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Symbolic dynamics for active sets of a class of constrained nonlinear optimal control and MPC problems [1]

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Abstract: Solutions to optimal control problems are usually understood to provide optimal trajectories. In this paper, we show that the optimal state-space system dynamics induce a dynamics of the active sets. More specifically, given the optimal active set at the solution obtained at the current time, its successor optimal active set (which, in turn, defines the successor solution) can be found with index set operations. These operations do not involve any optimal control (or other optimization or integration) problem, but they can be described with simple rules. These rules constitute the symbolic dynamics for active sets. The present paper treats a particular constrained nonlinear problem class, extending earlier results for the constrained linear-quadratic case.

Keywords: optimal control, nonlinear model predictive control

1. INTRODUCTION

Constrained optimal control problems are more demanding than their unconstrained counterparts. For example, discrete-time finite-horizon constrained linear optimal control problems, such as those that arise in linear model predictive control (MPC) formulations, admit a simple analytical linear state-feedback solution in the absence of constraints, while they require more substantial online or offline computation effort in the presence of linear constraints.

It has proven to be useful to characterize the solution to constrained optimal control problems with the set of active sets, where we call a subset of the indices to all constraints *active set* if this combination of constraints is active for at least one feasible initial state. The *set of active sets* then is the set of all active sets that may appear when solving the optimal control problem (OCP). Constructing the set of active sets is often associated with constructing explicit solutions (Gupta et al., 2011; Herceg et al., 2013; Mitze and Mönnigmann, 2020; Oberdieck et al., 2017; Herceg et al., 2015; Feller et al., 2013), specifically for the constrained linear-quadratic regulator (Bemporad et al., 2002; Tøndel et al., 2003; Seron et al., 2002).

Even if no explicit feedback law is ultimately constructed, the set of active sets of an OCP is still useful, as it defines and characterizes its solution. We claim that this characterization is not only interesting *per se* but also in practice. If an explicit solution is not computable, either because it is too expensive to store and evaluate (even for moderately complex linear-quadratic problems), or not computable at all (as in the general nonlinear case), knowing the optimal active set can greatly simplify online optimization. For example in the simple case of upper and lower bounds only, the OCP becomes unconstrained and with a smaller or equal number of optimization variables to determine. Even in case an explicit solution is predetermined offline, knowing the optimal active set can greatly simplify the solution of the *point-location problem* online.

The characterization with active sets has led to several interesting insights for the linear-quadratic case. Neighboring regions of the piecewise solution to the OCP can be identified from analyzing the active sets (Ahmadi-Moshkenani et al., 2018). Similarly, it has been shown that dynamic programming can be carried out with the active sets (Mönnigmann, 2019), which helps to accelerate the construction of all active sets (Mitze and Mönnigmann, 2020). Finally, symmetries of the solution to the OCP can be found in the set of active sets (Mitze et al., 2023). All these results are essentially based on *geometric* relations of the regions defined by the active sets without requiring geometric calculations.

In the present paper, we focus on *dynamic relations* of active sets for a particular *nonlinear* problem class. Specifically, we show that the successor active set can be found with very simple operations on the current active set (such as index shifts of index sets). In particular, no

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optimal control problems (or other optimization problems) need to be solved to obtain successor active sets. Successor active sets, which define the successor optimal solution, can be constructed both for open-loop optimal and closed-loop optimal solutions, where "closed-loop optimal" refers to the usual MPC use of the OCP solution on a receding horizon. We note that *dynamic* relations of active sets have been used to determine robust MPC solutions for the linear-quadratic case (Mönnigmann and Pannocchia, 2020) and a shrinking horizon nonlinear case (Dyrska and Mönnigmann, 2024) before.

Section 2 introduces the problem class. Section 3 states the main results. It first introduces the symbolic dynamics informally (see (I), (II), (III) in subsection 3.3), illustrates these ideas with a sample problem, and states formal results. Brief conclusions and an outlook are stated in Section 4.

2. PROBLEM STATEMENT

We consider discrete-time, nonlinear systems of the form

$$x(k+1) = f(x(k), u(k)),\ k = 0, 1, \dots \tag{1a}$$

with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}^m$, and a nonlinear state-update function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We assume states and inputs are subject to constraints

$$\begin{aligned} x(k) &\in \mathcal{X} \subset \mathbb{R}^n, \\ u(k) &\in \mathcal{U} \subset \mathbb{R}^m \end{aligned} \tag{1b}$$

for all time steps k, where $\mathcal{X}$ and $\mathcal{U}$ are compact sets that contain the origin in their interior and that can be described as the intersection of a finite number of sublevel sets. Furthermore, we assume f is twice continuously differentiable on an open superset of $\mathcal{X} \times \mathcal{U}$, and $f(0, 0) = 0$.

We are interested in the infinite-horizon nonlinear optimal control problem for (1)

$$\min_{u(k),x(k+1),k=0,1,\dots} \sum_{k=0}^{\infty} \ell(x(k), u(k)) \tag{2a}$$

subject to

$$\begin{aligned} &x(k+1) = f(x(k), u(k)),\ k = 0, 1, \dots \\ &x(k) \in \mathcal{X},\ k = 0, 1, \dots \\ &u(k) \in \mathcal{U},\ k = 0, 1, \dots \end{aligned} \tag{2b}$$

for given current state $x(0)$, and the stage cost $\ell(x, u)$, which is specified later. The finite-horizon problem over N steps

$$\min_{u(k),x(k+1),k=0,\dots N-1} \ell_{\mathcal{T}}(x(N)) + \sum_{k=0}^{N-1} \ell(x(k), u(k)) \tag{3a}$$

subject to

$$\begin{aligned} &x(k+1) = f(x(k), u(k)),\ k = 0, ..., N-1 \\ &x(k) \in \mathcal{X},\ k = 0, ..., N-1 \\ &u(k) \in \mathcal{U},\ k = 0, ..., N-1 \\ &x(N) \in \mathcal{T}. \end{aligned} \tag{3b}$$

is used as an auxiliary problem, where $\ell_{\mathcal{T}}(x)$ and $\mathcal{T} \subseteq \mathcal{X}$ with $0 \in \operatorname{int} \mathcal{T}$ are a terminal cost and constraint, respectively, which are described in more detail in Assumption 1. Let $\mathcal{F}_N$ refer to the set of initial states $x(0)$ for which (3) with horizon N has a solution.

The following Assumption 1 allows to relate solutions of (2) and (3). The assumption is mild for linear systems but strong for nonlinear systems. It therefore deserves some comments, which are stated below.

Assumption 1. Assume there exist a set $\mathcal{T} \subseteq \mathcal{X}$, a feedback law $\kappa : \mathcal{T} \to \mathcal{U}$ and a terminal cost $\ell_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}$ such that $\operatorname{int} \mathcal{T}$ is positive invariant for the closed-loop system and such that, for any $x(0) \in \mathcal{T}$, evaluating

$$\begin{aligned} u(k) &= \kappa(x(k)), \\ x(k+1) &= f(x(k), \kappa(x(k))) \end{aligned} \tag{4}$$

for $k = 0, 1, \dots$ yields a minimum for (2) for initial condition $x(0)$ with cost $\sum_{k=0}^{\infty} \ell(x(k), u(k)) = \ell_{\mathcal{T}}(x(0))$.

Assumption 1 is strong because it essentially states we know an optimal solution to the *infinite-horizon* problem for all initial conditions from $\mathcal{T}$ and know the corresponding infinite-horizon cost. This assumption can easily be fulfilled for linear-quadratic problems under mild additional conditions (Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998), (Bemporad et al., 2002, Sect. 3). A nonlinear class that respects Assumption 1 results for switching cost functions of the form

$$\ell(x, u) = \begin{cases} \tilde{\ell}(x, u) \text{ if } x \in \mathcal{T} \\ \hat{\ell}(x, u) \text{ otherwise.} \end{cases}$$

This switching cost function class yields the following relation of the infinite-horizon cost (2a) to the finite-horizon cost (3a):

$$\sum_{k=0}^{\infty} \ell(x(k), u(k) = \underbrace{\sum_{k=N}^{\infty} \tilde{\ell}(x(k), u(k))}_{\ell_{\mathcal{T}}(x(N))} + \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

where only $\tilde{\ell}$ appears in $\ell_{\mathcal{T}}$ because $x(N+k) \in \mathcal{T}$ for all $k \geq 0$ if $\operatorname{int} \mathcal{T}$ is positive invariant. Also note that $\ell_{\mathcal{T}}(x(N)) = \sum_{k=N}^{\infty} \tilde{\ell}(x(k), u(k))$ can be expressed in terms of only $x(N)$, because $x^+ = f(x, \kappa_\infty(x))$ and $u(x) = \kappa(x)$ uniquely determine the sequences $u(N), u(N+1), \dots$ and $x(N+1), x(N+2), \dots$ for a given $x(N)$. The required dual-mode controller results if we determine a control law $u = \kappa(x)$ that results in the desired properties on some set $\mathcal{T}$ and then treat $x^+ = f(x, \kappa(x) + u)$ by setting $u = 0$ on $\mathcal{T}$.

The following example illustrates Assumption 1. The example is used for all illustrations in the paper.

Example 1. Consider the system (1) with

$$f(x, u) = Ax + bu + \frac{1}{4}\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix},\ A = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

constraint sets

$$\begin{aligned} \mathcal{X} &= \left\{x \in \mathbb{R}^2 | -1 \leq x_i \leq 1,\ i = 1, 2\right\} \\ \mathcal{U} &= \{u \in \mathbb{R} | -1 \leq u \leq 1\} \end{aligned}$$

stage cost

$$\ell(x, u) = \frac{1}{2} x^\top Q x + \frac{1}{2} R u^2, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1/10$$

terminal cost $\ell_{\mathcal{T}}(x) = 0$, and terminal set $\mathcal{T}$ and controller $\kappa_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}$

$$\mathcal{T} = \left\{x \in \mathbb{R}^2 | \|x\|_2^2 \leq 1\right\},\ \kappa_{\mathcal{T}}(x) = 0.$$

It is easy to show that $\mathcal{T}$ is positive invariant for $x(k+1) = f(x(k), \kappa_{\mathcal{T}}(x(k)))$. The proof is given in Appendix A for completeness.

3. SYMBOLIC DYNAMICS FOR ACTIVE SETS

3.1 Preliminaries

We need to state a clear relation between solutions of the problems with finite and infinite horizons.

Lemma 1. Consider a constrained system (1) and suppose Assumption 1 holds.

(a) Let $x(0)$ be an arbitrary initial condition such that the horizon-N problem (3) has an optimal solution, which we denote

$$(u(k), x(k+1))_{k=0}^{N-1}\,. \tag{5}$$

If $x(N) \in \operatorname{int}\mathcal{T}$, then (5) extended by (4) for all $k \geq N$ is an optimal solution for the infinite-horizon problem (2) with initial condition $x(0)$.

(b) Conversely, let $x(0)$ be an arbitrary initial condition such that the infinite-horizon problem (2) has an optimal solution, which we denote

$$(u(k), x(k+1))_{k=0}^{\infty}\,. \tag{6}$$

If there exists an N such that $x(N) \in \operatorname{int}\mathcal{T}$, then (6) truncated after N steps is an optimal solution for (3) with horizon N and initial condition $x(0)$.

The proof of Lemma 1 is stated in Appendix B. Note that Lemma 1 cannot be stated as an equivalence. Problems (2) and (3) are not equivalent because the set of admissible initial conditions is in general larger for (2) than for (3).

Remark 1. The condition $x(N) \in \operatorname{int}\mathcal{T}$ can neither be omitted nor replaced by $x(N) \in \mathcal{T}$ in part (a) of Lemma 1. An example where all conditions of Lemma 1 but $x(N) \in \operatorname{int}\mathcal{T}$ hold and the implication of Lemma 1 does not hold is given in (Mönnigmann, 2019, Example 2 and Fig. 3) for a linear example to which the lemma applies. We stress this counterexample is not a pathological or otherwise artificial case. We must expect full-dimensional regions of initial conditions to exist such that (3) has solutions with $x(N) \in \operatorname{int}\mathcal{T}$ that cannot be extended to solutions for the infinite-horizon problem (2) even if Assumption 1 holds.

3.2 Ordering the constraints

We assume without restriction the constraints are ordered stage by stage, i.e., in the order

$$\begin{aligned}
&x(0) \in \mathcal{X},\ u(0) \in \mathcal{U}, \quad (q_{\text{stage}} \text{ constraints})\\
&x(1) \in \mathcal{X},\ u(1) \in \mathcal{U}, \quad (q_{\text{stage}} \text{ constraints})\\
&\quad\vdots\\
&x(N-1) \in \mathcal{X},\ u(N-1) \in \mathcal{U}, \quad (q_{\text{stage}} \text{ constraints})\\
&x(N) \in \mathcal{T} \quad (q_{\mathcal{T}} \text{ constraints})
\end{aligned} \tag{7}$$

where the number of constraints per line is given in parentheses. We refer to the stages evident from (7) as stage 0, stage $1, \ldots,$ stage $N-1$, and the terminal stage, respectively.

It turns out to be convenient to denote sets of active constraints with bit sequences. For example, consider a problem with horizon $N = 2$, $q_{\text{stage}} = 5$, $q_{\mathcal{T}} = 3$, $q = Nq_{\text{stage}} + q_{\mathcal{T}} = 13$ constraints in the order (7) and assume the active set $\mathcal{A} = \{1, 3, 12\}$ appears. Let q denote the total number of constraints $q = N\,q_{\text{stage}} + q_{\mathcal{T}}$.

We identify $\mathcal{A}$ with

$$\underbrace{10100}_{\alpha_0}\cdot\underbrace{00000}_{\alpha_1}\cdot\underbrace{010}_{\alpha_2} \tag{8}$$

where dots are introduced to separate stages and the α_i refer to the bit tupels of the stages. We call (8) an active set to avoid phrases like 'the bit sequence identified with the active set'. Recall $\mathcal{A}$ is an active set of (3) if there exists an $x(0) \in \mathcal{F}_N$ with active set $\mathcal{A}$ at the optimal solution.

3.3 Informal summary

We intend to characterize the solution of the horizon-$N+1$ problem (3) by carrying over as much information from the horizon-N problem as possible. We stress that the solution for horizon N is in general *not* contained in the solution for horizon $N+1$. This can easily be illustrated with explicit solutions in the linear-quadratic case, where the explicit control law for horizon $N+1$ does not coincide with the law for N (wherever they both exist, i.e., on the feasible set for N; see Mönnigmann (2019), Fig. 1 for an example).

In fact, the solution for an initial condition $x(0)$ and horizon N *does* remain the same for $N+1$ if $x(N) \in \operatorname{int}\mathcal{T}$. Solutions for which $x(N) \in \mathcal{T}$ is fulfilled with $x(N) \in \partial\mathcal{T}$, where $\partial\mathcal{T}$ refers to the boundary of T, do not have this property. It may appear pedantic to even pay attention to solutions with $x(N) \in \partial\mathcal{T}$, since $\partial\mathcal{T}$ is a set with measure zero. Solutions with this property do in general exist, however, for full-dimensional regions of initial conditions (see Mönnigmann, 2019, Fig. 3, for an example again) and therefore are not negligible, because they constitute parts of $\mathcal{F}_N$ with nonzero measure.

Rather than analyzing the effect of an increasing horizon point-by-point, i.e., $x(0)$ by $x(0)$, it proved useful to carry out this analysis with the active sets and the regional optimal feedback laws they define. Essentially, we need three simple operations on active sets, which are referred to as I, II and III below. Let "x" refer to an unspecified bit in an active set. Color is used as guide to the eye. In (I) and (III) (but not in (II)) the terminal constraints are assumed to be inactive as evident from the trailing zeros:

(I) Extending an $\mathcal{A}$ with an inactive stage
An active set for (3) with horizon N may be *extended to the right* with a zero stage to obtain an active set for (3) with horizon $N+1$:

$$\begin{aligned}
\mathcal{A}_N &= \alpha_0.\alpha_1.\cdots.\alpha_{N-1}.0\ldots0\\
&\downarrow\\
\mathcal{A}_{N+1} &= \alpha_0.\alpha_1.\ldots.\alpha_{N-1}.0\ldots0.0\ldots0
\end{aligned}$$

If $\mathcal{A}_N$ is an active set for horizon N, then $\mathcal{A}_{N+1}$ is an active set for horizon $N+1$. Specifically, $\mathcal{A}_{N+1}$ defines the same region as $\mathcal{A}_N$ or a superset thereof, and the optimal solution is equal on the common region. (See Prop. 1 and Example 2 below for a concise statement of I and an example, respectively.)

(II) Deleting the first stage of an $\mathcal{A}$
Deleting a stage *on the left* in an active set for (3) with horizon N results in an active set for (3) with horizon $N-1$:

$$\begin{aligned}
\mathcal{A}_N &= \mathtt{x}\ldots\mathtt{x}.\mathtt{x}\ldots\mathtt{x}.\cdots.\mathtt{x}\ldots\mathtt{x}.0\ldots0\\
&\downarrow\\
\mathcal{A}_{N-1} &= \qquad\quad \mathtt{x}\ldots\mathtt{x}.\cdots.\mathtt{x}\ldots\mathtt{x}.0\ldots0
\end{aligned}$$

If $\mathcal{A}_N$ is an active set for horizon N, then $\mathcal{A}_{N-1}$ is an active set for horizon $N-1$. (See Prop. 2 and Example 3.)

While (I) and (II) relate active sets for *different horizons*, combining the two results in a simple relation between active sets for the *same horizon*.

(III) Symbolic dynamics for active sets Applying (I) and (II) to an active set for horizon N results in a new active set for horizon N that is part of the solution:

$$\mathcal{A}_N = \texttt{x}\ldots\texttt{x}.\texttt{x}\ldots\texttt{x}.\cdots.\texttt{x}\ldots\texttt{x}.0\ldots 0$$
$$\downarrow$$
$$\mathcal{A}'_N = \texttt{x}\ldots\texttt{x}.\cdots.\texttt{x}\ldots\texttt{x}.0\ldots 0.0\ldots 0$$

Moreover, $\mathcal{A}'_N$ defines the successor region for the region defined by $\mathcal{A}_N$. In other words, $\mathcal{A}'_N$ defines the region to which the nominal system is driven when the first optimal control signal is applied. (See Prop. 3 and Example 4.)

3.4 Precise statements and illustrations

Proposition 1 states claim (I) more precisely. The proposition 1 extends Lemma 3 from Mönnigmann (2019) for linear-quadratic problems to the nonlinear system class treated here.

Proposition 1. (a) Consider (3) with constraint order (7). If

$$\alpha_0.\cdots.\alpha_{N-1}.\underbrace{0\cdots 0}_{q_{\mathcal{T}}} \tag{9}$$

is an active set of (3) with horizon N, then

$$\alpha_0.\cdots.\alpha_{N-1}.\underbrace{0\cdots 0}_{l(q_{\mathcal{X}}+q_{\mathcal{U}})}.\underbrace{0\cdots 0}_{q_{\mathcal{T}}} \tag{10}$$

is an active set for horizon $N+l$ for all $l \geq 0$.

(b) The set defined by (10) is equal to or a superset of the set defined by (9). Both active sets yield the same solution on the set defined by (9).

Proof. (a) According to Lemma 1 part (a), the active set (9) can be extended with (4) to a solution for the infinite-horizon problem. Since $\operatorname{int}\mathcal{T}$ is positive invariant for the system subject to $u(k) = \kappa(x(k))$ from (4) by assumption, all constraints on $u(k)$, $x(k)$, $k \geq N$ are inactive. According to part (b) of Lemma 1, the infinite-horizon solution can be truncated to any length $N+l$, $l \geq 0$. Since the constraints are inactive for all stages $l \geq 0$, the active set (10) exists for all $l \geq 0$. (b) Let $x(0) \in \mathbb{R}^n$ be an arbitrary point such that there exists a minimum for (3) with horizon N and with active set $\mathcal{A}_N$. Let (5) refer to this solution. According to Prop. 1 this solution can be extended to a solution (6) for the infinite-horizon problem. The positive invariance of $\operatorname{int}\mathcal{T}$ implies $x(N+l) \in \operatorname{int}\mathcal{T}$ for all $l \geq 0$. Therefore (6) can be truncated for any $l \geq 0$ according to part (b) of Lemma 1. Since this truncated solution yields the same active constraints for the first N stages as the horizon-N problem and solution, and since all additional stages $N+l$, $l \geq 0$ are inactive, the active set (10) results for horizon $N+l$ for any $l \geq 0$. We showed that any $x(0) \in \mathbb{R}^n$ that is a solution to (3) with horizon N and active set (9) also is a solution for (3) with horizon $N+l$ and active set (10) for any $l \geq 0$. □

We illustrate claim (I) and Prop. 1 with Example 2. The results shown in Figs. 1 to 4 are obtained by solving (3) on a grid of 100×100 points and recording all solutions including the active sets.

Example 2. Figure 1 shows all initial states $x(0)$ that result in the active sets

$$000001.000000.0 \quad (N = 2) \tag{11a}$$
$$000001.000000.000000.0 \quad (N = 3) \tag{11b}$$

for (3) with Example 1 and $N = 2$ and $N = 3$, respectively. It is evident from the figures that the set defined by (11b) and $N = 3$ is a superset of the set defined by (11a). Moreover, it is evident the optimal signal $u(0) = -1$ results in both cases. The remaining optimal input signals are also equal, which we claim without showing them. The active constraint in both active sets corresponds to $u(0) = -1$.

As a side-effect, Example 2 shows that sets of points defined by an active set may not be connected in the nonlinear case.

Proposition 2 and Example 3 belong to claim (II). Note that the terminal constraints are not required to be inactive in Prop. 2. Furthermore, note that claim (II) and Proposition 2 essentially state the principle of optimality for active sets.

Proposition 2. Consider (3) with constraint order (7). If

$$\mathcal{A}_N = \alpha_0.\alpha_1\cdots.\alpha_{N-1}.\alpha_N \tag{12}$$

is an active set for (3) with horizon N, then

$$\mathcal{A}_{N-1} = \quad \alpha_1\cdots.\alpha_{N-1}.\alpha_N \tag{13}$$

is an active set for (3) with horizon $N-1$. Moreover, any $x(0)$ from the set defined by $\mathcal{A}_N$ is mapped to a successor state in the set defined by $\mathcal{A}_{N-1}$ for the closed-loop nominal system.

Proof. Let $x(0)$ be any initial condition such that (3) for horizon N results in an optimal solution with active set (12). Let the optimal solution be denoted as in (5). Then the sequence that results from (5) after removing $u(0)$ and $x(1)$, i.e.,

$$(u(k), x(k+1))_{k=1}^{N-1} \tag{14}$$

is an optimal solution for (3) with horizon $N-1$ and initial condition $x(1)$ by the principle of optimality. Since the same constraints are active for (14) as for (5) the active set (13) results, which proves the first part of the claim. Since $x(0)$ was an arbitrary initial condition from the set defined by (12), and since $x(1)$ belongs to the set defined by (13), the second part of the claim holds. □

Example 3. Figure 2 shows all initial states $x(0)$ for which (3) with Example 1 results in the active sets

$$000001.000001.000000.0 \quad (N = 3)$$
$$000001.000000.0 \quad (N = 2)$$

for $N = 2$ and $N = 3$, respectively. The yellow region in the figure for $N = 2$ results from the yellow region for $N = 3$ with the principle of optimality.

Finally, Prop. 3 and Example 4 state respectively illustrate the main result, i.e., the symbolic dynamics for active sets stated in claim (III). It is the very point of the paper to show that sequences of active sets (e.g., (15)→(16) in Prop. 3 or (18a)→(18b)→(18c) and (19a)→(19b)→(19c) in Example 4) can be inferred from an active set with

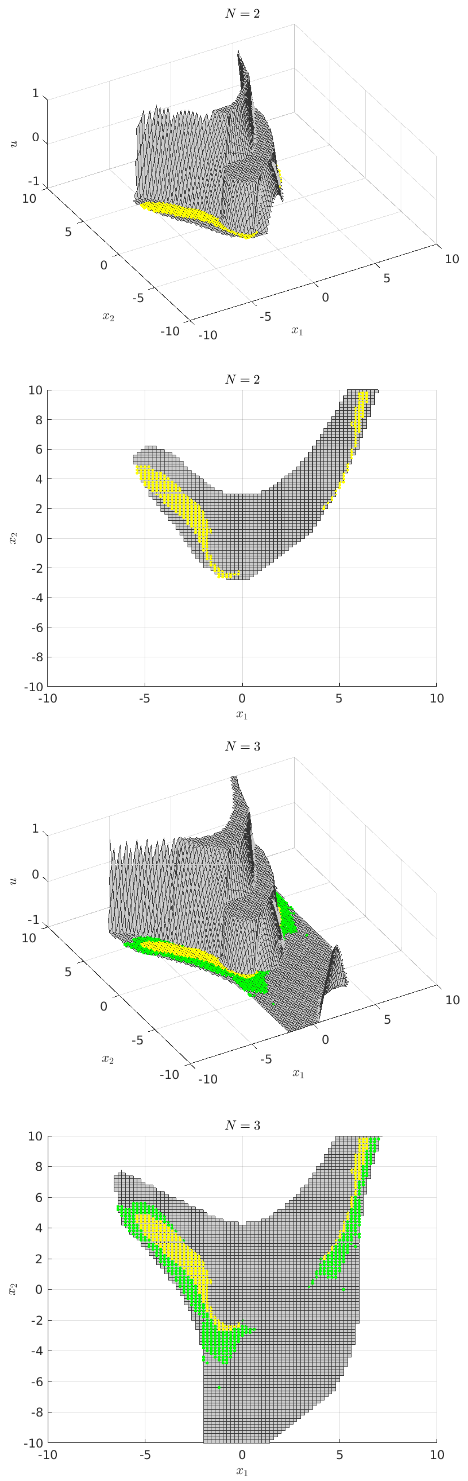

Fig. 1. Illustration of claim (I) and Prop. 1. The region defined by (11a) for $N = 2$ (yellow, top diagrams) is a subset of the region defined by (11b) for $N = 3$ (green and yellow, bottom diagrams, yellow used to highlight the region from $N = 2$). Grey area delineates $\mathcal{F}_N$.

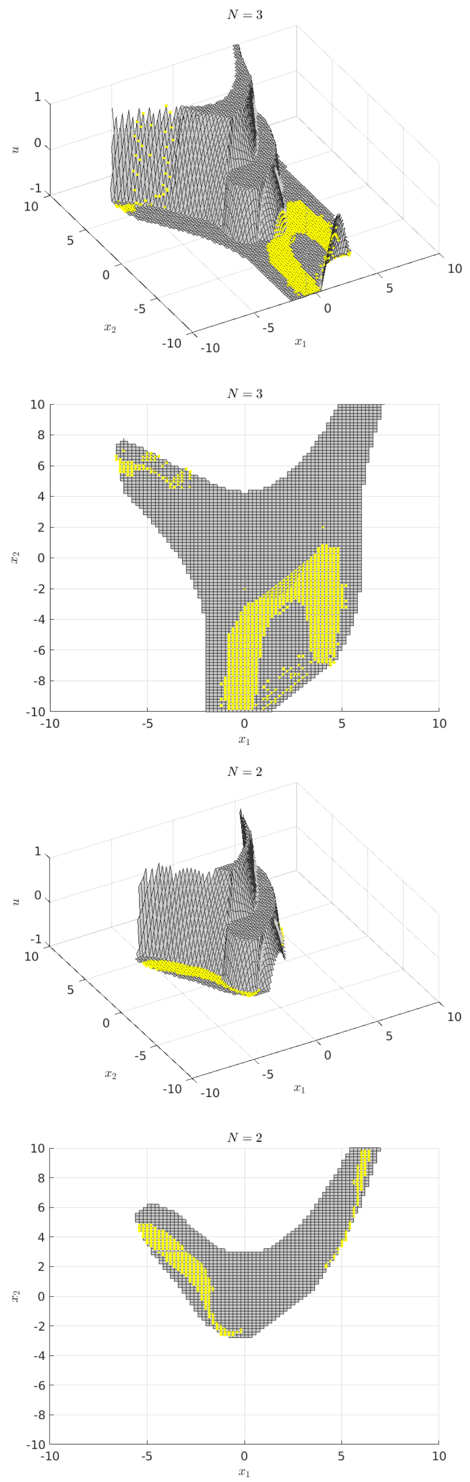

Fig. 2. Illustration of claim (II) and Prop. 2. Initial states from the region defined by (12) for $N = 3$ propagate to the region defined by (13) for $N = 2$.

inactive terminal constraints, where these sequences of active sets define sequences of regions through which the systems evolves under MPC. Obtaining these does not require solving optimal control problems, but the successor active sets can simply be constructed by deleting the first stage of the given active set and appending it by an inactive stage.

Proposition 3. Consider (3) with constraint order (7). If (15) is an active set, then (16) is an active set, where (16) results from deleting α_0 and inserting an inactive penultimate stage in (15).

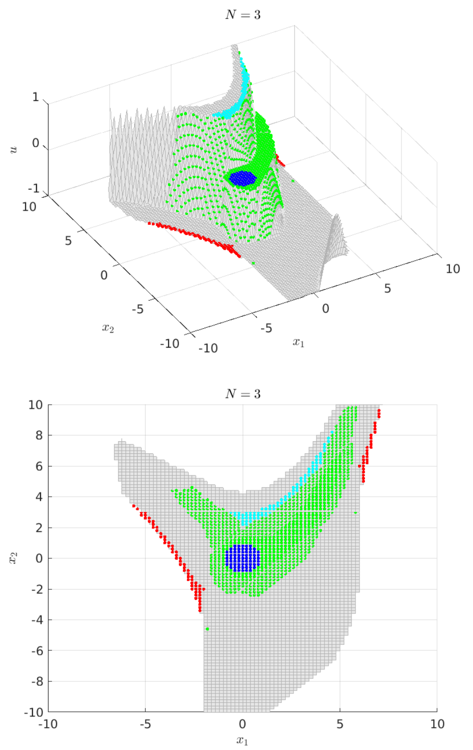

Fig. 3. Illustration of claim (III) and Prop. 3. The regions belong to the active sets (18). Any initial point $x(0)$ located in the red region passes through the cyan and green region and subsequently enters the blue region. The sequence of these active sets can be found from the first active set (18a) with claim (III) and Prop. 3 without solving any optimal control problem.

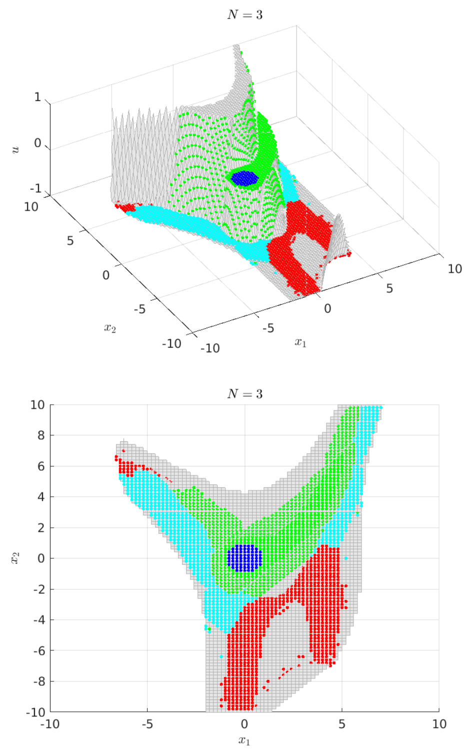

Fig. 4. Illustration of claim (III) and Prop. 3 with another example. The regions belong to the active sets (19). See the caption of Fig. 3 for further explanations.

$$\alpha_0.\alpha_1.\alpha_2.\cdots.\alpha_{N-1}.\underbrace{0\cdots0}_{q_{\mathcal{T}}} \tag{15}$$

$$\alpha_1.\alpha_2.\cdots.\alpha_{N-1}.0\cdots0.\underbrace{0\cdots0}_{q_{\mathcal{T}}} \tag{16}$$

Moreover, (16) defines the successor set to the set defined by (15) or a superset thereof. In other words, for any $x(0)$ that belongs to the set defined by (15), the successor state belongs to the set defined by (16).

Proof. If (15) is an active set for horizon N, then

$$\alpha_1.\alpha_2.\cdots.\alpha_{N-1}.0\cdots0 \tag{17}$$

is an active set for horizon N according to Prop. 2. Inserting an inactive penultimate inactive stage in (17) results in (16), which is an active set for horizon N according to Prop. 1. □

Example 4. Consider the optimal control problem (3) for Example 1 and $N = 3$.

Figure 3 shows the regions defined by active sets

$$000001.000010.000000.0 \quad \text{(red)} \tag{18a}$$
$$000010.000000.000000.0 \quad \text{(cyan)} \tag{18b}$$
$$000000.000000.000000.0 \quad \text{(green)} \tag{18c}$$

which result from applying claim (III) or Prop. 3 to (18a) once and twice. The colors stated in parentheses correspond to the colors of the regions in Fig. 3. To show another example, Fig. 4 shows the regions defined by active sets

$$000001.000001.000000.0 \quad \text{(red)} \tag{19a}$$
$$000001.000000.000000.0 \quad \text{(cyan)} \tag{19b}$$
$$000000.000000.000000.0 \quad \text{(green)} \tag{19c}$$

which result from applying claim (III) or Prop. 3 to (19a) once and twice.

4. CONCLUSIONS AND OUTLOOK

We showed how to infer active sets that characterize the optimal successor solution given the active set of the current optimal solution. It was the purpose of the paper to establish the operations that need to be carried out with the active sets. Because these rules do not involve optimal control (or other optimization) problems but are based on simple operations like index shifts, we coined the term *symbolic dynamics* for active sets for them.

Future work will focus on extending the problem class. Extensions may either attempt to drop some of the assumptions, or to combine the two-stage cost function with an advanced terminal controller such as an economic model predictive controller.

Appendix A. PROPERTIES OF EXAMPLE 1

The linear autonomous part of the dynamical system is locally asymptotically stable at $x = 0$, since its eigenvalues are $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{1}{2}$. Furthermore, it is easy to show that $\|\xi(k+1)\|^2 = \|A\xi(k)\|^2 = \frac{1}{2}\|\xi(k)\|^2$ for the linear autonomous part of the example. This implies $\|x(k+1)\|^2 = \|Ax(k) + \frac{1}{4}\begin{bmatrix}0\\ x_1^2\end{bmatrix}\|^2 \leq \frac{1}{2}\|x(k)\|^2 + \frac{1}{4^2}x_1^4$. Since the last expression is not larger than $\frac{1}{2} + \frac{1}{16}$ for all $x(k)$ the closed unit disk, the example is positive invariant on $\mathcal{T}$ for $u = \kappa_{\mathcal{T}}(x)$. It can be shown to be positive invariant on the interior of $\mathcal{T}$ by the same arguments.

Appendix B. PROOF OF LEMMA 1

Part (a): Consider the infinite-horizon problem (2) for $x(0)$, assume (5) extended by (4) is not an optimal solution and show a contradiction results. If (5) extended by (4) is not an optimal solution, then there exists, in any neighborhood of (5) extended by (4), a different sequence that respects the constraints of (2) and yields a lower cost function value for (2). Let

$$(\tilde{u}(k), \tilde{x}(k+1))_{k=0}^{\infty}$$

refer to this sequence and assume it is from a neighborhood of (4, 5) sufficiently small for $x(N) \in \operatorname{int} \mathcal{T}$ to imply

$$\tilde{x}(N) \in \operatorname{int} \mathcal{T}. \tag{B.1}$$

This implies at least one of the terms

$$\sum_{k=0}^{N-1} \ell(\tilde{x}(k), \tilde{u}(k)) + \sum_{k=N}^{\infty} \ell(\tilde{x}(k), \tilde{u}(k))$$

evaluates to a lower value than its counterpart in

$$\sum_{k=0}^{N-1} \ell(x(k), u(k)) + \sum_{k=N}^{\infty} \ell(x(k), u(k)).$$

If this is the case for the second term, this contradicts the optimality of (4). If this is the case for the first term, this contradicts the optimality of (5), where we used the fact that

$$(\tilde{u}(k), \tilde{x}(k+1))_{k=0}^{N-1}$$

is admissible for (3) because it respects (B.1) and the remaining constraints of (3) are fulfilled because (3) has them in common with (2). Thus, the desired contradiction results in any case.

Part (b): The truncated sequence (6) is feasible for the horizon-N problem (3), since problems (2) and (3) have the constraints for stages $k = 0, \ldots, N-1$ in common, and the only remaining constraint of (3), i.e., $x(N) \in \mathcal{T}$, is fulfilled by assumption. Now assume the truncated sequence is feasible but not optimal for (3). This implies there exists (in any, arbitrarily small, neighborhood $\mathcal{N} \subset \mathbb{R}^{mN} \times \mathbb{R}^{nN}$ of the truncated sequence) a different sequence

$$(\bar{u}(k), \bar{x}(k+1))_{k=0}^{N-1} \tag{B.2}$$

that results in a lower cost function value, with $\bar{x}(N) \in \mathcal{T}$ (and, in general, $\bar{x}(N) \neq x(N)$). By part (a) we can extend (B.2) to a solution for the infinite-horizon problem, which we denote $(\bar{u}(k), \bar{x}(k))_{k=0}^{\infty}$. Since

$$(\bar{u}(k), \bar{x}(k))_{k=0}^{N-1} \neq (u(k), x(k))_{k=0}^{N-1},$$

we also have

$$(\bar{u}(k), \bar{x}(k))_{k=0}^{\infty} \neq (u(k), x(k))_{k=0}^{\infty}, \tag{B.3}$$

and the optimality of the former does not contradict the optimality of the latter.

Since $\mathcal{N}$ can be chosen to be arbitrarily small, we can make the difference between the l.h.s. and r.h.s. in (B.3) arbitrarily small (e.g., in the obvious 2-norm). This implies there exists a solution $(\bar{u}(k), \bar{x}(k+1))_{k=0}^{\infty}$ in any neighborhood of $(u(k), x(k+1))_{k=0}^{\infty}$ such that the former results in a lower cost function value than the latter, which is the desired contradiction.

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