425 & 0 \\ 0 & 0.2425 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#2)} \\ -1 & \text{if } \begin{bmatrix} -2.2356 & -1.4077 \\ 0 & -0.2425 \\ 0 & 0.2425 \\ 0.2425 & 0 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#3)} \\ 1 & \text{if } \begin{bmatrix} 1.3618 & 0.4618 \\ 2.2356 & 1.4078 \\ -2.2356 & -1.4078 \\ 0 & 0.2425 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#4)} \\ -1 & \text{if } \begin{bmatrix} -1.3618 & -0.4618 \\ 2.2356 & 1.4078 \\ -2.2356 & -1.4078 \\ 0 & -0.2425 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#5)} \\ \begin{bmatrix} -0.2425 & 0 \end{bmatrix} x & \text{if } \begin{bmatrix} -0.1751 & -1.1118 \\ -1.3618 & -0.4618 \\ 2.2356 & 1.4078 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#6)} \\ \begin{bmatrix} -0.2425 & 0 \end{bmatrix} x & \text{if } \begin{bmatrix} 0.1751 & 1.1118 \\ 1.3618 & 0.4618 \\ -2.2356 & -1.4078 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(Region \#7)} \end{cases}$$




(a) Closed-loop RHC

(b) Polyhedral partition of the state-space: approximate solution

Figure 3: Explicit RHC controller: approximate solution for $\epsilon_i = 10000$, $i = 1, \ldots, 4$.

| $\rho$ | $e_{\rm abs}$ | # regions |
|---|---|---|
| 0.03 | 0.01927 | 53 |
| 0.05 | 0.03211 | 47 |
| 0.1 | 0.05794 | 43 |
| 0.15 | 0.08690 | 39 |
| 0.25 | 0.12639 | 37 |
| 0.5 | 0.25278 | 25 |
| 1 | 0.28158 | 21 |
| 5 | 0.28158 | 21 |

Table 2: A priori error bound $\rho$, a posteriori error bound $e_{\rm abs}$, and number of regions in the suboptimal partition

was computed in 13.4 s and is depicted in Figure 3. Region #1 corresponds to the unconstrained linear controller, regions #2, #3, #4, #5, correspond to the saturated controller, and regions #6 and #7 are transition regions between the unconstrained and the saturated controller. The control law is stabilizing, as $L = \left[\begin{smallmatrix} 19.5936 & 5.7937 \\ 5.7937 & 19.6299 \end{smallmatrix}\right]$ provides a common Lyapunov function for the closed-loop piecewise affine system. The maximum absolute error is $e_{\rm abs} = 1.9369$, while the maximum relative error is $e_{\rm rel} = 0.97431$ (attained at $\left[\begin{smallmatrix} -0.1351 \\ 0.9249 \end{smallmatrix}\right]$). Note that, by construction, the control law is exact in the central region.

In Figure 4 we show different approximate explicit RHC laws obtained by varying $\epsilon_1 = \cdots = \epsilon_4$, when also the region corresponding to the unconstrained case is relaxed. In all cases the closed-loop system is quadratically stable, and the corresponding matrices $L$ are reported in Table 3. The cpu time needed to solve the approximate mp-QP problem ranges from 35 s to 14 m.

**Example 6.2**

We synthesize the suboptimal RHC law *sRHC* with a priori stability guarantees for the double integrator $1/s^2$, according to Theorem 6. We sample the dynamics with $T_s = 1$ s to obtain the discrete time state-space model with state-transition matrices $\mathcal{A} = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, $\mathcal{B} = \left[\begin{smallmatrix} 1 \\ 0.5 \end{smallmatrix}\right]$. We constrain inputs and states within the range $-1 \leq u(t) \leq 2$, $\|x(t)\|_\infty \leq 100$, and choose the MPC parameters $T = 6$, $Q = I$, $R = 0.01$. By choosing $P$ as the solution to the resulting Riccati equation, using the algorithm of [17] and in accordance with

| $\epsilon_3$ | $L$ | $e_{\text{abs}}$ |
|---|---|---|
| 0.01 | $\begin{bmatrix} 9.2334 & 4.1382 \\ 4.1382 & 16.9986 \end{bmatrix}$ | 0.17174 |
| 0.05 | $\begin{bmatrix} 13.1923 & 5.3354 \\ 5.3354 & 19.1048 \end{bmatrix}$ | 0.73860 |
| 0.1 | $\begin{bmatrix} 15.2848 & 5.7500 \\ 5.7500 & 19.6389 \end{bmatrix}$ | 1.02863 |
| 0.5 | $\begin{bmatrix} 17.4725 & 5.9318 \\ 5.9318 & 19.7260 \end{bmatrix}$ | 2.00000 |

Table 3: Relaxation tolerance, common quadratic Lyapunov function, and maximum absolute error for the solutions shown in Fig. 4

Theorem 6 we obtain that

$$\Omega = \left\{ x : \begin{bmatrix} -0.6630 & -0.3304 \\ 1.3261 & 0.6609 \\ 0.1048 & 0.2169 \\ -0.2097 & -0.4338 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is the corresponding maximum output admissible set. The first critical region generated by the suboptimal multiparametric solver, associated with the void combination of active constraints, is $CR_\emptyset = \Omega$, for which $\gamma = 0.4555$ is the maximum positive number such that the ellipsoid $\mathcal{E} \triangleq \{x : x'Qx \leq \gamma\}$ is contained in $CR_\emptyset$. The exact explicit RHC and the suboptimal *sRHC* controllers consists of 53 regions and 39 regions, respectively, which are depicted in Fig. 5. For *sRHC*, the relaxation parameter $\epsilon(N_h)$ ranges between 0.0304 and 64.9584. The closed-loop trajectories for the exact and the suboptimal controllers are indistinguishable.

## 7 Conclusions

In this paper we addressed the problem of reducing the number of polyhedral cells associated with explicit solutions to RHC problems. Such number tends to increase exponentially with the number of constraints involved in the optimization problem. Our solution consists of finding an *approximate* solution to mp-QP by relaxing the KKT conditions for optimality (except primal feasibility). Error bounds for the errors on the optimal value and the optimizer are provided, and constraint fulfillment and closed-loop stability of the resulting suboptimal RHC law are guaranteed by explicit formulas. The degree of approximation is, in principle, arbitrary and allows to trade off between optimality and a comparatively small number of cells in the piecewise affine solution. Clearly, the choice of the degree of relaxation also depends on stability requirements, although it may be more a reflection of the particular Lyapunov function chosen to prove closed-loop stability properties than fundamental limitations on the proposed procedure. This is a point which is worthy further study. Future work will be also devoted to extend the approach of this paper to multiparametric Linear Programming (mp-LP) and to multiparametric Mixed Integer Linear Programming (mp-MILP).

## 8 Acknowledgements

We thank Domenico Mignone for providing us the LMI-based routine for computing common quadratic and piecewise-quadratic Lyapunov functions. We also thank the anonymous referees and the associate editor Professor David Q. Mayne, whose comments greatly helped to improve the quality of the paper.

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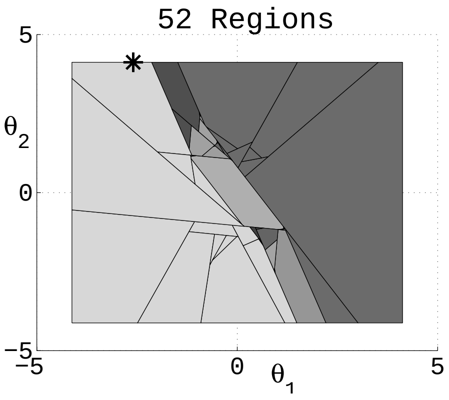
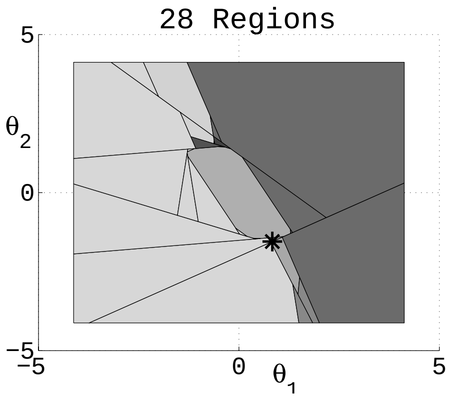
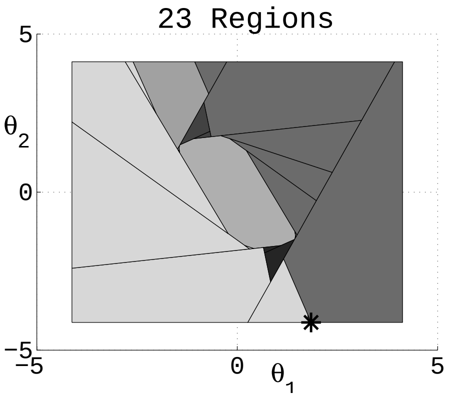


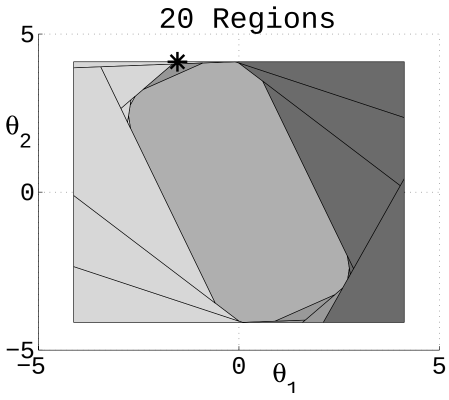

(a) $\epsilon_i = 0.01$




(b) $\epsilon_i = 0.05$




(c) $\epsilon_i = 0.1$




(d) $\epsilon_i = 0.5$

Figure 4: Approximate mp-QP solutions for different values of $\epsilon_1 = \cdots = \epsilon_4$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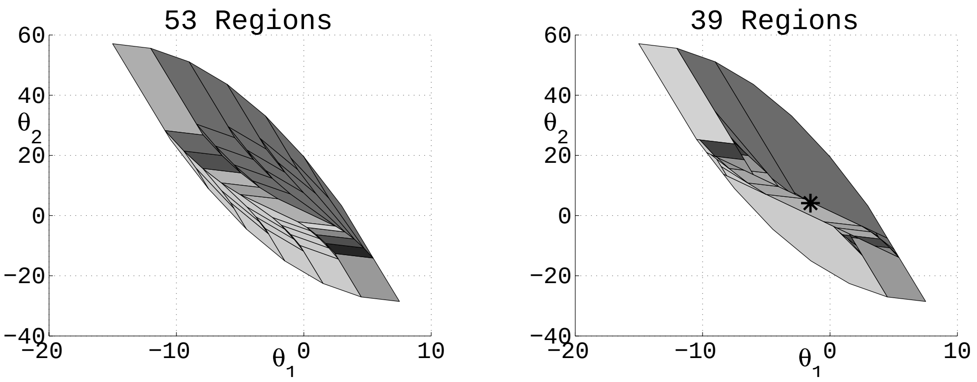


(a) exact RHC law (b) *sRHC* law

Figure 5: Exact and approximate RHC controllers with stability guarantees for Example 6.2


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# A Proof of Proposition 2

By following the approach of [21, Sect. 3.1], the idea is to represent the piecewise affine function $U^*$ as a set of linear equalities and mixed-integer inequalities. Let $CR_i = \{x \in$

$\mathbb{R}^n : S^i x \leq T^i\}$ be the polyhedral critical regions, $\cup_{i=1}^{\ell} CR_i = X_f$, and $U^*(x) = Z_U^i x + \zeta_U^i$ for $x \in CR_i$ $(i = 1, \dots, \ell)$. Introduce the integer variables $\delta_i \in \{0, 1\}$

$$[\delta_i = 1] \leftrightarrow [x \in CR_i], \quad i = 1, \dots, \ell. \tag{29}$$

For every $i = 1, \dots, \ell$, define the vector $M^i \triangleq \max_{x \in X_f \cap \overline{X}} \{S^i x - T^i\}$ (whose evaluation requires the solution of an LP for each entry). Since the critical regions $CR_i$ have mutually disjoint interiors, (29) is equivalent to

$$\begin{array}{l} S^i x - T^i \leq M^i(1 - \delta_i), \quad i = 1, \dots, \ell, \\ \sum_{i=1}^{\ell} \delta_i = 1, \end{array} \tag{30}$$

and $I^1 U^*(x)$ can be rewritten as $I^1 U^*(x) = \sum_{i=1}^{\ell} z^i$, where $z^i \triangleq I^1(Z_U^i x + \zeta_U^i)\delta_i$, or, equivalently,

$$\begin{array}{l} z^i \leq M\delta_i, \quad z^i \geq m\delta_i, \\ z^i \leq I^1(Z_U^i x + \zeta_U^i) - m(1 - \delta_i), \\ z^i \geq I^1(Z_U^i x + \zeta_U^i) - M(1 - \delta_i), \end{array} \tag{31}$$

where $M \triangleq \max_i \max_{x \in X_f \cap \overline{X}} I^1(Z_U^i x - \zeta_U^i)$, $m \triangleq \min_i \min_{x \in X_f \cap \overline{X}} I^1(Z_U^i x - \zeta_U^i)$. Similarly, $\widehat{U}(x)$ can be represented by introducing integer variables $\widehat{\delta}_i$, $\widehat{z}^i$, $i = 1, \dots, \widehat{\ell}$, and by introducing conditions analogous to (30) and (31), say $(\widehat{30})$ and $(\widehat{31})$.

The absolute error can then be computed by solving for all $j = 1, \dots, n_u$ the pairs of MILPs

$$\max_{x, \delta, z, \widehat{\delta}, \widehat{z}} \quad \pm \left( \sum_{i=1}^{\widehat{\ell}} \widehat{z}_j^i - \sum_{i=1}^{\ell} z_j^i \right) \tag{32a}$$

$$\text{subj. to} \quad (30)+(31)+(\widehat{30})+(\widehat{31}), \tag{32b}$$

where $z_j^i$ is the $j$th entry of $z^i$, and by setting $e_{\text{abs}}$ as the maximum of the optimal values found by solving the MILPs (32).

For the relative error $e_{\text{rel}}$, the cost function (32a) becomes $\pm \left( \sum_{i=1}^{\widehat{\ell}} \widehat{z}_j^i - \sum_{i=1}^{\ell} z_j^i \right) / \|x\|_1$. Rather than treating the resulting optimization problem as a mixed-integer nonlinear program, most efficiently we can introduce a scalar parameter $\alpha > 0$ and consider the following parametric problems:

$$\sigma_{\pm}^j(\alpha) = \max_{x, \delta, z, \widehat{\delta}, \widehat{z}} \quad \frac{\pm \left( \sum_{i=1}^{\widehat{\ell}} \widehat{z}_j^i - \sum_{i=1}^{\ell} z_j^i \right)}{\alpha} \tag{33a}$$

$$\text{subj. to} \quad \|x\|_1 = \alpha \tag{33b}$$

$$(30)+(31)+(\widehat{30})+(\widehat{31}). \tag{33c}$$

Indeed, if we let $\overline{\alpha} \triangleq \max_{x \in X_f \cap \overline{X}} \|x\|_1$, then we have

$$e_{\text{rel}} = \max_{\alpha \in (0, \overline{\alpha}]} \left\{ \max_{j=1,\dots,n_u} \left\{ \max\{\sigma_-^j(\alpha), \sigma_+^j(\alpha)\} \right\} \right\}. \tag{34}$$

By following an approach similar to [22, Lemma 1], problem (33) can be cast as a parametric MILP (p-MILP). Denote by $x_j$ the $j$th entry of $x \in \mathbb{R}^n$. Introduce $n$ binary variables $\mu_j \in \{0, 1\}$ satisfying $[x_j < 0] \rightarrow [\mu_j = 1]$, $[x_j > 0] \rightarrow [\mu_j = 0]$ which is equivalent to

$$x_j \leq K(1 - \mu_j), \quad x_j \geq -K\mu_j, \tag{35}$$

where $K \geq \max_{x \in X_f \cap \overline{X}} \|x\|_\infty$ (determining $K$ requires $n$ LPs, or no LP if, for instance, $X_f$

is a box). By letting $v_j \triangleq \min(x_j, 0) = x_j\mu_j$, the condition $\|x\|_1 = \alpha$ can be represented as

$$\begin{array}{l} \alpha = \sum_{i=1}^{m}(x_j - 2v_j), \\ v_j \leq K\mu_j, \quad v_j \geq -K\mu_j, \\ v_j \leq x_j + K(1-\mu_j), \\ v_j \geq x_j - K(1-\mu_j). \end{array} \tag{36}$$

So, problem (33) is equivalent to the following p-MILP

$$\sigma^j_{\pm}(\alpha) = \max_{x,\delta,z,\widehat{\delta},\widehat{z},\mu,v} \quad \frac{\pm\left(\sum_{i=1}^{\widehat{\ell}} \widehat{z}^i_j - \sum_{i=1}^{\ell} z^i_j\right)}{\alpha} \tag{37a}$$

$$\text{subj. to} \quad (32b), (35), (36). \tag{37b}$$

In order to solve problem (37), we define $\overline{\sigma}^j_{\pm}(\alpha) \triangleq \alpha \cdot \sigma^j_{\pm}(\alpha)$ for all $j = 1, \ldots, n_u$. Then we have

$$\overline{\sigma}^j_{\pm}(\alpha) = \max_{x,\delta,z,\widehat{\delta},\widehat{z},\mu,v} \quad \pm\left(\sum_{i=1}^{\widehat{\ell}} \widehat{z}^i_j - \sum_{i=1}^{\ell} z^i_j\right) \tag{38a}$$

$$\text{subj. to} \quad (32b), (35), (36). \tag{38b}$$

and so $\overline{\sigma}^j_{\pm}(\alpha)$ is a (possibly noncontinuous) piecewise affine function [23], for all $j = 1, \ldots, n_u$. From (34) we have $e_{\text{rel}} = \max_{\alpha\in(0,\overline{\alpha}]} \{(1/\alpha)\overline{\sigma}(\alpha)\}$, where

$$\overline{\sigma}(\alpha) \triangleq \max_{j=1,\ldots,n_u} \left\{\max\{\overline{\sigma}^j_{-}(\alpha), \overline{\sigma}^j_{+}(\alpha)\}\right\}$$

is a piecewise affine function, which may be explicitly computed as in [23]. It follows that $e_{\text{rel}}$ can be computed as the maximum of a scalar piecewise hyperbolic function over a bounded interval. □