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Stochastic Programming Applied to Model Predictive Control

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Abstract— **Many robust model predictive control (MPC) schemes are based on min-max optimization, that is, the future control input trajectory is chosen as the one which minimizes the performance due to the worst disturbance realization. In this paper we take a different route to solve MPC problems under uncertainty. Disturbances are modelled as random variables and the expected value of the performance index is minimized. The MPC scheme that can be solved using Stochastic Programming (SP), for which several efficient solution techniques are available. We show that this formulation guarantees robust constraint fulfillment and that the expected value of the optimum cost function of the closed loop system decreases at each time step.**

Keywords: Stochastic systems, Robust control, Predictive control for linear systems

I. INTRODUCTION

Model predictive control (MPC) is a popular strategy originated in the late seventies. The basic idea of MPC is the explicit use of a model of the process to predict the output at future time instants and to obtain the control signal by minimizing a cost function that depends on such predictions. The control inputs are implemented in accordance with a receding horizon scheme.

Standard MPC algorithms, however, do not take directly into account model uncertainties and disturbances. Although the feedback mechanism itself is able to partially compensate for them, robust control designs that cope with uncertainties in an explicit way are of interest in modern MPC theory.

Many robust MPC schemes are based on the min-max strategy originally proposed in [1], where the performance index due to the worst possible disturbance realization is minimized. Several strategies may be found in the literature, see [2], [3], [4], [5], [6], [7] and the references therein. In all cases, the resulting min-max optimization problems are computationally very demanding and, in general, it is common feeling the control laws are too conservative.

Stochastic MPC takes a different route to solve MPC problems under uncertainty. Disturbances are modelled as random variables and the expected value of the cost function is minimized. As in the min-max case, feedback predictions are taken into account (see [5]). The stochastic view of the disturbance in MPC could be traced back to Clarke's Generalized Predictive Control [8]. Like in many approaches that follow the same line of thinking, the results are valid only in the unconstrained case. Recent works in SMPC deal with input constraints for different classes of models, see e.g. [9], [10]. However, state constraints are not tackled, and efficient algorithms for evaluating the control law are not provided.

In this paper, we formulate robust MPC schemes that can be solved by Stochastic Programming (SP) techniques as in [11]. Stochastic programming is a special class of mathematical programming that involves optimization under uncertainty (see [12], [13], [14]). The original applications were agricultural economics, aircraft route planning and production of heating oil back in the 50's. Nowadays SP is becoming a mature theory that is successfully applied in several other application domains (see the survey [15]). For other contributions in control theory of SP techniques the reader is referred to [16], [17], [11]. From the computational viewpoint specific efficient algorithms for stochastic LP and QP are available in the literature (see for example [18], [19], [20], [21]) and commercial solutions to SP were announced recently [22].

In Section II the problem formulation is presented. In Section III some properties of the controller are presented. Some simulation results are given in Section IV.

II. PROBLEM FORMULATION

Consider the following discrete time linear uncertain system

$$\begin{aligned} x_{k+1} &= A(w_{k+1})x_k + B(w_{k+1})u_k + D(w_{k+1}) \\ x_{k+1} &= \phi(x_k, u_k, w_{k+1}) \end{aligned} \tag{1}$$

subject to state and input linear constraints defined by $x_k \in \mathbb{X} \subset R^{n_x}$ and $u_k \in \mathbb{U} \subset R^{n_u}$, where $x_k \in R^{n_x}$ is the state of the system, $u_k \in R^{n_u}$ the input vector, and $w_{k+1} \in R^{n_w}$ is an unknown uncertainty that we model as a random variable that lies in a bounded set $W \subseteq R^{n_w}$.

We assume that the predicted state can be always defined as the convex combination of q vertices θ_l in the following way

$$\begin{aligned} \phi(x,u,w) &= \sum_{l=1}^{q} \mu_l(w)\phi(x,u,\theta_l), \\ \mu_l(w) &\geq 0, \ \sum_{l=1}^{q} \mu_l(w) = 1. \end{aligned} \tag{2}$$

The functions $\mu_l(w)$ may not be univocally determined and a definition criterion must be provided as for example minimizing the norm of the vector $\mu(w) = [\mu_1(w), \ldots, \mu_l(w)]^T$. Typical models that satisfy this assumption are uncertain FIR

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systems [2] and polytopic systems given as the convex hull of l matrices [4].

The stage cost $L_p(x,u)$ and the terminal cost $F_p(x)$ are defined as

$$L_\infty(x,u) = \|Qx\|_\infty + \|Ru\|_\infty,$$
$$F_\infty(x) = \|Px\|_\infty,$$

for $p = \infty$ and as

$$L_2(x,u) = x^T Q x + u^T R u,$$
$$F_2(x) = x^T P x,$$

for $p = 2$.

The Stochastic Model Predictive Control (SMPC) problem proposed in this work is defined as:

$$\begin{aligned} J^*(x_0) = \min_{u_0,\ldots,u_{N-1}} & \; J(x_0) \\ \text{s.t.} \quad & x_{k+1} = \phi(x_k, u_k, w_{k+1}) \\ & x_{k+1} \in \mathbb{X}, \; \forall w_{k+1} \in W \\ & x_N \in \Omega, \; \forall w_{k+1} \in W \\ & u_k \in \mathbb{U} \\ & k = \{0,\ldots,N-1\} \end{aligned} \tag{3}$$

where

$$\begin{aligned} J(x_0) = \; & L_p(x_0,u_0) + E_1[L_p(x_1,u_1) + E_2[L_p(x_2,u_2) \\ & + E_2[\ldots + E_{N-1}[L_p(x_{N-1},u_{N-1}) \\ & + E_N[F_p(x_N)]]\ldots]]], \end{aligned}$$

and x_0 is the current state vector of the system under control. In (3),

$$\begin{aligned} & u_k \equiv u_k(w_1,\ldots,w_k) \\ & x_k \equiv x_k(x_0,u_0,\ldots,u_{k-1},w_1,\ldots,w_k) \\ & E_k[\,.\,] \equiv E_{w_k|w_1,\ldots,w_{k-1}}[\,.\,]. \end{aligned}$$

Note that the expectation evaluated for time step k considers the previous disturbances $w_1 \ldots w_{k-1}$ as known parameters, and the expectation is evaluated with respect to the future unknown disturbance w_k.

Vector x_k is the predicted state at time k. Thus, x_k is a random vector that depends on all inputs and disturbances trajectories, as well as the initial state x_0.

The stochastic model predictive control (SMPC) controller presented here has an equivalent formulation to that of the Closed-Loop Constrained Robust Optimal Control problem (CL-CROC) formulated in [6], [5], [7]. The main difference is that rather than minimizing the maximum of the cost function with respect to all the possible disturbance realizations, we minimize the expected cost over a given horizon N.

Problem (3) is a multi-stage SP problem. Stochastic programming (SP) is a special class of mathematical programming that involves optimization under uncertainty (see [12]). Such problems occur in various streams of industry and economical theory. Multi-stage problems are made of a succession of random events and recourse decisions. Each decision is a different stage and stages are divided by random events. In the proposed controller, SMPC, the decisions are the future control inputs while the random events are the disturbances acting on the model. This way of structuring the problem provides the desired "closed-loop prediction" or "feedback prediction" feature, a highly desirable property for reducing the conservatism of the control action (for instance, see examples in [23], [6]).

A. Scenario Generation

In general, uncertainty is best modelled as a continuous random variable. However, in practice, continuous probability distributions are too difficult to handle from a computational viewpoint. Therefore, discrete probability measures have a prominent role in approaches based on stochastic programming. Besides turning integrals into sums, discrete distributions allow equivalent representations of optimization models as block-structured large scaled deterministic optimization mathematical programs.

By resorting to sampling of the continuous distribution, the number q of possible values of the uncertainty at each time step determines the complexity of the controller.

In this paper is proposed to use a discrete probability distribution $\mathbf{W}(w)$ of the q vertices of W each with a probability p_l defined by

$$p_l = \int_{w \in W} \mu_l(w)\mathbf{W}(w)dw. \tag{4}$$

It is important to note that although only the vertices θ_l are taken into account, robust constraint satisfaction can still be assured.

With w_k discretized, the realized values of the uncertainty give rise to what is called a *scenario tree*, see Figure 1. The root node of the tree represents the initial time step $k = 0$. Each new level of the tree stands for a new time step and contains all possible uncertainty trajectories.

Each node has q children, one for each possible realization. The conditional probability of visiting the n^{th} node in the k^{th} time step from its parent node is denoted p_k^n and is equal to the probability of the corresponding uncertainty realization w_k^n, this is, if $w_k^n = \theta_l$ then $p_k^n = p_l$.

Each node represents a possible uncertainty and input trajectory and is assigned a set of state and input variables $\{\hat{x}_k^n, u_k^n\}$, with $k = 0,\ldots,N$ and $n \in \mathcal{N}(k)$ where $\mathcal{N}(k)$ is the set of nodes at level k of the scenario tree. The uncertainty w_k^n is the uncertainty realization that defines the node state vector from the Father node, namely,

$$\hat{x}_{k+1}^n = \phi(\hat{x}_k^{n'}, u_k^{n'}, w_{k+1}^n),$$
$$Node(k,n') = Father(Node(k+1,n)).$$

Each node has also an unconditional probability of being visited P_k^n, which is equal to the product of conditional probabilities along the path to that node. By definition $\sum_{n \in \mathcal{N}(k)} P_k^n = 1$.

B. Deterministic Model

Problem (3) can be formulated as a (large) mathematical optimization program if the probability distribution $\mathbf{W}(w)$ of the uncertainty is discrete. As mentioned before, each node

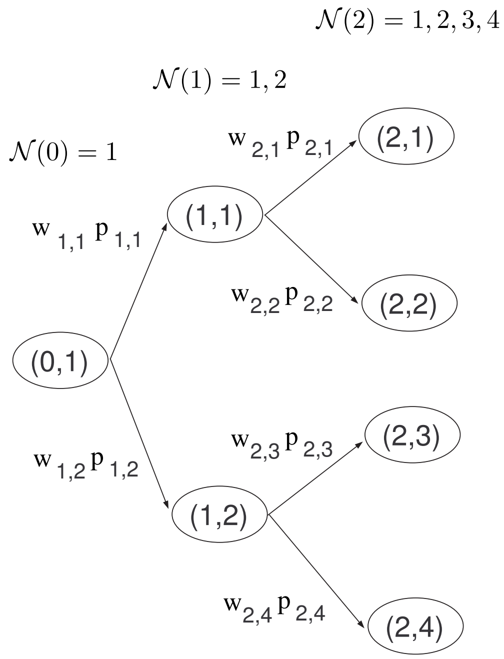

Fig. 1. Scenario tree with $N = 2$ and $q = 2$

of the scenario tree is assigned a set of variables $\hat{x}_k^n$ and u_k^n.

$$\begin{array}{ll} J^*(x_0) = \min\limits_U J(x_0, U) & \\ \text{s.t.} \quad \hat{x}_{k+1}^n = \phi(\hat{x}_k^{n'}, u_k^{n'}, w_{k+1}^n), & \\ \quad\quad Node(k, n') = Father(Node(k+1, n)), & \\ \quad\quad \hat{x}_0^1 = x_0, \hat{x}_k^n \in \mathbb{X}, \hat{x}_N^n \in \Omega, u_k^n \in \mathbb{U} & \\ \quad\quad \forall k = 0, \ldots, N-1, \ \forall n \in \mathcal{N}(k). & \end{array} \tag{5}$$

where the optimization vector is given by

$$U = \{u_k^n : k = 0, \ldots, N-1; \ n \in \mathcal{N}(k)\},$$

and the objective function is

$$\begin{aligned} J(x_0, U) = & \sum_{k=0}^{N-1} \sum_{n \in \mathcal{N}(k)} P_k^n L_p(\hat{x}_k^n, u_k^n) \\ & + \sum_{n \in \mathcal{N}(N)} P_N^n F_p(\hat{x}_N^n). \end{aligned}$$

The objective function is optimized with respect to the whole set of input variables. Note that the predicted state vectors are linear functions of the input and disturbance trajectories of the corresponding node, therefore, the (large scale) optimization program that defines the SMPC controller can be formulated either as an LP program, if the ∞-norm or the 1-norm is used in the cost function, (see, e.g., [24]) or as a QP program, if squared Euclidean norms are used.

While the size of a stochastic program is linear with the number of scenarios, the number of scenarios is exponential with the number of stages, which makes the solution of the deterministic program (5) a difficult task. However, stochastic programs have a very definite structure which can be exploited to solve the problems in an efficient way. Different algorithms can be found in the literature for exploiting this structure, see for example [18], [20], [21]).

It is important to note that as the stochastic programs have an LP or an QP large scale equivalent, the optimal cost function $J^*(x)$ is a convex function on x.

III. Properties of the proposed controller

In this section some properties of the proposed controller are presented.

Lemma 1: If the terminal region Ω is chosen to be a robust admissible invariant set for a given linear feedback law K then, given a feasible solution for x_0, U, for any possible realization of the uncertainty $w_1 \in W$, it is possible to build a new feasible set of input variables $U_s(w_1)$ that satisfies the problem constraints for $\phi(x_0, u_0^1, w_1)$.

Proof: For each vertex θ_l of W a subtree S_l is defined as all the nodes with $w_1 = \theta_l$. The nodes of the subtree l at level k are defined as

$$\mathcal{N}_l(k) = \{n \in \mathcal{N}(k) : Node(k, n) \in S_l\}$$

and the control inputs of the nodes on level N are given by the local control law K, namely

$$u_N^n = K\hat{x}_N^n, \ \forall n \in \mathcal{N}(N).$$

For the new leaf nodes, no new variable $\hat{x}_{N+1}^n$ are added. The new states are given by the prediction equation depending on the last state, the known input and a given uncertainty realization, namely as $\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)$.

A set of feasible input variables can then be obtained as

$$U_s(\theta_l) = \{u_k^n : k = 1, \ldots, N; \ n \in \mathcal{N}_l(k)\} \tag{6}$$

This set of variables are defined for $k = 1, \ldots, N$ as we consider $x_1 = \phi(x_0, u_0^1, \theta_l)$. This change of initial time step is used in the proof of the next theorem. Figure 2 shows the two subtrees for the scenario tree of Figure 1.

As Ω is chosen to be a robust admissible invariant set for K, $\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l) \in \Omega$ and $K\hat{x}_N^n \in \mathbb{U}$ so (5) is also satisfied for $k = N+1$ and $U_s(\theta_l)$ satisfies (5) for $\phi(x_0, u_0^1, \theta_l)$.

As all the constraints are linear, using (2)

$$U_s(w_1) = \sum_{l=1}^{q} \mu_l(w_1) U_s(\theta_l)$$

satisfies (5) for $\phi(x_0, u_0^1, w_1)$. ■

It is important to note that in this way, robust constraint satisfaction is assured.

The value of the cost function for a given vertex of the uncertainty for the feasible set of variables (6) can be posed as

$$\begin{aligned} & J(\phi(x_0, u_0^1, \theta_j), U_s(\theta_j)) = \\ & \sum_{k=1}^{N-1} \sum_{n \in \mathcal{N}_j(k)} \frac{P_k^n}{p_j} L_p(\hat{x}_k^n, u_k^n) \\ & + \sum_{n \in \mathcal{N}_j(N)} \frac{P_N^n}{p_j} L_p(\hat{x}_N^n, K\hat{x}_N^n) \\ & + \sum_{n \in \mathcal{N}_j(N)} \sum_{l=1}^{q} \frac{P_N^n}{p_j} p_l F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)). \end{aligned} \tag{7}$$

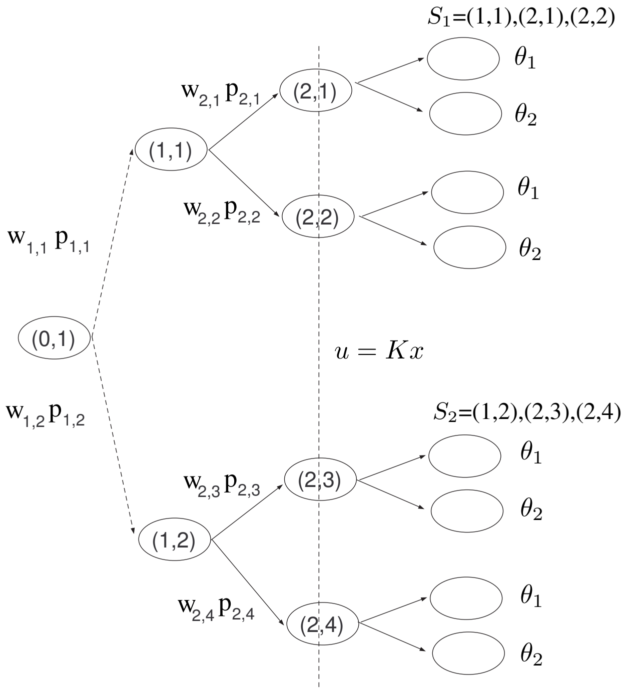

Fig. 2. Expanded tree with $N = 2$ and $q = 2$

It is important to note that both the cost function for x_0 and the cost function for the proposed feasible set of inputs, are defined using the variables of the original scenario tree.

Theorem 1: If the terminal region Ω is chosen to be a robust admissible invariant set for a given linear feedback law K and the terminal cost $F_p(x)$ satisfies

$$F_p(\phi(x, Kx, w)) - F_p(x) \leq -L_p(x, Kx) + \gamma, \forall w \in W,$$

then the following inequality holds

$$E[J^*(\phi(x_0, u_0^1, w))] - J^*(x_0) \leq -L_p(x_0, u_0^1) + \gamma,$$

where u_0^1 is the first control input of the optimal solution $U^*(x_0)$.

Proof: Taking into account that $J^*(x)$ is a convex function, (2) and (4), the following inequalities hold

$$\begin{aligned}
&E[J^*(\phi(x_0, u_0^1, w))] = \int_{w\in W} J^*(\phi(x_0, u_0^1, w))\mathbf{W}(w)dw \\
&\leq \int_{w\in W} \sum_{j=1}^{q} \mu_j(w) J^*(\phi(x_0, u_0^1, \theta_j))\mathbf{W}(w)dw \\
&\leq \sum_{j=1}^{q} J^*(\phi(x_0, u_0^1, \theta_j)) \int_{w\in W} \mu_j(w)\mathbf{W}(w)dw \\
&\leq \sum_{j=1}^{q} p_j J^*(\phi(x_0, u_0^1, \theta_j))
\end{aligned}$$

From the optimum solution $U^*(x_0)$, q feasible solution sets $U_s(\theta_j)$ are builded, one for each possible realization of the uncertainty θ_j. It is clear to see that because of optimality,

$$E[J^*(\phi(x_0, u_0^1, w))] \leq \sum_{j=1}^{q} p_j J(\phi(x_0, u_0^1, \theta_j), U_s(\theta_j))$$

Taking into account this inequality, (5) and (7), and that $\sum_{j\in\mathcal{N}(N)} P_N^n = 1$, the following inequalities holds

$$\begin{aligned}
&E[J^*(\phi(x_0, u_0^1, w))] - J^*(x_0) + L_p(x_0, u_0^1) \\
&\leq \sum_{n\in\mathcal{N}(N)} P_N^n L_p(\hat{x}_N^n, K\hat{x}_N^n) - P_N^n F_p(\hat{x}_N^n) \\
&+ \sum_{n\in\mathcal{N}(N)} \sum_{l=1}^{q} P_N^n p_l F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)) \\
&\leq \max_{n\in\mathcal{N}(N)} [L_p(\hat{x}_N^n, K\hat{x}_N^n) - F_p(\hat{x}_N^n) \\
&+ \sum_{l=1}^{q} p_l F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l))] \sum_{n\in N(N)} P_N^n \\
&\leq \max_{n\in\mathcal{N}(N)} [L_p(\hat{x}_N^n, K\hat{x}_N^n) - F_p(\hat{x}_N^n) \\
&+ \sum_{l=1}^{q} p_l F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l))]
\end{aligned}$$

Let us denote $\Gamma(\hat{x}_N^n) = L_p(\hat{x}_N^n, K\hat{x}_N^n) - F_p(\hat{x}_N^n)$, for all $n \in \mathcal{N}(N)$, as $\sum_{l=1}^{q} p_l = 1$, the following inequalities are satisfied

$$\begin{aligned}
&\Gamma(\hat{x}_N^n) + \sum_{l=1}^{q} p_l F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)) \\
&\leq \Gamma(\hat{x}_N^n) + \max_{l=1\ldots q} F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)) \sum_{l=1}^{q} p_l \\
&= \Gamma(\hat{x}_N^n) + \max_{l=1\ldots q} F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l))
\end{aligned}$$

Taking into account that

$$F_p(\phi(x, Kx, w)) - F_p(x) + L_p(x, Kx) \leq \gamma, \forall w \in W,$$

is satisfied for any possible state vector, it holds

$$\Gamma(\hat{x}_N^n) + \max_{l=1\ldots q} F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)) \leq \gamma,$$

so

$$\Gamma(\hat{x}_N^n) + \sum_{l=1}^{q} p_l F_p(\phi(\hat{x}_N^n, K\hat{x}_N^n, \theta_l)) \leq \gamma,$$

As this inequality is satisfied for any $\hat{x}_N^n$, then

$$E[J^*(\phi(x_0, u_0^1, w))] - J^*(x_0) \leq -L_p(x_0, u_0^1) + \gamma.$$

■

Theorem 1 and Lemma 1 guarantee robust constraint satisfaction, but do not guarantee convergence in a deterministic way. However, the expected value of the optimal cost function is shown to decrease on each step so a certain degree of convergence is proved. This is related to the definition of *stochastic stability*, see [25].

IV. EXAMPLE

Consider the discrete-time equivalent of

$$G(s) = \frac{-20(s - 10)}{(s + 2)(s^2 + 5s + 100)}, \tag{8}$$

for a sampling time $T_s = 0.3s$. We use a CARIMA model with integrated uncertainty

$$\Delta A(z^{-1})y_k = B(z^{-1})\Delta u_{k-1} + w_k,$$

with $|w_k| \leq \epsilon$. While no output constraints are present, Δu must be in $[-0.5\ 0.5]$.

TABLE I

AVERAGE COSTS CUMULATED ALONG ACTUAL TRAJECTORIES.

	SMPC	Nom	FMM
$\epsilon = 0.5$	40.4	47.0	49.5
$\epsilon = 1$	56.9	64.6	65.3
$\epsilon = 2$	79.4	88.8	92.9

TABLE II

COMPUTATIONAL ASPECTS FOR DIFFERENT PREDICTION HORIZONS.

	var	con	FMM(s)	SMPC(s)
$N = 5$	481	+2000	0.04	0.01
$N = 6$	1153	+5000	0.11	0.02
$N = 7$	2688	+11000	0.25	0.04
$N = 8$	6145	+25000	1.05	0.12
$N = 9$	13825	+60000	2.35	0.35

The stage and terminal costs have the form

$$\begin{aligned} L_\infty(x, \Delta u) &= |Q(Cx - y_r)| + |R\Delta u| \\ F_\infty(x) &= |P(Cx - y_r)|, \end{aligned}$$

respectively, where y_r is the desired output set-point, $Q = P = 1$ and $R = 2$. The following controllers are taken into account:

- SMPC: SMPC controller with $N = 5$;
- Nom: Nominal MPC controller with $N = 5$;
- FMM: Min-max MPC controller with $N = 5$;

Different simulations for different values of the disturbance amplitude ϵ were done over 50 time steps and for a set-point change from $y_r = 0$ to $y_r = 4$. The total cumulated costs, which are computed by summing up $L_p(x_k, u_k)$ for $k = 0, \ldots, 50$ and averaging over a hundred simulations with different random disturbances, are reported in Table I.

The min-max controller has been evaluated as a single large scale linear problem. The problem is defined as the SMPC problem, but instead of minimizing the weighted sums of the cost of each possible trajectory, the maximum one is minimized.

A. Computational Aspects

Table II shows the size of the large scale optimization programs in number of optimization variables (var), and number of constraints (con). The entry FMM shows the time required to evaluate the Min-max MPC while using the MOSEK solver offered in [26]. Entry SMPC shows the time required to evaluate the SMPC using the MSLiP solver offered in [26]. This solver implements a nested Benders decomposition method for the multistage stochastic linear programming problem [18].

It is important to note that in the case of quadratic cost criterions, the SMPC optimization problem is a quadratic program that can be solved by specific methods like nested Benders decomposition, while the min-max formulation can not be solved using quadratic programming techniques and nowadays is regarded as too complex for real implementation.

V. CONCLUSIONS

In this paper we have investigated a robust MPC formulation that copes with model uncertainties and disturbances based on stochastic programming ideas. Rather than minimizing the worst case of the cost function for all possible disturbance realizations, we minimize an approximation of the expected value. The resulting stochastic MPC control action is obtained by solving a stochastic programming problem, for which several efficient solution techniques are available.

Robust constraint satisfaction is proved and also that the expected value of the cost function decreases at each time step. Although this does not assure convergence to the origin, this is related to the definition of *stochastic stability*, see [25]. Further works include studies on this issue.

VI. ACKNOWLEDGEMENTS

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