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Approximate Explicit MPC on Simplicial Partitions with Guaranteed Stability for Constrained Linear Systems ⋆

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Abstract: This paper proposes an approximate explicit model predictive control design approach for regulating linear time-invariant systems subject to both state and control constraints. The proposed control law is implemented as a piecewise-affine function defined on a regular simplicial partition, and has two main positive features. First, the regularity of the simplicial partition allows a very efficient implementation of the control law on digital circuits, with computation performed in tens of nanoseconds. Second, the asymptotic stability of the closed-loop system is enforced a priori by design.

1. INTRODUCTION

Model predictive control (MPC) is becoming increasingly popular both in academia and in industry due to its ability to solve control problems optimally while satisfying constraints on state and control variables (see, *e.g.*, Rawlings and Mayne (2009) and the references therein). The main drawback of MPC is the computation time required for solving on-line an optimization problem, which has historically prevented its application to fast processes (Qin and Badgwell, 2003).

To circumvent this problem, two main research directions were pursued in the last decade (we limit our overview to the control of linear time-invariant (LTI) systems, that are the subject of this paper). The first relates to fast algorithms for on-line optimization (see, *e.g.*, Bemporad and Patrinos (2012); Ferreau et al. (2008); Richter et al. (2011); Wang and Boyd (2010)). The second regards computing the control law off-line as an explicit piecewise-affine (PWA) function of the state vector (see, *e.g.*, Bemporad et al. (2002), Alessio and Bemporad (2009)): the off-line computation employs a multiparametric programming solver, thus achieving the same solution obtained by solving an optimization problem on-line. The on-line computation in explicit MPC relies on determining the region of the PWA partition where the current state value is located (usually referred to as the *point location* problem, which takes usually a high percentage of the overall on-line computation time), and then on evaluating an affine function from a pre-stored lookup table. The number of regions of the PWA partition which defines the explicit MPC control law depends typically exponentially on the number of constraints included in the multiparametric program. For this reason, explicit MPC is usually applied to control processes of relatively small size (approximatively, up to 2 manipulated inputs and 10 state variables). However, it permits to use short sampling periods, such as 1-50 ms. To simplify the complexity of explicit MPC controllers, approximate explicit MPC techniques have been considered (Alessio and Bemporad (2009); Christophersen et al. (2007); Grieder and Morari (2003); Johansen and Grancharova (2003)). In these approaches, optimality is sacrificed for a control law defined over a smaller number of regions. In order to obtain even faster controllers, explicit MPC has been recently implemented on hardware devices, such as *field programmable gate arrays* (FPGAs) in Mariéthoz et al. (2009), pushing the time needed to compute the control law (latency) down to hundreds of nanoseconds.

In a recent work (Bemporad et al., 2011), an approximate MPC controller for LTI systems is proposed, based on a special class of functions, hereafter referred to as *piecewise-affine simplicial (PWAS) functions*, proposed by Julián et al. (2000). The choice of PWAS functions leads to a regular partition, where the point-location problem is solved with a negligible effort if compared to explicit MPC defined on generic PWA partitions. For this reason, PWAS functions can be efficiently implemented on digital circuits (Storace and Poggi, 2010), which allows an extremely fast computation of the control law. More precisely, for an example with two state variables and two control inputs (Bemporad et al., 2011), the latency for an FPGA implementation could be reduced by one order of magnitude with respect to the implementation of the exact controller on the same FPGA (from hundreds to tens of nanoseconds). The control law proposed in Bemporad et al. (2011) presents feasibility and local optimality properties, but the asymptotic stability of the

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origin of the closed-loop system and the evaluation of its domain of attraction can be determined only a-posteriori, using Lyapunov-based techniques (see also Rubagotti et al. (2011)). In order to define stabilizing approximate MPC controllers on regular partitions, two approaches based on the use of PWA hyper-rectangular partitions have been recently proposed in Genuit et al. (2011); Lu et al. (2011), based on ISS Lyapunov functions and control Lyapunov functions, respectively.

In this paper, we propose an approximate explicit MPC control approach for LTI systems based on PWAS functions, which can be implemented on a digital circuit exactly as in Bemporad et al. (2011). However, we guarantee the asymptotic stability of the resulting closed-loop system *a-priori*, also obtaining the domain of attraction in which hard constraints on state and input variables are satisfied.

More specifically, a suitable bound is imposed a-priori on the approximation error. Then, a robust MPC control law $u^*(x)$ based on tightened constraints is defined, based on ideas of Chisci et al. (2001); Kolmanovsky and Gilbert (1998). The obtained optimal control law can be implemented as an explicit PWA control law, similarly to the nominal case in Bemporad et al. (2002). After obtaining the optimal control law, an approximation procedure is carried out in order to find an approximate PWAS control law $u(x)$, such that the approximation error $u(x) - u^*(x)$ satisfies the previously-defined bound, which implies the asymptotic stability of the closed-loop system.

The paper is organized as follows: the main notation used throughout the paper and the formulation of the control problem are introduced in Sections 2 and 3, respectively. Section 4 describes the structure of the PWAS control law, and gives a brief overview of its implementation on digital circuits. In Section 5, the synthesis of the robustly stabilizing MPC control law is described, while Section 6 deals with the approximation procedure leading to the stabilizing PWAS control law. In Section 7 the proposed control law is synthesized for a simple system and tested in simulation. Finally, conclusions are drawn in Section 8.

2. NOTATION

The sets $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$, $\mathbb{R}_{>0}$ are the sets of positive integers, non-negative integers, real, and positive real numbers, respectively. Given a set $\mathcal{A} \subset \mathbb{R}^n$, its interior is referred to as $\mathrm{int}(\mathcal{A})$. Given two sets $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} \oplus \mathcal{B} \triangleq \{a + b \,:\, a \in \mathcal{A},\ b \in \mathcal{B}\}$ and $\mathcal{A} \sim \mathcal{B} = \{a \,:\, a + b \in \mathcal{A},\ \forall b \in \mathcal{B}\}$ be their Minkowski addition and Pontryagin difference, respectively. Given two vectors $u, v \in \mathbb{R}^n$, the notation $u \leq v$ refers to component-wise inequalities. Given , its Positive definiteness of a square matrix $M \in \mathbb{R}^{n\times n}$ is indicated as $M \succ 0$. Given a vector $v \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, $\|v\|_M^2 \triangleq v'Mv$. Given a matrix $M \in \mathbb{R}^{n\times m}$ and a compact set $\mathcal{W} \subset \mathbb{R}^m$, the product $M\mathcal{W}$ denotes the image of $\mathcal{W}$ under the mapping defined by M, $M\mathcal{W} \triangleq \{v \in \mathbb{R}^n \,:\, v = Mw,\ w \in \mathcal{W}\}$. In case $\mathcal{W}$ is a polytope, $M\mathcal{W}$ can be computed as the convex hull of the images of the vertices of $\mathcal{W}$.

When convenient, the explicit dependence on time of the dynamic variables will be omitted for the sake of readability of the paper.

3. PROBLEM STATEMENT

The controlled plant is described by the following LTI state space model

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $t \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. It is assumed that the whole state vector x is available for feedback. The state and input values are required to satisfy

$$x \in \mathcal{X},\ \mathcal{X} \triangleq \{x \in \mathbb{R}^n \;:\; C_x x \leq g_x\} \tag{2}$$

$$u \in \mathcal{U},\ \mathcal{U} \triangleq \{u \in \mathbb{R}^m \;:\; C_u u \leq g_u\} \tag{3}$$

with $C_x \in \mathbb{R}^{s_x\times n}$, $C_u \in \mathbb{R}^{s_u\times m}$, $g_x \in \mathbb{R}^{s_x}$, $g_u \in \mathbb{R}^{s_u}$.

Assumption 1. The following holds for system (1):

(1) the pair (A, B) is stabilizable;
(2) $\mathcal{X}$ and $\mathcal{U}$ are nonempty, compact, and contain the origin in their interior. □

The objective is to regulate the state x to the origin without violating the constraints (2)-(3), using a control profile $u(x)$ defined on a PWAS partition, whose structure is described in the next section.

4. CONTROL LAW ON A SIMPLICIAL PARTITION

This section is devoted to the description of the control law $u(x)$ as a PWAS function, and on its practical implementation on hardware devices.

4.1 Description of the control law

The function $u(x)$ is defined on a closed hyper-rectangle $\mathcal{S} \subset \mathbb{R}^n$, $\mathcal{S} = \{x \in \mathbb{R}^n \,:\, x_{min} \leq x \leq x_{max}\}$, which is partitioned as $\mathcal{S} = \bigcup_{i=0}^{L_S-1} S_i$, where $\{S_i\}_{i=0}^{L_S-1}$ are simplices. A simplex S_i in the Euclidean space $\mathbb{R}^n$ is a polytope given by the convex hull of its $n+1$ vertices $x_i^0, x_i^1, \ldots, x_i^n \in \mathbb{R}^n$. The partitioning of $\mathcal{S}$ is performed as follows:

(1) Divide every dimensional component x_j of $\mathcal{S}$ into p_j subintervals of length $(x_{max,j} - x_{min,j})/p_j$. These intervals define a number $\prod_{j=1}^n p_j$ of hyper-rectangles, and $\mathcal{S}$ contains $N_v \triangleq \prod_{j=1}^n (p_j+1)$ vertices v_k, collected into a set named $\mathcal{V}_S$.
(2) Partition every rectangle into $n!$ simplices with non-overlapping interiors. The set $\mathcal{S}$ contains $L_S \triangleq n! \prod_{j=1}^n p_j$ simplices S_i, such that $\mathcal{S} = \cup_{i=0}^{L_S-1} S_i$ and $\mathrm{int}(S_i) \cap \mathrm{int}(S_j) = \emptyset$, $\forall i, j = 0, \ldots, L_S - 1$.

Since the partitioning of the hyper-rectangles into simplices is univocally determined, the resulting number of simplices is determined by $p_1, ..., p_n$. After defining the sets S_i, it is possible to introduce the related PWAS function. We choose to define each component of $u(x)$, namely $u_j(x)$, $j = 1, ..., n$, as the weighted sum of N_v linearly independent basis functions. Even though different basis functions can be used (see, *e.g.*, Repetto et al. (2003); Storace et al. (2003)), in this paper we refer to the so-called α-basis (Julián et al. (2000)). Every element of the j-th basis is affine over each simplex and satisfies

$$\alpha_{j,k}(v_h) = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k. \end{cases}$$

After ordering the functions of the α-basis, we can consider them as an N_v-length vector $\phi(x)$. Then, each component of $u(x)$, namely $u_j(x)$, is a scalar PWAS function defined as

$$u_j(x) \triangleq \sum_{k=1}^{N_v} \theta_{j,k}\, \phi_k(x) = \phi(x)'\theta_j \tag{4}$$

where $\theta_j = [\theta_{j,1} \ \dots \ \theta_{j,N_v}]' \in \mathbb{R}^{N_v}$ is the weight vector. Note that the coefficients $\theta_{j,k}$ coincide with the values of the PWAS function $u_j(x)$ at the vertices of the simplicial partition. The PWAS vector function $u : \mathbb{R}^n \to \mathbb{R}^m$ is defined by the weight vector $\theta = [\theta_1' \ \theta_2' \ \dots \ \theta_m']' \in \mathbb{R}^{mN_v}$, as

$$\begin{aligned} u(x) &= \begin{bmatrix} u_1(x) \\ \vdots \\ u_m(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi(x)'\theta_1 \\ \vdots \\ \phi(x)'\theta_m \end{bmatrix} \\ &= \begin{bmatrix} \phi'(x) & 0 & \cdots & 0 \\ 0 & \phi'(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi'(x) \end{bmatrix} \theta = \Phi(x)\theta. \end{aligned} \tag{5}$$

4.2 Implementation on digital circuits

The main reason for defining $u(x)$ as in (5) is that PWAS functions can be implemented in digital circuits using linear interpolators. In fact, by exploiting the regularity of the partition, the point location problem becomes much easier than for the case of generic PWA partitions. The value of $u(x)$ can be obtained, for any $x \in \mathcal{S}$, as a linear interpolation of the values $u(v_{i,j})$ at the $n+1$ vertices $v_{i,0}, ..., v_{i,n}$ of the simplex S_i that contains x,

$$u(x) = \sum_{j=0}^{n} \mu_j u(v_{i,j}). \tag{6}$$

With reference to FPGA implementations (Storace and Poggi, 2010), the circuit realization is given by three functional blocks:

(a) an internal memory that stores the N_v weights θ_k;
(b) a block that finds the simplex S_i to which x belongs (using the algorithm proposed in Rovatti et al. (1998)) and computes the coefficients μ_j (this step requires a sorting algorithm designed for digital implementation, see, *e.g.*, Pedroni (2004));
(c) a block that computes the weighted sum in (6), using high-speed multipliers and accumulators in the FPGA.

The latency of the described circuit (*i.e.*, the time needed to obtain $u(x)$ once the measurement of x is available) does not depend on the number of intervals p_j (which influences the area occupancy). However, the latency grows linearly with n and the number of bits used to encode x.

5. ROBUSTLY STABILIZING OPTIMAL MPC

The next step is to obtain a PWAS function $u(x)$ as in (5) using a procedure that leads to asymptotic convergence to the origin for the closed-loop system. The proposed approach consists in expressing the control variable $u(x)$ as

$$u(x) = u^*(x) + w(x)$$

where $u^*(x)$ is an optimal control which satisfies

$$u^* \in \mathcal{U}, \tag{7}$$

while $w(x)$ represents an approximation error (a priori unknown), and is considered as a bounded disturbance. System (1) can therefore be expressed as

$$x(t+1) = Ax(t) + Bu^*(t) + Bw(t) \tag{8}$$

5.1 Definition of a linear auxiliary control law

In order to formulate the MPC control law $u^*(x)$, we define an auxiliary control law $\kappa(x) = Kx$, where K is such that the closed-loop system

$$x(t+1) = (A + BK)x(t) \tag{9}$$

is asymptotically stable. The value of K can be determined by pole-placement techniques, or it can be the solution for system (9) of the infinite-horizon linear quadratic regulator, given weight matrices $Q = Q' \in \mathbb{R}^{n\times n}$ on the state and $R = R' \in \mathbb{R}^{m\times m}$ on the input, with $Q, R \succ 0$. The resulting closed-loop system obtained by applying $u^*(x) = \kappa(x)$ in (8) is

$$x(t+1) = A_\kappa x(t) + Bw(t), \tag{10}$$

where $A_\kappa \triangleq A + BK$. This control law will be used in the remainder of the paper as baseline to design the MPC controller.

5.2 Definition of invariant sets

We want to define a robust MPC control law $u^*(x)$, mainly based on the approach described in Chisci et al. (2001), which leads to robust convergence of the state to the origin without violating constraints (2) and (7). Let $\bar{w} \in \mathbb{R}_{\geq 0}$ be a fixed scalar such that

$$w(x) \in \mathcal{W} \triangleq \{w \in \mathbb{R}^m : \|w\|_\infty \leq \bar{w}\},\ \forall x \in \mathcal{X}. \tag{11}$$

We now find the minimal *robust positively invariant* (RPI) set (see, e.g., Blanchini and Miani (2008)) for the closed-loop system (10). We denote by

$$\mathcal{R}_j \triangleq \bigoplus_{i=0}^{j-1} A_\kappa^i B\mathcal{W} \tag{12}$$

the set of states of system (10) reachable in j steps from the origin. Define $\mathcal{R}_\infty \triangleq \lim_{j\to\infty} \mathcal{R}_j$ as the minimal RPI set for system (10) with $w \in \mathcal{W}$. Relying for instance on (Blanchini and Miani, 2008, Proposition 6.9), one can prove that $\mathcal{R}_\infty$ is bounded and convex in our case, as (A, B) is a reachable pair, A_κ is asymptotically stable, $\mathcal{W}$ is convex, compact, and includes the origin in its interior. Nonetheless, an explicit computation of $\mathcal{R}_\infty$ is in general impossible (apart from the very specific case of A_κ nilpotent, as stated by Mayne and Schroeder (1997)). Therefore, it is useful to compute a convex polytopic over-approximation (not necessarily RPI) $\hat{\mathcal{R}}_\infty$ such that $\mathcal{R}_\infty \subseteq \hat{\mathcal{R}}_\infty$. There exist different methods to compute $\hat{\mathcal{R}}_\infty$, and the reader is referred to (Blanchini, 1999, Secs. 6.4-6.5), and Rakovic et al. (2005) for an overview.

Let the MPC control law acting on system (1) be

$$u^*(x) \triangleq Kx + \zeta^*(x), \tag{13}$$

Note that $\zeta^*(x) \in \mathbb{R}^m$ represents the difference between the MPC control move and the baseline linear control law $\kappa(x)$. In the following, we will make use of tightened constraints on the nominal evolution of (10) to ensure the fulfillment of the actual constraints for the perturbed system. Given $x(t) = x$, the nominal evolution of (10) is denoted by $\hat{x}(t+k|t)$, while the actual system with the same initial condition is $x(t+k|t)$. Both evolutions are obtained by applying the corresponding control sequence denoted by $\zeta^*(t|t), \ldots, \zeta^*(t+k-1|t)$. It is well known from the set-theoretical analysis in Chisci et al. (2001) and Kolmanovsky and Gilbert (1998), that, given $\mathcal{X}_k \triangleq \mathcal{X} \sim \mathcal{R}_k$ and $\mathcal{U}_k \triangleq \mathcal{U} \sim K\mathcal{R}_k$, one has that, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\hat{x}(t+k|t) \in \mathcal{X}_k \Leftrightarrow x(t+k|t) \in \mathcal{X},\ \forall w \in \mathcal{W},$$
$$K\hat{x}(t+k|t) \in \mathcal{U}_k \Leftrightarrow Kx(t+k|t) \in \mathcal{U},\ \forall w \in \mathcal{W}.$$

The next step is to find the maximal output admissible robust set for system (10), defined as

$$\mathcal{X}_f \triangleq \{x(0) \in \mathbb{R}^n : x(k|0) \in \mathcal{X},\ Kx(k|0) \in \mathcal{U},\ \forall k \in \mathbb{Z}_{\geq 0},\ \forall w \in \mathcal{W}\}. \quad (14)$$

The same set can be conveniently expressed, using tightened sets, as

$$\mathcal{X}_f = \{x : A_\kappa^k x \in \mathcal{X}_k,\ KA_\kappa^k x \in \mathcal{U}_k,\ \forall k \in \mathbb{Z}_{\geq 0}\}. \quad (15)$$

The set $\mathcal{X}_f$ can be computed by Algorithm 6.1 in Kolmanovsky and Gilbert (1998) using linear programming. In particular, exploiting the results in Theorems 6.2 and 6.3 in Kolmanovsky and Gilbert (1998), $\mathcal{X}_f$ is finitely generated if $0 \in \text{int}(\mathcal{X} \sim \mathcal{R}_\infty)$ and $0 \in \text{int}(\mathcal{U} \sim K\mathcal{R}_\infty)$. If $\mathcal{R}_\infty$ is not computable, one can use the above-mentioned over-approximation $\hat{\mathcal{R}}_\infty$ instead.

Assumption 2. It is supposed that $0 \in \text{int}(\mathcal{X} \sim \mathcal{R}_\infty)$ and $0 \in \text{int}(\mathcal{U} \sim K\mathcal{R}_\infty)$ (which ensures the computability of $\mathcal{X}_f$). Moreover, we assume that $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{X}_f)$. □

Remark 1. The condition $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{X}_f)$ represents a slightly stronger requirement with respect to condition $\mathcal{R}_\infty \subseteq \mathcal{X}_f$, which always holds. It is important to note that, if $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{X}_f)$, being $\hat{\mathcal{R}}_\infty$ a closed set, any state trajectory that converges to $\hat{\mathcal{R}}_\infty$ asymptotically, converges to $\mathcal{X}_f$ in finite time. □

Recalling the sets S_i defined in Section 4, we introduce the set

$$\mathcal{S}_f \triangleq \bigcup S_i \ :\ S_i \subseteq \mathcal{X}_f,\ i = 0, ..., L-1 \quad (16)$$

which will be useful to formulate the subsequent results.

5.3 MPC with tightened constraints

For the proposed robust MPC strategies, the prediction of the system trajectory on the finite *prediction horizon* $N \in \mathbb{Z}_{>0}$ will make use of the nominal trajectory of the system, but the fulfillment of the constraints will be required for all realizations of the disturbance $w \in \mathcal{W}$. The vector of optimization variables (inputs) to be determined at time t is $Z \triangleq [\zeta'(t|t) \ \cdots \ \zeta'(t|t+N-1)]' \in \mathbb{R}^{mN}$. The definition of the optimal sequence $\zeta^*(x)$ is based on the solution of the following finite-horizon optimal control problem (FHOCP) at each time t, with $x(t) = x$:

$$Z^*(x) = \arg\min_Z \sum_{k=0}^{N-1} \|\zeta(k)\|_\Psi^2,\ \Psi = \Psi' \succ 0 \quad (17a)$$
$$\text{s.t. } \hat{x}(k) \in \mathcal{X}_k, \qquad k = 0, \ldots, N-1 \quad (17b)$$
$$K\hat{x}(k) + \zeta(k) \in \mathcal{U}_k,\ k = 0, \ldots, N-1 \quad (17c)$$
$$\hat{x}(N) \in \mathcal{X}_f \sim \mathcal{R}_N \quad (17d)$$

For ease of notation, implying that the solution of the FHOCP is computed at time t, we set $\zeta(k) \triangleq \zeta(t+k|t)$ and $\hat{x}(k) \triangleq \hat{x}(t+k|t)$. Note that (17b) and (17c) lead to the fulfillment of (2) and (7), respectively, along the prediction horizon. Finally, (17d) guarantees that $x(t+k|t) \in \mathcal{X}_f$ for all possible disturbance sequences.

The FHOCP (17) is quadratic with respect to the decision variable Z, and is subject to linear constraints. Also, the current state x can be considered as a parameter. Therefore, (17) can be recast as a multi-parametric quadratic program (mpQP), where the set of parameters x for which a feasible solution exists is called $\mathcal{D}_N$. Since $\mathcal{X}, \mathcal{U}$ and $\mathcal{W}$ are polytopes, $\mathcal{D}_N$ is also a polytope and can be easily computed using linear programming and projections. Also, an increase of the prediction horizon leads to a larger set $\mathcal{D}_N$, *i.e.* $\mathcal{D}_N \supseteq \mathcal{D}_{N-1} \supseteq \ldots \supseteq \mathcal{D}_1 \supseteq \mathcal{X}_f$. Note that solving the given robust problem is not computationally more involved than solving the nominal problem obtainable with $\mathcal{W} = 0$, since the same number of decision variables and constraints are obtained. The nominal case (*i.e.*, $\mathcal{W} = 0$) can be seen as a limit case of the robust case, and $\mathcal{D}_N$ is always included in the corresponding set obtained for the nominal case, namely $\mathcal{D}'_N$.

The application of the *receding horizon* principle leads to defining the MPC control law $\zeta^*(x)$ as $\zeta^*(x) \triangleq [I \ 0 \ \ldots \ 0]Z^*(x)$. Following the development in Bemporad et al. (2002), explicit expressions for the optimal value of the cost function in (17a), namely $J^*(x)$, and for $Z^*(x)$, can be obtained solving an mpQP. In particular, both $J^*(x)$ and $Z^*(x)$ are Lipschitz continuous, and more precisely $J^*(x)$ is piecewise-quadratic, while $Z^*(x)$ is piecewise-affine. This implies that also $\zeta^*(x)$ and $u^*(x)$ are piecewise-affine function defined in $\mathcal{D}_N$. The set $\mathcal{D}_N$ is then partitioned as $\mathcal{D}_N = \bigcup_{i=0}^{L_D-1} D_i$, where $\{D_i\}_{i=0}^{L_D-1}$ are polytopes (not necessarily simplices) with non-overlapping interiors.

Theorem 1. Let Assumptions 1-2 hold for system (8) with $w \in \mathcal{W}$, and let $u^*(x)$ be defined in (13). Assume also that $0 \in \text{int}(\mathcal{S}_f)$ (this latter being defined in (16)), $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{S}_f)$, and

$$w(x) = 0,\ \forall\ x \in \mathcal{S}_f. \quad (18)$$

If $x(0) \in \mathcal{D}_N$, the origin is an asymptotically stable equilibrium for system (8), with domain of attraction equal to $\mathcal{D}_N$. Moreover, $x(t) \in \mathcal{X}$ and $u^*(t) \in \mathcal{U}$ for all $t \geq 0$.

Proof: We recall that Assumptions (A1)-(A5) in Chisci et al. (2001) are automatically satisfied if Assumptions 1-2 and (11) hold. Therefore, according to Lemma 7 and Theorem 8 in Chisci et al. (2001), recursive feasibility is ensured if $x(0) \in \mathcal{D}_N$. As a consequence, $x(t) \in \mathcal{X}$ and $u^*(t) \in \mathcal{U}$ for all $t \in \mathbb{Z}_{\geq 0}$. Also, $x(t) \to \mathcal{R}_\infty$ as $t \to \infty$, since K is stabilizing for the nominal system (9). On the other hand, according to the expression of $\mathcal{X}_f$ in (15), the evolution of the nominal system given by $\hat{x}(k)$ with initial condition $x \in \mathcal{X}_f$ and $\zeta(t+k|t) = 0,\ \forall k = 1, ..., N-1$, fulfills the constraints (17b)-(17c). Also, as noticed in Chisci et al. (2001), the constraints $\hat{x}(k) \in \mathcal{X}_k$ and $K\hat{x}(k) \in \mathcal{U}_k$ for $k \geq N$ are equivalent to the terminal constraint (17d). Then, we conclude that $Z = [0\ \cdots\ 0]'$ is a feasible solution for (17) whenever $x \in \mathcal{X}_f$, and is the minimizer of (17), since it is the global minimum of the objective function, *i.e.*, $x \in \mathcal{X}_f \implies Z^*(x) = [0\ \cdots\ 0]'$. Since $\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{S}_f)$, then there exists $\epsilon \in \mathbb{R}_{>0}$ arbitrary small, such that $(1+\epsilon)\hat{\mathcal{R}}_\infty \subset \text{int}(\mathcal{S}_f)$. Considering that $\hat{\mathcal{R}}_\infty$ is a RPI set for system (10), it is a RPI set for system (9) as well. Therefore, by linearity of the system, $(1+\epsilon)\hat{\mathcal{R}}_\infty$ is also a RPI set for (9). Considering now the actual dynamics (8), from the trivial relation $\hat{\mathcal{R}}_\infty \subset (1+\epsilon)\hat{\mathcal{R}}_\infty$ it follows that (for all initial conditions $x(0) \in \mathcal{D}_N$) there exists $t_1 \in \mathbb{Z}_{\geq 0}$ such that $x(t_1) \in (1+\epsilon)\hat{\mathcal{R}}_\infty$. Since it is assumed that $w(x) = 0$ for all $x \in \mathcal{S}_f$, and $(1+\epsilon)\hat{\mathcal{R}}_\infty$ is positively invariant for the system (9), one has that the system dynamics is equal to (9) for all $t \geq t_1$, which leads to the asymptotic convergence of the state of system (8) to the origin for all $x(0) \in \mathcal{D}_N$. ■

6. APPROXIMATION PROCEDURE

First of all, we assume that a control law $u^*(x)$ has been computed for system (1), therefore obtaining the domain of attraction $\mathcal{D}_N$. We define $\mathcal{S}$ as described in Section 4 as the smallest hyper-rectangle such that $\mathcal{D}_N \subseteq \mathcal{S}$. Then, we partition the (not necessarily convex) set $\mathcal{S} \setminus \mathcal{D}_N$ as $\mathcal{S} \setminus \mathcal{D}_N = \bigcup_{i=0}^{\hat{L}_D - 1} D_i$, where $\{D_i\}_{i=0}^{\hat{L}_D-1}$ are polytopes with non-overlapping interiors. In this way, a generic partition of $\mathcal{S}$ as $\mathcal{S} = \bigcup_{i=0}^{\tilde{L}_D-1} D_i$ is obtained, where $\tilde{L}_D \triangleq L_D + \hat{L}_D$, while we denote its set of vertices as $\tilde{\mathcal{V}}_D$. In order to introduce the approximation procedure, we use the concept of *mixed partition* (see, *e.g.*, Bemporad et al. (2011)), as the partition of $\mathcal{S}$ induced by the facets of both simplicial (S_i) and generic (D_i) partitions. As a result, $\mathcal{S}$ is further partitioned into convex polytopes, and the partition is completely defined by the sets of vertices $\mathcal{V}_S$, $\tilde{\mathcal{V}}_D$ and $\mathcal{V}_M$, the latter representing the set of vertices given by the intersection of the two partitions and belonging neither to $\mathcal{V}_S$ nor to $\tilde{\mathcal{V}}_D$. Finally, let

$$\mathcal{V}_I \triangleq \left\{ v \in \left(\mathcal{V}_S \cup \tilde{\mathcal{V}}_D \cup \mathcal{V}_M \right)\ :\ v \in \mathcal{D}_N \right\}$$

and note that $\mathcal{D}_N$ is the convex hull of all $v \in \mathcal{V}_I$.

In this paper, we choose to find $u(x)$ that minimizes the maximum discrepancy with respect to $u^*(x)$ for all $x \in \mathcal{D}_N$ (note that $u^*(x)$ is not defined on $\mathcal{S} \setminus \mathcal{D}_N$), that is

$$F_\infty \triangleq \max_{j=1,\ldots,m} \sup_{x \in \mathcal{D}_N} \left\{ \left| u_j(x) - u_j^*(x) \right| \right\} \tag{19}$$

When minimizing F_∞ in (19), some constraints have to be imposed for all $x \in \mathcal{D}_N$. Since the minima and maxima of the PWA function $w(x) = u(x) - u^*(x)$ on any of the regions of the mixed partition are on vertices, it is sufficient to impose constraints only on the vertices of $\mathcal{V}_I$. In particular:

(1) The control law $u(x)$ must satisfy the constraint (3), which is already satisfied by $u^*(x)$. This can be done imposing $C_u u(v) \leq g_u$ for all $v \in \mathcal{V}_I$, which implies $C_u u(x) \leq g_u$ for all $x \in \mathcal{D}_N$.
(2) The value of $u(x)$ must be computed such that $\|u(x) - u^*(x)\|_\infty \leq \bar{w}$, in order for system (1) to satisfy (11). This can be obtained simply imposing $\|u(v) - u^*(v)\|_\infty \leq \bar{w}$ for all $v \in \mathcal{V}_I$;
(3) In order to obtain (18), we impose that $u(v) = u^*(v)$ for all $v \in \mathcal{V}_I \cap \mathcal{S}_f$.

Therefore, after recalling the relationship between the vector θ and the control variable $u(x)$ in (4)-(5), our proposal is to obtain $u(x)$ by solving the following linear program:

$$\min_{\theta,\eta} \quad \eta \tag{20a}$$

$$\text{s.t.} \quad \eta \geq \pm \left(\phi(v)'\theta_j - u_j^*(v) \right),\ v \in \mathcal{V}_I,\ j = 1, \ldots, m \tag{20b}$$

$$C_u \Phi(v)\theta \leq g_u,\ v \in \mathcal{V}_I \tag{20c}$$

$$\Phi(v)\theta = u^*(v),\ v \in \mathcal{V}_I \cap \mathcal{S}_f \tag{20d}$$

$$\eta \leq \bar{w} \tag{20e}$$

The formulation of the cost function (20a) together with the constraint (20b) leads to finding the vector θ that minimizes the maximum difference between $u_j(x)$ and $u_j^*(x)$ for all $x \in \mathcal{D}_N$ and all components j. Conditions (20c) and (20d) lead to the fulfillment of (3) and (18), respectively. Condition (20e) ensures the fulfillment of (11). Once a feasible solution to (20) has been found, vector θ determines the control law $u(x)$ for all $x \in \mathcal{S}$.

6.1 Properties of the PWAS control law

The following result holds when the approximate control law $u(x)$ is applied to system (1).

Theorem 2. Assume that system (8) fullfills all the assumptions required in Theorem 1, and that a feasible solution for the FHOCP (17) has been determined as $u^*(x)$, together with the set $\mathcal{D}_N$. Also, assume that there exists a realization of $u(x)$ obtained through a feasible solution of (20). Applying the obtained control law $u(x)$ to system (1), if $x(0) \in \mathcal{D}_N$, one has $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ for all $t \geq 0$. Moreover, the origin of system (1) is an asymptotically stable equilibrium point, with domain of attraction equal to $\mathcal{D}_N$.

Proof: Conditions (20d)-(20e) allow to consider $w(x) = u(x) - u^*(x)$ as a disturbance term that satisfies all the requirements to synthesize $u^*(x)$ in (13). Therefore, by application of Theorem 1, all the mentioned properties are proved. ■

Considering that the feasibility of both (17) and (20) is not guaranteed, we give some guidelines on choosing the design parameters. We assume that the number of vertices N_v of the simplicial partition is fixed a priori, taking into account the available hardware device. Given the sets $\mathcal{X}$ and $\mathcal{U}$, the tuning parameter on which the designer can act to design $u^*(x)$ is $\bar{w}$. It is easy to see that, if (17) is feasible for a given $\bar{w} = \bar{w}_1$, then the same problem will be feasible for all $\bar{w} \leq \bar{w}_1$. Then, one can find by bisection the maximum feasible value of $\bar{w}$, namely $\bar{w}_{max}$, and then (17) will be feasible for all $\bar{w}$ such that $0 \leq \bar{w} \leq \bar{w}_{max}$. We also know that a smaller value of $\bar{w}$ leads to a larger set $\mathcal{D}_N$. On the other hand, a small value of $\bar{w}$ could impose a too tight approximation in problem (20), making it infeasible.

The designer can start obtaining a feasible realization of the PWAS control for a value of $\bar{w}$ close to $\bar{w}_{max}$. Then, this value can be decreased in order to enlarge the set $\mathcal{D}_N$. If a feasible solution cannot be obtained, then one can increase the number of vertices of the simplicial partition and restart the procedure.

7. SIMULATION EXAMPLE

Consider the problem of regulating to the origin the unstable plant defined by

$$A = \begin{bmatrix} 1.1 & -1.4 \\ 0.9 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

subject to the constraints

$$\mathcal{X} = \{x \in \mathbb{R}^2 \ : \ \|x\|_\infty \leq 1\}$$
$$\mathcal{U} = \{u \in \mathbb{R} \ : \ |u| \leq 1\}.$$

Assumption 1 is satisfied. We impose $\bar{w} = 7 \cdot 10^{-2}$. A fixed gain for the auxiliary control law is defined as $K = [\,-1.4299 \ -0.3588\,]$, as the infinite-horizon linear quadratic regulator using the weight matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.1,$$

The auxiliary control law leads to the satisfaction of Assumption 2: the sets $\hat{\mathcal{R}}_\infty$, $\mathcal{S}_f$, and $\mathcal{X}_f$ are shown in Fig. 1.

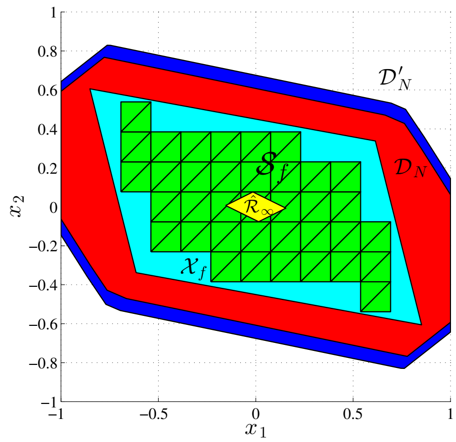

Fig. 1. Graphical representation of the sets (from the outer to the inner) $\mathcal{D}'_N$, $\mathcal{D}_N$, $\mathcal{X}_f$, $\mathcal{S}_f$, $\hat{\mathcal{R}}_\infty$.

The MPC control law $u^*(x)$ in (13) is computed with $\Psi = 1$ and $N = 4$, and its domain of attraction $\mathcal{D}_N$ is also shown in Fig. 1. The approximate control law $u(x)$ is computed with $p_1 = p_2 = 13$ (defined in Section 4.1), obtaining $N_v = 196$ vertices and $L_s = 338$ simplices, with a maximum approximation error $\eta = 6.75 \cdot 10^{-2}$. The PWAS control law so obtained is shown in Fig. 2. For comparison purposes, we obtained the optimal MPC control law described in (13) in case $\mathcal{W} = \{0\}$ (*i.e.*, no approximation error was considered), as previously mentioned, maximizes the domain of attraction. This latter, namely $\mathcal{D}'_N$, is also shown In Fig. 1. Using the recently developed MOBY-DIC toolbox (Oliveri et al. (2012)), the VHDL code defining the PWAS control law has been generated. The latency on a Xilinx Spartan 3 FPGA (xc3s200) board (using the architecture B in Storace and Poggi (2010) and encoding the state variables (circuit inputs) with 12 bits words) has been estimated to be approximately 65 ns. Finally, in Fig. 3, the time evolution of the state and control variables are shown starting from the initial condition $x(0) = [\,0.88 \ 0.2\,]$. Note that the approximation does not significantly affect the state and control trajectories, and keeps constraints satisfied.

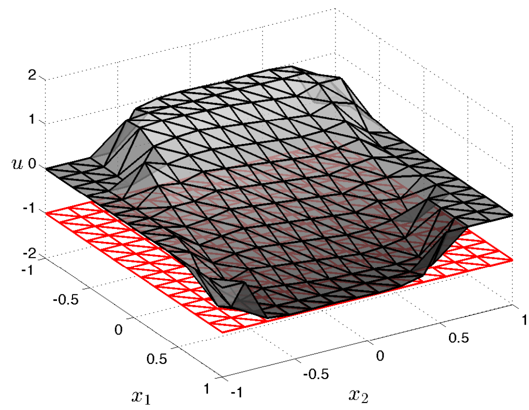

Fig. 2. Control function $u(x)$ on the simplicial partition of the set $\mathcal{S}$.

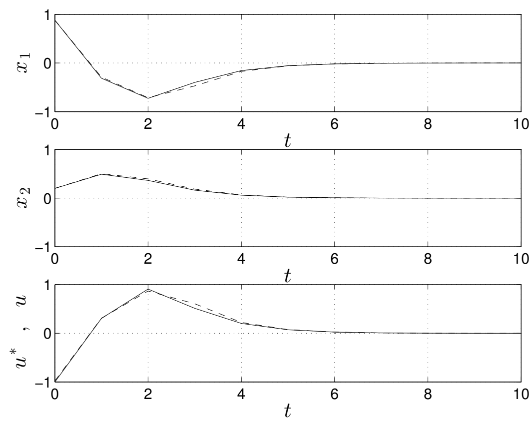

Fig. 3. Time evolution of state and control variables in the simulation example. The solid lines represent the variables ($x_1(t)$, $x_2(t)$, $u(t)$) that would result by applying the optimal control law $u^*(t)$, while the dashed lines represent the same variables obtained by applying the PWAS control law represented in Fig. 2.

8. CONCLUSIONS

In this paper an approximate MPC control law for LTI systems based on PWAS functions was proposed, that can guarantee both a-priori stability for the closed-loop system and efficient implementation on digital hardware. The applicability of the proposed control strategy is limited to the case of small-sized systems, similarly to standard explicit MPC. The theoretical properties of the control law have been proved based on robust MPC synthesis, and the simulation results on a second-order unstable plant have confirmed the expected results, both for the theoretical properties of the PWAS controller and for the performance of the related FPGA implementation.

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