On the Stability and Robustness of Non-Smooth Nonlinear Model Predictive Control

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Keywords : Non-smooth systems, Hybrid systems, Model predictive control, Lyapunov stability, Robust stability

This paper considers discrete-time nonlinear, possibly discontinuous, systems in closed-loop with Model Predictive Controllers (MPC). The aim of the paper is to provide a priori sufficient conditions for asymptotic stability in the Lyapunov sense and robust stability, while allowing for both the system dynamics and the value function of the MPC cost (the usual candidate Lyapunov function in MPC) to be discontinuous functions of the state. The motivation for this work lies in the recent development of MPC for hybrid systems, which are inherently discontinuous and nonlinear systems. As an application of the general theory, it is shown that Lyapunov stability is achieved in hybrid MPC. For a particular class of piecewise affine systems, a modified MPC set-up is proposed, which is proven to be robust to small additive disturbances via an input-to-state stability argument.

1 An introductory survey

One of the problems in Model Predictive Control (MPC) that has received an increased attention over the years consists in guaranteeing closed-loop stability for the controlled system. The usual approach to ensure stability in MPC is to consider the value function of the MPC cost as a candidate Lyapunov function. Then, if the system dynamics and the MPC value function are continuous, the classical Lyapunov stability theory [1] can be used to prove that the MPC control law is stabilizing [2]. The first results that weaken the requirement that the MPC value function must be continuous in the state were presented in [3,4], which consider terminal equality constraint MPC. Attractivity is proven in [3] for the closed-loop system without requiring that the MPC control (and hence the MPC value function) is continuous in the state. However, continuity of the system dynamics on a neighborhood of the origin is employed to prove Lyapunov stability. Although continuity of the system *is still assumed* in [4], which shows that MPC can generate discontinuous state feedbacks, the Lyapunov stability proof (Theorem 2 in [4]) *does not use* the continuity property. Later on, an exponential stability result is given in [5] and an asymptotic stability theorem is presented in [6], which considers sub-optimal MPC. The theorems of [5,6] explicitly point out that both the system dynamics and the candidate Lyapunov function only need to be continuous at the equilibrium. Stability of sub-optimal MPC is proven in [6] under the usual assumptions (existence of class $\mathcal{K}$ bounds on V and its forward difference) plus the extra requirement that the MPC optimal sequence of controls is upper bounded in norm by a $\mathcal{K}$ function of the norm of the state. A recent overview on stability of receding horizon control in discrete-time can be found in [7]. Although continuity of the system dynamics and local continuity of the candidate Lyapunov function are assumed in [7], the stability proof (Theorem 4.3.2 in [7]) only uses continuity of V at the equilibrium, as done in [4]. The issue of discontinuous state feedback stabilization and *continuous-time* receding horizon control was addressed in [8], where it was pointed out that only attractivity can be proven for the equilibrium of the closed-loop system.

Next to closed-loop stability, one of the most studied properties of MPC controllers is the so-called inherent robustness, which ensures that a nominally stabilizing controller is robust in the presence of perturbations. The importance of this property cannot be overstated, since all controllers designed to be nominally stable are usually affected by perturbations when applied in practice. Previous results developed for *smooth* nonlinear MPC, such as the ones in [5, 9], prove that robust asymptotic stability and Input-to-State Stability (ISS) [10] with respect to additive disturbance inputs is achieved, if the system dynamics, the MPC value function and the MPC control law are *Lipschitz continuous*. An important warning regarding robustness of MPC was issued in [11], which points out that even if the system is continuous, the absence of a continuous Lyapunov function or continuous control law may result in a closed-loop system that has no robustness. A novel approach, which no longer requires for the terminal cost to be a local Lyapunov function, is employed in [12] to achieve robust asymptotic stability for the MPC closed-loop system, under the assumption that the system dynamics and the MPC value function are *continuous*.

This paper is motivated by the recent development of MPC for hybrid systems, which are inherently discontinuous and nonlinear systems. Many efficient tools for solving hybrid MPC optimization problems already exist, e.g. the Hybrid Toolbox [13] and the MPT Toolbox [14], and attractivity was proved for the equilibrium of the closed-loop system in [15, 16]. However, proofs of Lyapunov stability only appeared in the hybrid MPC literature very recently, e.g. [17–20]. Note that [17] and [18] consider *continuous* Piecewise Affine (PWA) systems (affected by additive disturbances in [17]) and the (robust) Lyapunov stability result of Theorem 2 in [17] is obtained by forcing the MPC value function ($1, \infty$-norms based) to be zero in the terminal set and by taking certain assumptions that are hard to be *a priori* guaranteed in general. The stability result of Theorem 5 in [18] (quadratic cost based MPC) uses the assumption that the origin lies in the interior of one of the polyhedral regions on which the PWA system is defined, while Theorem 14.1.5 in [19] addresses exponential stability of quadratic cost MPC of PWA systems, but it relies on the survey [21], where continuity of the MPC value function is assumed. In [20], the authors provide *a priori sufficient conditions* for asymptotic stability in the Lyapunov sense for discontinuous PWA systems in closed-loop with ∞-norms based MPC controllers. In this paper we consider discrete-time nonlinear, *possibly discontinuous*, systems in closed-loop with MPC controllers and we aim at providing a general theorem on asymptotic stability in the Lyapunov sense that unifies most of the previously-mentioned results. Besides closed-loop stability, the issue of *inherent robustness* is particularly relevant for hybrid systems and MPC because, in this case, the system dynamics, the MPC value function and the MPC control law are generally discontinuous. Based on the result of [10], we present an ISS theorem for MPC. For a class of constrained PWA systems, a modified ∞-norms MPC set-up is proposed, which is proven to be robust to small additive disturbances via the general ISS result.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{S}$ the boundary of $\mathcal{S}$, by $\mathrm{int}(\mathcal{S})$ its interior and by $\mathrm{cl}(\mathcal{S})$ its closure. A polyhedron is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. Consider the nominal and perturbed discrete-time autonomous nonlinear systems described by

$$x_{k+1} = G(x_k), \quad k \in \mathbb{Z}_+, \tag{1a}$$

$$\tilde{x}_{k+1} = \tilde{G}(\tilde{x}_k, w_k), \quad k \in \mathbb{Z}_+, \tag{1b}$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$ are the states at sampling instant k, $G : \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{G} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ are nonlinear, possibly discontinuous, functions. For simplicity of notation, we assume that the origin is an equilibrium in (1), meaning that $G(0) = 0$ and $\tilde{G}(0, 0) = 0$. The vector $w \in \mathcal{W} \subseteq \mathbb{R}^l$ represents an unknown disturbance and the set $\mathcal{W}$ is assumed to be known and with $0 \in \mathrm{int}(\mathcal{W})$. For a sequence $\{w_j\}_{j\in\mathbb{Z}_+}$ let $\|\{w_j\}_{j\in\mathbb{Z}_+}\| := \sup\{\|w_j\| \mid j \in \mathbb{Z}_+\}$. Also, let $w_{[k]}$ denote the truncation of $\{w_j\}_{j\in\mathbb{Z}_+}$ at time k, i.e. $w_{[k]}(j) = w_j$ if $j \leq k$, and $w_{[k]}(j) = 0$ if $j > k$. Due to space limitations, we refer to [22] for definitions regarding Lyapunov stability, attractivity, asymptotic stability in the Lyapunov sense and exponential stability of the origin for the nominal system (1a).

Definition 2.1 A real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. It *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and it is radially unbounded (i.e. $\varphi(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed k, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed s, $\beta(s, \cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Definition 2.2 [10] The perturbed system (1b) is (globally) *Input-to-State Stable (ISS)* for $w \in \mathcal{W}$ if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each initial condition $x_0 \in \mathbb{R}^n$ and each $w_{[k-1]} \in \mathcal{W}^k$ it holds that $\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|w_{[k-1]}\|)$ for all $k \in \mathbb{Z}_+ \setminus \{0\}$.

3 Non-smooth nonlinear MPC: problem set-up

Consider the following nominal and perturbed discrete-time dynamical nonlinear systems:

$$x_{k+1} = g(x_k, u_k), \quad k \in \mathbb{Z}_+, \tag{2a}$$

$$\tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, u_k, w_k), \quad k \in \mathbb{Z}_+, \tag{2b}$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the states and the control input, respectively, at sampling instant k and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\tilde{g} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$ are nonlinear, possibly discontinuous, functions with $g(0,0) = 0$ and $\tilde{g}(0,0,0) = 0$. In the sequel we will investigate the stability and the robust stability of the nominal system (2a) and the perturbed system (2b), respectively, in the case when MPC is used to generate the control input. We assume that the state and the input vectors are constrained for both systems (2a) and (2b), in a convex, compact subset $\mathbb{X}$ of $\mathbb{R}^n$ and a convex, compact subset $\mathbb{U}$ of $\mathbb{R}^m$, respectively, which contain the origin in their interior. For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) := (x_{1/k}, \ldots, x_{N/k})$ denote the state sequence generated by the nominal system (2a) from initial state $x_{0/k} := x_k$ and by applying the input sequence $\mathbf{u}_k := (u_{0/k}, \ldots, u_{N-1/k}) \in \mathbb{U}^N$. Furthermore, let $\mathcal{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin. The class of *admissible input sequences* defined with respect to $\mathcal{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\, x_{N/k} \in \mathcal{X}_T\}$. Now consider the following constrained optimization problem.

Problem 3.1 Let the target set $\mathcal{X}_T \subseteq \mathbb{X}$ and $N \geq 1$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0,0) = 0$ be mappings. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost function $J(x_k, \mathbf{u}_k) := F(x_{N/k}) + \sum_{i=0}^{N-1} L(x_{i/k}, u_{i/k})$ over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

In the MPC literature [21], F, L and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x) \neq \emptyset$. Similarly, Problem 3.1 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathcal{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible initial states* with respect to Problem 3.1 and let

$$V_{\text{MPC}} : \mathcal{X}_f(N) \to \mathbb{R}_+, \quad V_{\text{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{3}$$

denote the MPC value function corresponding to Problem 3.1. The existence of a minimum in (3) is usually guaranteed by assuming continuity of the dynamics (2a) and of the stage and terminal costs [21]. However, it is known that the global optimum is also achieved in various cases of hybrid MPC [15, 16], although in these cases the system and the terminal cost can be discontinuous. Since hybrid MPC is one of the main motivations for this work, we assume in the sequel that there exists an optimal sequence of controls $\mathbf{u}_k^* := (u_{0/k}^*, u_{1/k}^*, \ldots, u_{N-1/k}^*)$ for Problem 3.1 and any state $x_k \in \mathcal{X}_f(N)$. Hence, the infimum in (3) is a minimum and $V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, the *MPC control law* is defined as

$$u_k^{\text{MPC}} = u_{0/k}^*; \quad k \in \mathbb{Z}_+. \tag{4}$$

The following stability analysis holds in the case when the optimum is not unique in Problem 3.1, i.e. all results apply irrespective of which optimal sequence is selected.

4 Non-smooth nonlinear MPC: stability and ISS

Let $h_{\text{aux}} : \mathbb{R}^n \to \mathbb{R}^m$ denote an arbitrary, possibly discontinuous, nonlinear function which is zero at zero and let $\mathcal{X}_{\mathbb{U}} := \{x \in \mathbb{X} \mid h_{\text{aux}}(x) \in \mathbb{U}\}$ denote the safe set with respect to *state and input* constraints for this control law. Furthermore, let $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}}$ denote a positively invariant set [22] for system (2a) in closed-loop with $u_k = h_{\text{aux}}(x_k)$.

The following theorem was obtained as a kind of general and unifying result by putting together the previous results on stability of discrete-time MPC that were mentioned in the introductory survey.

Assumption 4.1 *Terminal cost and constraint set* There exist $\alpha_1, \alpha_2 \in \mathcal{K}$, a neighborhood of the origin $\mathcal{N} \subseteq \mathcal{X}_f(N)$ and a feedback control law h_{aux} such that $L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in \mathcal{X}_f(N)$ and all $u \in \mathbb{U}$, $F(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathcal{N}$ and

$$F(g(x, h_{\text{aux}}(x))) - F(x) + L(x, h_{\text{aux}}(x)) \leq 0 \quad \text{for all} \quad x \in \mathcal{X}_T. \tag{5}$$

Assumption 4.2 *Terminal equality constraint* $\mathcal{X}_T = \{0\}$, $F(x) = 0$ for all $x \in \mathbb{X}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ and a neighborhood of the origin $\mathcal{N} \subseteq \mathcal{X}_f(N)$ such that $L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in \mathcal{X}_f(N)$ and all $u \in \mathbb{U}$ and $L(x_{i/k}^*, u_{i/k}^*) \leq \alpha_2(\|x_k\|)$, for any optimal $\mathbf{u}_k^*$, initial state $x_k =: x_{0/k}^* \in \mathcal{N}$ and $i = 0, \ldots, N-1$, where $(x_{1/k}^*, \ldots, x_{N/k}^*) := \mathbf{x}_k(x_k, \mathbf{u}_k^*)$.

Theorem 4.3 *Fix $N \geq 1$ and suppose that Assumption 4.1 holds and that $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}}$ is a positively invariant set for system* (2a) *in closed-loop with $u_k = h_{\text{aux}}(x_k)$ that contains the origin in its interior, or, that Assumption 4.2 holds. Then it holds that:*

(i) *If Problem 3.1 is feasible at time* $k \in \mathbb{Z}_+$ *for state* $x_k \in \mathbb{X}$*, then Problem 3.1 is feasible at time* $k+1$ *for state* $x_{k+1} = g(x_k, u_k^{MPC})$ *and, moreover, Problem 3.1 is feasible for all* $x \in \mathcal{X}_T$*;*

(ii) *The origin of the MPC closed-loop system* (2a)-(4) *is asymptotically stable in the Lyapunov sense for all states in* $\mathcal{X}_f(N)$*;*

(iii) *If Assumption 4.1 or Assumption 4.2 holds with* $\alpha_1(\|x\|) := a\|x\|^\lambda$, $\alpha_2(\|x\|) := b\|x\|^\lambda$ *for some constants* $a, b, \lambda > 0$*, then the origin of the MPC closed-loop system* (2a)-(4) *is locally exponentially stable.*

Proof Due to space limitations we only present the Lyapunov stability proof (note that our proof differs from the one of Theorem 4.3.2 in [7]) for the terminal cost and constraint case (i.e. when Assumption 4.1 holds) and we refer to [22] for the complete proof. By optimality and Assumption 4.1 it follows that $V_{\text{MPC}}(x) \geq \alpha_1(\|x\|)$ for all $x \in \mathcal{X}_f(N)$ and $V_{\text{MPC}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \tilde{\mathcal{N}}$, where $\tilde{\mathcal{N}} := \mathcal{X}_T \cap \mathcal{N}$, and that $V_{\text{MPC}}(g(x, u_k^{\text{MPC}})) - V_{\text{MPC}}(x) \leq -\alpha_1(\|x\|)$ for all $x \in \mathcal{X}_f(N)$. Since $\mathbb{X}$ is assumed to be compact and $\mathcal{X}_f(N) \subseteq \mathbb{X}$, it follows that $\mathcal{X}_f(N)$ is bounded. From (i) it follows that $\mathcal{X}_f(N)$ is a positively invariant set for the MPC closed-loop system (2a)-(4). Let x_k be the solution of (2a)-(4), obtained from the initial condition x_0 at time $k = 0$. Choose an $\eta > 0$ such that the ball $\mathcal{B}_\eta := \{x \in \mathbb{R}^n \mid \|x\| \leq \eta\}$ satisfies $\mathcal{B}_\eta \subseteq \tilde{\mathcal{N}}$. Due to $\alpha_1, \alpha_2 \in \mathcal{K}$ we can choose for any $0 < \varepsilon \leq \eta$ a $\delta \in (0, \varepsilon)$ such that $\alpha_2(\delta) < \alpha_1(\varepsilon)$. For any $x_0 \in \mathcal{B}_\delta \subseteq \mathcal{X}_f(N)$, due to positive invariance of $\mathcal{X}_f(N)$, it follows:

$$\ldots \leq V_{\text{MPC}}(x_{k+1}) \leq V_{\text{MPC}}(x_k) \leq \ldots \leq V_{\text{MPC}}(x_0) \leq \alpha_2(\|x_0\|) \leq \alpha_2(\delta) < \alpha_1(\varepsilon). \quad (6)$$

Since we have that $V_{\text{MPC}}(x) \geq \alpha_1(\varepsilon)$ for all $x \in \mathcal{X}_f(N) \setminus \mathcal{B}_\varepsilon$ it follows that $x_k \in \mathcal{B}_\varepsilon$ for all $k \in \mathbb{Z}_+$. Hence, the origin of the MPC closed-loop system (2a)-(4) is *Lyapunov stable*. □

Note the following aspects regarding Theorem 4.3: (i) The hypothesis of Theorem 4.3 only requires that both V and g are continuous at $x = 0$, while allowing for both of them to be discontinuous for $x \neq 0$; (ii) We only use $\alpha_1, \alpha_2 \in \mathcal{K}$ locally, in an arbitrary neighborhood of the origin. Outside this neighborhood it is sufficient that $\alpha_1, \alpha_2 \in \mathcal{M}$ (a real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{M}$ ($\varphi \in \mathcal{M}$) if it is non-decreasing and if $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$). Also, it is sufficient that $\alpha_1 \in \mathcal{M}$ in Assumption 4.1 for proving attractivity only, as shown in [3]. Allowing for class $\mathcal{M}$ bounds, which can be discontinuous, might be convenient from a synthesis point of view, e.g. when dealing with hybrid systems; (iii) Exponential stability can be guaranteed on the basis of the hypothesis alone only locally for *non-smooth nonlinear MPC*. Exponential stability in $\mathcal{X}_f(N)$ is obtained if, for example, $\alpha_1(\|x\|) := a\|x\|^\lambda$, $\alpha_2(\|x\|) := b\|x\|^\lambda$ for some $a, b, \lambda > 0$ and Assumption 4.2 holds with $\mathcal{N}$ replaced by $\mathcal{X}_f(N)$. In [7] (Theorem 4.4.2-(iv)) it is proven that *continuity of* V_{MPC} *on* $\mathcal{X}_f(N)$ and Assumption 4.1 are sufficient for exponential stability in $\mathcal{X}_f(N)$.

Next, we state a version of the global ISS result of Lemma 3.5 from [10] applied for the MPC set-up, for the case when the disturbance input satisfies $w \in \mathcal{B}_\mu := \{w \in \mathcal{W} \mid \|w\| \leq \mu\}$ with $\mu > 0$ sufficiently small.

Theorem 4.4 [10] *Let* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ *and let* $\mathcal{V}_r := \{x \in \mathbb{R}^n \mid V_{MPC}(x) \leq r\}$ *be such that* $\mathcal{V}_r \subseteq \mathcal{X}_f(N)$*. Suppose* $L(x, u) \geq \alpha_1(\|x\|)$ *for all* $x \in \mathcal{X}_f(N)$ *and all* $u \in \mathbb{U}$, $V_{MPC}(x) \leq \alpha_2(\|x\|)$ *for all* $x \in \mathcal{V}_r$ *and that there exists a* $\mu \in (0, r)$ *such that*

$$V_{MPC}(\tilde{g}(x_k, u_k^{MPC}, w_k)) - V_{MPC}(x_k) \leq -\alpha_3(\|x_k\|) + \sigma(\|w_k\|), \quad \forall x_k \in \mathcal{V}_r, w_k \in \mathcal{B}_\mu, k \in \mathbb{Z}_+. \quad (7)$$

Then, the perturbed system (2b) *in closed-loop with the nominal MPC control* (4) *is input-to-state stable for all initial conditions* $x_0 \in \mathcal{V}_r$ *and all disturbance inputs* $w_k \in \mathcal{B}_\mu$, $k \in \mathbb{Z}_+$.

Although in Lemma 3.5 of [10] it is assumed that V is continuous, the continuity property is not used in the proof. Hence, if one can ensure a priori that inequality (7) is satisfied at all times $k \in \mathbb{Z}_+$, without requiring continuity of V_{MPC}, ISS can be achieved for the closed-loop system. Note that if V_{MPC} is uniformly continuous on $\mathcal{X}_f(N)$, the disturbance acts additively on the state and the hypothesis of Theorem 4.3 holds, then it is easy to prove that the hypothesis of Theorem 4.4 is satisfied, which ensures ISS. For other robust stability results that use continuous candidate Lyapunov (ISS) functions see [11, 12]. For the sake of completeness we give a proof for the following particular case of Theorem 4.4, i.e. when $\alpha_1(\|x\|) := a\|x\|^\lambda$, $\alpha_2(\|x\|) := b\|x\|^\lambda$ and $\alpha_3(\|x\|) := c\|x\|^\lambda$ for some $a, b, c, \lambda > 0$, which was also used in [5, 9] for Lipschitz continuous nonlinear systems.

Lemma 4.5 *Suppose that there exist some* $a, b, c, \lambda > 0$ *such that the hypothesis of Theorem 4.4 holds with* $\alpha_1(\|x\|) := a\|x\|^\lambda$, $\alpha_2(\|x\|) := b\|x\|^\lambda$ *and* $\alpha_3(\|x\|) := c\|x\|^\lambda$*. Then, the perturbed system* (2b) *in closed-loop with the nominal MPC control* (4) *is ISS for all* $x_0 \in \mathcal{V}_r$ *and all* $w_k \in \mathcal{B}_\mu$, $k \in \mathbb{Z}_+$.

Proof From the hypothesis of Theorem 4.4 we have that $\alpha_1(\|x\|) \leq V_{\text{MPC}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathcal{V}_r$. For $x \in \mathcal{V}_r \setminus \{0\}$, due to $V_{\text{MPC}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathcal{V}_r$ we have that

$$V_{\text{MPC}}(x) - \alpha_3(\|x\|) \leq (1 - \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)})V_{\text{MPC}}(x) = \rho V_{\text{MPC}}(x), \quad \forall x \in \mathcal{V}_r \setminus \{0\},$$

where $\rho := 1 - \frac{c}{b}$. Next, we show that $\rho \in [0,1)$. Since inequality (7) holds for $w_k = 0$ for all $k \in \mathbb{Z}_+$ it follows that $0 \leq V_{\text{MPC}}(\tilde{g}(x_k, u_k^{\text{MPC}}, 0)) \leq V_{\text{MPC}}(x_k) - c\|x_k\|^\lambda \leq (b-c)\|x_k\|^\lambda$. Hence, it follows that $b \geq c > 0$ and thus $\rho \in [0,1)$. Since $V_{\text{MPC}}(0) - \alpha_3(\|0\|) = \rho V_{\text{MPC}}(0) = 0$, we have that $V_{\text{MPC}}(x) - \alpha_3(\|x\|) \leq \rho V_{\text{MPC}}(x)$ for all $x \in \mathcal{V}_r$. Then,

$$V_{\text{MPC}}(g(\tilde{x}_k, u_k^{\text{MPC}}, w_k)) = V_{\text{MPC}}(\tilde{x}_{k+1}) \leq \rho V_{\text{MPC}}(\tilde{x}_k) + \sigma(\|w_k\|), \quad \forall \tilde{x}_k \in \mathcal{V}_r,\, w_k \in \mathcal{B}_\mu, k \in \mathbb{Z}_+. \tag{8}$$

Next, choose $\mu \in (0,r)$ such that $\sigma(\mu) \leq (1-\rho)r$. Then, from (8) it follows that

$$V_{\text{MPC}}(\tilde{x}_{k+1}) \leq \rho V_{\text{MPC}}(\tilde{x}_k) + \sigma(\mu) \leq \rho r + (1-\rho)r = r, \quad \forall \tilde{x}_k \in \mathcal{V}_r,\, w \in \mathcal{B}_\mu,\, k \in \mathbb{Z}_+.$$

Hence, $\mathcal{V}_r$ is a robustly positively invariant set [22] for the perturbed system (2b)-(4) and inequality (8) yields:

$$V_{\text{MPC}}(\tilde{x}_{k+1}) \leq \rho^{k+1} V_{\text{MPC}}(x_0) + \rho^k \sigma(\|w_0\|) + \rho^{k-1}\sigma(\|w_1\|) + \ldots + \sigma(\|w_k\|),$$

for all $x_0 \in \mathcal{V}_r$, $w_k \in \mathcal{B}_\mu$, $k \in \mathbb{Z}_+$. Then, it follows that:

$$\alpha_1(\|\tilde{x}_{k+1}\|) \leq V_{\text{MPC}}(\tilde{x}_{k+1}) \leq \rho^{k+1}\alpha_2(\|x_0\|) + \sum_{i=0}^{k} \rho^i \sigma(\|w_{k-i}\|) \leq \rho^{k+1}\alpha_2(\|x_0\|) + \sigma(\|w_{[k]}\|)\frac{1}{1-\rho},$$

for all $x_0 \in \mathcal{V}_r$, $w_{[k]} \in \mathcal{B}_\mu^{k+1}$, $k \in \mathbb{Z}_+$. One can easily check that $\alpha_1 \in \mathcal{K}_\infty$ implies $\alpha_1^{-1} \in \mathcal{K}_\infty$. Then, $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ yields:

$$\|\tilde{x}_{k+1}\| \leq \alpha_1^{-1}(\rho^{k+1}\alpha_2(\|x_0\|) + \sigma(\|w_{[k]}\|)\frac{1}{1-\rho}) \leq \alpha_1^{-1}(2\max(\rho^{k+1}\alpha_2(\|x_0\|), \sigma(\|w_{[k]}\|)\frac{1}{1-\rho})) \leq$$
$$\leq \alpha_1^{-1}(2\rho^{k+1}\alpha_2(\|x_0\|)) + \alpha_1^{-1}(2\sigma(\|w_{[k]}\|)\frac{1}{1-\rho}), \quad \forall x_0 \in \mathcal{V}_r,\, w_{[k]} \in \mathcal{B}_\mu^{k+1},\, k \in \mathbb{Z}_+.$$

Let $\beta(s,k) := \alpha_1^{-1}(2\rho^k\alpha_2(s))$. For a fixed $k \in \mathbb{Z}_+$, we have that $\beta(\cdot,k) \in \mathcal{K}$ due to $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\rho \in (0,1)$[1]. For a fixed s, it follows that $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty}\beta(s,k) = 0$, due to $\rho \in (0,1)$ and $\alpha_1^{-1} \in \mathcal{K}_\infty$. Thus, it follows that $\beta \in \mathcal{KL}$. Now let $\gamma(s) := \alpha_1^{-1}(2\sigma(s)\frac{1}{1-\rho})$. Since $\frac{1}{1-\rho} > 0$, it follows that $\gamma \in \mathcal{K}$ due to $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$. Hence, the closed-loop system (2b)-(4) is ISS in the sense of Definition 2.2 for all $x_0 \in \mathcal{V}_r$ and all disturbance inputs $w_k \in \mathcal{B}_\mu$, $k \in \mathbb{Z}_+$. □

5 ∞-norms based MPC of constrained PWA systems

In this section we consider the class of time-invariant discrete-time piecewise affine systems, i.e.

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \Omega_j,\, j \in \mathcal{S},\, k \in \mathbb{Z}_+; \tag{9a}$$
$$\tilde{x}_{k+1} = A_j \tilde{x}_k + B_j u_k + f_j + w_k \quad \text{when} \quad \tilde{x}_k \in \Omega_j, w_k \in \mathcal{B}_\mu \subset \mathbb{R}^n,\, j \in \mathcal{S},\, k \in \mathbb{Z}_+. \tag{9b}$$

Also, we take the auxiliary controller $h_{\text{aux}}(x) := K_j x$ when $x \in \Omega_j$, $j \in \mathcal{S}$. Here, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$ with $\mathcal{S} := \{1,2,\ldots,s\}$ a *finite set* of indices. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \text{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \text{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (9) with $u = 0$ and we require that $f_j = 0$ for all $j \in \mathcal{S}_0$. This includes PWA systems which *may be discontinuous over the boundaries*. Next, consider the case when ∞-norms are used to define the MPC cost function, i.e. $F(x) = \|P_j x\|_\infty$ when $x \in \Omega_j$ and $L(x,u) = \|Qx\|_\infty + \|Ru\|_\infty$. Here $P_j \in \mathbb{R}^{p_j \times n}$, $j \in \mathcal{S}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{r\times n}$ are assumed to be matrices that have full-column rank.

In [20] the authors developed ways to compute (off-line) the terminal weight matrices $\{P_j \mid j \in \mathcal{S}\}$ and the feedbacks $\{K_j \mid j \in \mathcal{S}\}$ such that inequality (5) holds. Then, it can be shown that PWA systems in closed-loop with MPC controllers calculated as in (4) and using an ∞-norms based cost in Problem 3.1 satisfy the hypothesis of Theorem 4.3, thereby establishing Lyapunov stability for the origin of the closed-loop system. A similar result

[1] If $\rho = 0$ we have that $\|\tilde{x}_k\| \leq \alpha_1^{-1}(\sigma(\|w_{k-1}\|)) \leq \beta(\|x_0\|,k) + \alpha_1^{-1}(\sigma(\|w_{[k-1]}\|))$ for any $\beta \in \mathcal{KL}$, $k \in \mathbb{Z}_+ \setminus \{0\}$.

for quadratic cost based MPC and PWA prediction models can be found in [22]. However, due to the fact that in hybrid MPC both the system and the MPC value function may be discontinuous in general, it follows, as pointed out in [11], that the closed-loop system may have no robustness, despite the fact that nominal asymptotic stability is guaranteed. Since most of the work that has been done in hybrid MPC addresses the nominal case, it is relevant to investigate this robustness issue.

In this paper we aim at modifying hybrid MPC so as to ensure *a priori* a certain level of inherent robustness to perturbations. The approach is based on the ISS result of Lemma 4.5. The key idea is to constrain the nominal predicted state and the corresponding sequence of controls such that the mode sequence corresponding to a perturbed initial state remains the same as the nominal predicted mode sequence. This is done by not allowing the predicted state to take values in the regions of the state space where the effect of the disturbance might trigger a different mode sequence. In the sequel we illustrate this approach for a particular class of PWA systems (i.e. the case when $0 \in \operatorname{int}(\Omega_{j^*})$ for some $j^* \in \mathcal{S}$) that was also considered in [18] in order to ensure nominal stability for the MPC closed-loop system. Note that for this class of PWA systems there exists a neighborhood of the origin $\mathcal{N} \subseteq \Omega_{j^*}$ where V_{MPC} is uniformly continuous due to the fact that the PWA system is linear in Ω_{j^*}. This property together with nominal stability of the MPC closed-loop system ensures local ISS, i.e. for all $x_0 \in \mathcal{V}_{r^*}$, where $r^* > 0$ is such that $\mathcal{V}_{r^*} \subseteq \mathcal{N} \subseteq \Omega_{j^*}$. Next, we develop an MPC set-up that yields an ISS closed-loop system for all initial conditions $x_0 \in \mathcal{V}_r := \{x \in \mathcal{X}_f(N) \mid V_{\text{MPC}}(x) \leq r\}$. Let $\mathcal{S} \sim \mathcal{P} := \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote the Pontryagin difference of two arbitrary sets $\mathcal{S}$ and $\mathcal{P}$, let $\eta := \max_{j\in\mathcal{S}} \|A_j\|_\infty$ and let $\mathcal{L}^i_\mu := \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq \mu \sum_{p=0}^{i-1} \eta^p\}$ for all $i \in \mathbb{Z}_{\geq 1}$. Also, let $\gamma > 0$ be such that $\|Qx\|_\infty \geq \gamma\|x\|_\infty$ for all $x \in \mathbb{R}^n$, let $F(x) = \|Px\|_\infty$ for all $x \in \mathbb{X}$, where $P \in \mathbb{R}^{p\times n}$ has full column rank, and let $\xi := \|P\|_\infty$. Consider now the following (tightened) set of constraints:

$$\tilde{\mathcal{U}}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid x_{i/k} \in \mathbb{X}_i,\ i = 1, \ldots, N-1,\ x_{N/k} \in \mathcal{X}_T\},\ \ k \in \mathbb{Z}_+, \tag{10}$$

where $\mathbb{X}_i := \cup_{j\in\mathcal{S}}\{\Omega_j \sim \mathcal{L}^i_\mu\} \subseteq \mathbb{X}$ for all $i = 1, \ldots, N-1$ and $(x_{1/k}, \ldots, x_{N/k})$ is a state sequence generated from initial state $x_{0/k} := x_k$ and by applying the input sequence $\mathbf{u}_k$ to the nominal PWA model (9a). Note that the idea of a tightened set of constraints is not new in robust MPC, e.g. it was used in [9] for Lipschitz continuous nonlinear systems. Here we employ this set-up for PWA systems.

Theorem 5.1 *Assume that* $0 \in \operatorname{int}(\Omega_{j^*})$ *for some* $j^* \in \mathcal{S}$ *and let* $c_0, \ldots, c_N$ *denote positive constants. Take* $r > \mu > 0$, $N \geq 1$ *and* $\theta > \theta_1 > 0$ *such that* $\mu \leq \min(\frac{(1-\gamma/N \max_{i=0,\ldots,N} c_i)r}{\xi\eta^{N-1}+\|Q\|_\infty \sum_{p=0}^{p=N-2}\eta^p}{}^2, \frac{\theta-\theta_1}{\xi\eta^{N-1}})$, $\mathcal{F}_\theta := \{x \in \mathbb{R}^n \mid F(x) \leq \theta\} \subseteq \Omega_{j^*} \cap \mathcal{X}_{\mathbb{U}} \cap \mathbb{X}_{N-1}$ *and* $g(x, h_{\text{aux}}(x)) \in \mathcal{F}_{\theta_1}$ *for all* $x \in \mathcal{F}_\theta$ *(g denotes here the PWA system* (9a)*). Set* $\mathcal{X}_T = \mathcal{F}_{\theta_1}$*. Furthermore, suppose that Assumption 4.1 holds for the nominal PWA system* (9a) *and the state feedback* h_{aux} *and that inequality* (5) *is satisfied for all* $x \in \mathcal{F}_\theta$*. Then it holds that:*

(i) *If Problem 3.1 with the set of constraints* $\tilde{\mathcal{U}}_N(x_k)$ *is feasible at time* $k \in \mathbb{Z}_+$ *for state* $x_k \in \mathbb{X}$*, then Problem 3.1 is feasible at time* $k+1$ *for state* $\tilde{x}_{k+1} = A_j x_k + B_j u_k^{MPC} + f_j + w_k$ *for all* $w_k \in \mathcal{B}_\mu$*;*

(ii) *The perturbed PWA system* (9b) *in closed-loop with the MPC control* (4) *calculated using the nominal PWA model* (9a) *to obtain the predicted state trajectory and by solving Problem 3.1 with the modified set of constraints* (10) *at each sampling instant is ISS for all* $x_0 \in \mathcal{V}_r$ *and all* $w_k \in \mathcal{B}_\mu$, $k \in \mathbb{Z}_+$*.*

Proof Let $(x^*_{1/k}, \ldots, x^*_{N/k})$ denote the optimal predicted state sequence obtained at time k from initial state $x_{0/k} := x_k$ and by applying the input sequence $\mathbf{u}^*_k$ to the PWA model (9a). Let $(\tilde{x}_{1/k+1}, \ldots, \tilde{x}_{N/k+1})$ denote the state sequence obtained from the perturbed initial state $\tilde{x}_{0/k+1} := \tilde{x}_{k+1} = x_{k+1} + w_k = x^*_{1/k} + w_k$ and by applying the input sequence $\tilde{\mathbf{u}}_{k+1} := (u^*_{1/k}, \ldots, u^*_{N-1/k}, h_{\text{aux}})$ to the nominal PWA model (9a).

(i) The state constraints imposed in (10) ensure: (P1) $(\tilde{x}_{i/k+1}, x^*_{i+1/k}) \in \Omega_{j_{i+1}} \times \Omega_{j_{i+1}}$, $j_{i+1} \in \mathcal{S}$ for all $i = 0, \ldots, N-2$ and that $\|\tilde{x}_{i/k+1} - x^*_{i+1/k}\|_\infty \leq \eta^i\mu$ for $i = 0, \ldots, N-2$. Pick the indices $j_{i+1} \in \mathcal{S}$ such that $x^*_{i+1/k} \in \Omega_{j_{i+1}}$ for all $i = 0, \ldots, N-2$. Then, due to $x^*_{i+1/k} \in \Omega_{j_{i+1}} \sim \mathcal{L}^{i+1}_\mu$, it follows by Lemma 2 of [9] that $\tilde{x}_{i/k+1} \in \Omega_{j_{i+1}} \sim \mathcal{L}^i_\mu \subset \mathbb{X}_i$ for $i = 0, \ldots, N-2$. Property (P1) for $i = N-2$ implies that $\tilde{x}_{N-1/k+1} = x^*_{N/k} + \prod_{i=1}^{N-1} A_{j_i} w_k$. Then, it follows that $F(\tilde{x}_{N-1/k+1}) - F(x^*_{N/k}) \leq \xi\eta^{N-1}\mu$, which implies that $F(\tilde{x}_{N-1/k+1}) \leq \theta_1 + \xi\eta^{N-1}\mu \leq \theta$ due to $x^*_{N/k} \in \mathcal{X}_T = \mathcal{F}_{\theta_1}$ and $\mu \leq \frac{\theta-\theta_1}{\xi\eta^{N-1}}$. Hence, $\tilde{x}_{N-1/k+1} \in \mathcal{F}_\theta \subset \mathcal{X}_{\mathbb{U}} \cap \mathbb{X}_{N-1}$ and then $h_{\text{aux}}(\tilde{x}_{N-1/k+1}) \in \mathbb{U}$ and $\tilde{x}_{N/k+1} \in \mathcal{X}_T = \mathcal{F}_{\theta_1}$. Thus, the sequence of inputs $\tilde{\mathbf{u}}_{k+1}$ is feasible at time $k+1$.

(ii) For ∞-norms based hybrid MPC it is known [16] that all the elements of the MPC optimal sequence of controls $\mathbf{u}^*_k$ are PWA functions of the state x_k. Moreover, $f_{j^*} = 0$ and R has full-column rank implies that there exists a neighborhood of the origin where these controls are Piecewise Linear (PWL). Then, it can be shown that there exist constants $\beta_i > 0$ such that $\|u^*_{i/k}\|_\infty \leq \beta_i\|x_k\|_\infty$ for $i = 0, \ldots, N-1$. Using Lemma 1 of [20]

[2] This ensures that $\mathcal{V}_r$ is a robustly positively invariant set [22], as shown in the proof of Lemma 4.5.

it follows that there exist constants $c_i > 0$ such that $F(x^*_{N/k}) \leq c_N \|x_k\|_\infty$ and $L(x^*_{i/k}, u^*_{i/k}) \leq c_i \|x_k\|_\infty$ for all $x_k \in \mathcal{X}_f(N)$ and $i = 0, \ldots, N-1$, $k \in \mathbb{Z}_+$. Hence, $V_{\text{MPC}}(x) \leq \alpha_2(\|x\|_\infty)$ for all $x \in \mathcal{X}_f(N)$, where $\alpha_2(\|x\|_\infty) := N \max_{i=0,\ldots,N} c_i \|x\|_\infty$. Then, $L(x,u) \geq \|Qx\|_\infty \geq \gamma \|x\|_\infty$ for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ implies $\alpha_1(\|x\|_\infty) \leq V_{\text{MPC}}(x) \leq \alpha_2(\|x\|_\infty)$ for all $x \in \mathcal{X}_f(N)$. Let $\tilde{x}_{k+1}$ denote the solution of the perturbed system (9b)-(4) obtained as indicated before part (i) of the proof. Then, by optimality, property (P1), $\tilde{x}_{N-1/k+1} \in \mathcal{F}_\theta$ and from inequality (5) it follows that:

$$V_{\text{MPC}}(\tilde{x}_{k+1}) - V_{\text{MPC}}(x_k) \leq J(\tilde{x}_{k+1}, \tilde{\mathbf{u}}_{k+1}) - J(x_k, \mathbf{u}^*_k) = -L(x^*_{0/k}, u^*_{0/k}) + F(\tilde{x}_{N/k+1}) - F(\tilde{x}_{N-1/k+1}) +$$

$$+ L(\tilde{x}_{N-1/k+1}, h_{\text{aux}}(\tilde{x}_{N-1/k+1})) + F(\tilde{x}_{N-1/k+1}) - F(x^*_{N/k}) + \sum_{i=0}^{N-2} (L(\tilde{x}_{i/k+1}, \tilde{\mathbf{u}}_{k+1}(i+1)) - L(x^*_{i+1/k}, u^*_{i+1/k})) \leq$$

$$\leq -L(x^*_{0/k}, u^*_{0/k}) + F(\tilde{x}_{N/k+1}) - F(\tilde{x}_{N-1/k+1}) + L(\tilde{x}_{N-1/k+1}, h_{\text{aux}}(\tilde{x}_{N-1/k+1})) + (\xi \eta^{N-1} + \|Q\|_\infty \sum_{p=0}^{N-2} \eta^p) \|w_k\|_\infty \leq$$

$$\leq -\alpha_3(\|x_k\|_\infty) + \sigma(\|w_k\|_\infty),$$

where $\alpha_3(\|x\|_\infty) = \alpha_1(\|x\|_\infty) := \gamma \|x\|_\infty$ and $\sigma(\|w_k\|_\infty) := (\xi \eta^{N-1} + \|Q\|_\infty \sum_{p=0}^{N-2} \eta^p) \|w_k\|_\infty$. Hence, it follows that V_{MPC} satisfies the hypothesis of Lemma 4.5, thereby establishing ISS for the closed-loop system. □

Note the following aspects regarding Theorem 5.1: (i) In the above proof we showed that there exist $a, b > 0$ such that $a\|x\|_\infty \leq V_{\text{MPC}}(x) \leq b\|x\|_\infty$ for all $x \in \mathcal{X}_f(N)$. This implies that the hypothesis of Theorem 4.3 is sufficient for *exponential stability in* $\mathcal{X}_f(N)$ for ∞-norms MPC and PWA prediction models, even though V_{MPC} is not continuous on $\mathcal{X}_f(N)$ in general; (ii) ISS is no longer achieved via the above reasoning if the tightened set of constraints (10) is employed for general PWA systems, i.e. when $0 \notin \text{int}(\Omega_j)$ for all $j \in \mathcal{S}$, due to the fact that the sets $\mathbb{X}_i$, $i \geq 1$, do not contain the origin in this case. Future work deals with the implementation of this idea for general PWA systems via a dual-mode MPC scheme. (iii) One has to make a trade off in ensuring that the disturbance does not affect the predicted mode sequence on one hand, and in keeping the tightening of the constraints (10) as mild as possible on the other hand. This problem may be tackled by making the state constraints time varying, i.e. such that $\mathbb{X}_i \to \mathbb{X}$ for all $i \geq 1$ when $w_k \to 0$ as $k \to \infty$ (an estimate of the disturbance may be required) or by incorporating a local feedback which ensures that $\|A_j + B_j K_j\|_\infty$ is small for all $j \in \mathcal{S}$ (and hence η is small) and by using the MPC control in order to ensure constraint satisfaction for the local controller in $\mathcal{X}_f(N)$. These approaches make the object of future research.

6 Conclusions

In this paper we have presented an overview of the stability and robust stability theory for nonlinear MPC while focusing on the application and the extension of the classical results to non-smooth nonlinear systems. A stability theorem has been developed, which unifies many previous results. Robust stability issues have also been addressed and the input-to-state stability result of [10] was applied to non-smooth nonlinear MPC. The potential of these results for hybrid MPC has been illustrated for a particular class of PWA systems.

Acknowledgements Research supported by the Dutch Science Foundation (STW), Grant “Model Predictive Control for Hybrid Systems” (DMR. 5675), the European 6th Framework Network of Excellence HYCON (contract number FP6-IST-511368) and the European IST project SICONOS (IST-2001-37172).

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