Discrete-time Non-smooth Nonlinear MPC: Stability and Robustness

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Summary. This paper considers discrete-time nonlinear, possibly discontinuous, systems in closed-loop with model predictive controllers (MPC). The aim of the paper is to provide a priori sufficient conditions for asymptotic stability in the Lyapunov sense and input-to-state stability (ISS), while allowing for both the system dynamics and the value function of the MPC cost to be discontinuous functions of the state. The motivation for this work lies in the recent development of MPC for hybrid systems, which are inherently discontinuous and nonlinear. For a particular class of discontinuous piecewise affine systems, a new MPC set-up based on infinity norms is proposed, which is proven to be ISS to bounded additive disturbances. This ISS result does not require continuity of the system dynamics nor of the MPC value function.

Key words: Discontinuous nonlinear systems, Hybrid systems, Model predictive control, Lyapunov stability, Input-to-state stability

1 An introductory survey

One of the problems in model predictive control (MPC) that has received an increased attention over the years consists in guaranteeing closed-loop stability for the controlled system. The usual approach to ensure stability in MPC is to consider the value function of the MPC cost as a candidate Lyapunov function. Then, if the system dynamics is continuous, the classical Lyapunov stability theory [1] can be used to prove that the MPC control law is stabilizing [2]. The requirement that the system dynamics must be continuous is (partially) removed in [3,4], where terminal equality constraint MPC is considered. In [3], continuity of the system dynamics on a neighborhood of the origin is still used to prove Lyapunov stability, but not for proving attractivity. Although continuity of the system is still assumed in [4], the Lyapunov stability proof (Theorem 2 in [4]) does not use the continuity property. Later on, an exponential stability result is given in [5] and an asymptotic stability theorem is presented in [6], where sub-optimal MPC is considered. The theorems of [5,6] explicitly point out that both the system dynamics and the candidate Lyapunov function only need to be continuous at the equilibrium.

Next to closed-loop stability, one of the most studied properties of MPC controllers is robustness. Previous results developed for *smooth* nonlinear MPC, such as the ones in [5, 7], prove that robust asymptotic stability is achieved, if the system dynamics, the MPC value function and the MPC control law are *Lipschitz continuous*. Sufficient conditions for input-to-state stability (ISS) [8] of smooth nonlinear MPC were presented in [9, 10] based on Lipschitz continuity of the system dynamics. A similar result was obtained in [11], where the Lipschitz continuity assumption was relaxed to basic continuity. An important warning regarding robustness of smooth nonlinear MPC was issued in [12], where it is pointed out that the absence of a *continuous Lyapunov function* may result in a closed-loop system that has no robustness.

This paper is motivated by the recent development of MPC for hybrid systems, which are inherently discontinuous and nonlinear systems. Attractivity was proven for the equilibrium of the closed-loop system in [13, 14]. However, proofs of Lyapunov stability only appeared in the hybrid MPC literature recently, e.g. [15–18]. In [17], the authors provide *a priori sufficient conditions* for asymptotic stability in the Lyapunov sense for *discontinuous* piecewise affine (PWA) systems in closed-loop with MPC controllers based on ∞-norm cost functions. Results on robust hybrid MPC were presented in [15] and [19], where dynamic programming and tube based approaches were considered for solving feedback *min-max* MPC optimization problems for *continuous* PWA systems.

In this paper we consider discrete-time nonlinear, *possibly discontinuous*, systems in closed-loop with MPC controllers and we aim at providing a general theorem on asymptotic stability in the Lyapunov sense that unifies most of the previously-mentioned results. Besides closed-loop stability, the issue of *robustness* is particularly relevant for hybrid systems and MPC because, in this case, the system dynamics, the MPC value function and the MPC control law are typically discontinuous. We present an input-to-state stability theorem that can be applied to discrete-time non-smooth nonlinear MPC. For a class of *discontinuous* PWA systems, a new MPC set-up based on ∞-norm cost functions is proposed, which is proven to be ISS with respect to bounded additive disturbances.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. We define with $\mathbb{Z}^N$ the N-dimensional Cartesian product $\mathbb{Z} \times \ldots \times \mathbb{Z}$, for some $N \in \mathbb{Z}_{\geq 1}$. For a sequence $\{z_j\}_{j\in\mathbb{Z}_+}$ with $z_j \in \mathbb{R}^l$ let $\|\{z_j\}_{j\in\mathbb{Z}_+}\| := \sup\{\|z_j\| \mid j \in \mathbb{Z}_+\}$. For a sequence $\{z_j\}_{j\in\mathbb{Z}_+}$ with $z_j \in \mathbb{R}^l$, $z_{[k]}$ denotes the truncation of $\{z_j\}_{j\in\mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k]} = \{z_j\}_{j\in\mathbb{Z}_{[0,k]}}$. For a set $\mathcal{P} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{P}$ the boundary of $\mathcal{P}$, by $\mathrm{int}(\mathcal{P})$ its interior and by $\mathrm{cl}(\mathcal{P})$ its closure. Let $\mathcal{P}_1 \sim \mathcal{P}_2 \triangleq \{x \in \mathbb{R}^n \mid x + \mathcal{P}_2 \subseteq \mathcal{P}_1\}$ denote the Pontryagin difference of two arbitrary sets $\mathcal{P}_1$ and $\mathcal{P}_2$. A polyhedron is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces.

Consider now the following discrete-time autonomous nonlinear systems:

$$x_{k+1} = G(x_k), \quad k \in \mathbb{Z}_+, \tag{1a}$$

$$\tilde{x}_{k+1} = \widetilde{G}(\tilde{x}_k, w_k), \quad k \in \mathbb{Z}_+, \tag{1b}$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$ are the state, $w_k \in \mathbb{R}^l$ is an unknown disturbance input and, $G : \mathbb{R}^n \to \mathbb{R}^n$, $\widetilde{G} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ are nonlinear, possibly discontinuous, functions. For simplicity of notation, we assume that the origin is an equilibrium in (1), meaning that $G(0) = 0$ and $\widetilde{G}(0, 0) = 0$. Due to space limitations, we refer to [20] for definitions regarding Lyapunov stability, attractivity, asymptotic stability in the Lyapunov sense and exponential stability of the origin for the nominal system (1a).

Definition 1. *A real-valued scalar function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ *if it is continuous, strictly increasing and* $\varphi(0) = 0$. *A function* $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ *if for each fixed* $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ *and for each fixed* $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ *is non-increasing and* $\lim_{k\to\infty} \beta(s, k) = 0$.

Definition 2. (ISS) *Let* $\mathbb{X}$ *with* $0 \in \text{int}(\mathbb{X})$ *and* $\mathbb{W}$ *be subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^l$, *respectively. The perturbed system* (1b) *is called* ISS for initial conditions in $\mathbb{X}$ and disturbance inputs in $\mathbb{W}$ *if there exist a* $\mathcal{KL}$*-function* β *and a* $\mathcal{K}$*-function* γ *such that, for each* $x_0 \in \mathbb{X}$ *and all* $\{w_p\}_{p\in\mathbb{Z}_+}$ *with* $w_p \in \mathbb{W}$ *for all* $p \in \mathbb{Z}_+$, *it holds that the state trajectory satisfies* $\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|w_{[k-1]}\|)$ *for all* $k \in \mathbb{Z}_{\geq 1}$.

Note that the regional ISS property introduced in Definition 2 can be regarded as a local version of the global ISS property defined in [8] and it is similar to the robust asymptotic stability property employed in [11].

3 The MPC optimization problem

Consider the following nominal and perturbed discrete-time nonlinear systems:

$$x_{k+1} = g(x_k, u_k), \quad k \in \mathbb{Z}_+, \tag{2a}$$

$$\tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, u_k, w_k), \quad k \in \mathbb{Z}_+, \tag{2b}$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and the control input, respectively, and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\tilde{g} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$ are nonlinear, possibly discontinuous, functions with $g(0, 0) = 0$ and $\tilde{g}(0, 0, 0) = 0$. In the sequel we will consider the case when MPC is used to generate the control input in (2). We assume that the state and the input vectors are constrained for both systems (2a) and (2b), in a compact subset $\mathbb{X}$ of $\mathbb{R}^n$ and a compact subset $\mathbb{U}$ of $\mathbb{R}^m$, respectively, which contain the origin in their interior. For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by the nominal system (2a) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\ x_{N|k} \in \mathbb{X}_T\}$.

Problem 1. Let the target set $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \geq 1$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0,0) = 0$ be mappings, possibly discontinuous. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost function $J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k})$, with prediction model (2a), over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

In the MPC literature, $F(\cdot)$, $L(\cdot,\cdot)$ and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x) \neq \emptyset$. Similarly, Problem 1 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible initial states* with respect to Problem 1 and let

$$V_{\text{MPC}} : \mathbb{X}_f(N) \to \mathbb{R}_+, \quad V_{\text{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{3}$$

denote the MPC value function corresponding to Problem 1. We assume that there exists an optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ for Problem 1 and any state $x_k \in \mathbb{X}_f(N)$. Hence, the infimum in (3) is a minimum and $V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, *the MPC control law* is defined as

$$u^{\text{MPC}}(x_k) \triangleq u_{0|k}^*; \quad k \in \mathbb{Z}_+. \tag{4}$$

The following stability analysis also holds when the optimum is not unique in Problem 1, i.e. all results apply irrespective of which optimal sequence is selected.

4 General results on stability and ISS

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ denote an arbitrary, possibly discontinuous, nonlinear function with $h(0) = 0$ and let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$.

The following theorem was obtained as a kind of general and unifying result by putting together the previous results on stability of discrete-time nonlinear MPC that were mentioned in the introductory survey.

Assumption 1 *Terminal cost and constraint set*: There exist $\alpha_1, \alpha_2 \in \mathcal{K}$, a neighborhood of the origin $\mathcal{N} \subseteq \mathbb{X}_f(N)$ and a feedback control law $h(\cdot)$ such that $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$, with $0 \in \text{int}(\mathbb{X}_T)$, is a positively invariant set [20] for system (2a) in closed-loop with $u = h(x)$, $L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}_f(N)$ and all $u \in \mathbb{U}$, $F(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathcal{N}$ and

$$F(g(x, h(x))) - F(x) + L(x, h(x)) \leq 0 \quad \text{for all} \quad x \in \mathbb{X}_T. \tag{5}$$

Assumption 2 *Terminal equality constraint*: $\mathbb{X}_T = \{0\}$, $F(x) = 0$ for all $x \in \mathbb{X}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ and a neighborhood of the origin $\mathcal{N} \subseteq \mathbb{X}_f(N)$ such that

$L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}_f(N)$ and all $u \in \mathbb{U}$ and $L(x^*_{i|k}, u^*_{i|k}) \leq \alpha_2(\|x_k\|)$, for any optimal $\mathbf{u}^*_k \in \mathcal{U}_N(x_k)$, initial state $x_k =: x^*_{0|k} \in \mathcal{N}$ and $i = 0, \ldots, N-1$, where $(x^*_{1|k}, \ldots, x^*_{N|k}) =: \mathbf{x}_k(x_k, \mathbf{u}^*_k)$.

Theorem 1. (Stability of Non-smooth Nonlinear MPC) *Fix $N \geq 1$ and suppose that either Assumption 1 holds or Assumption 2 holds. Then:*

(i) *If Problem 1 is feasible at time $k \in \mathbb{Z}_+$ for state $x_k \in \mathbb{X}$, Problem 1 is feasible at time $k+1$ for state $x_{k+1} = g(x_k, u^{MPC}(x_k))$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$;*

(ii) *The origin of the MPC closed-loop system* (2a)-(4) *is asymptotically stable in the Lyapunov sense for initial conditions in $\mathbb{X}_f(N)$;*

(iii) *If Assumption 1 or Assumption 2 holds with $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$ for some constants $a, b, \lambda > 0$, the origin of the MPC closed-loop system* (2a)-(4) *is exponentially stable in $\mathbb{X}_f(N)$.*

The interested reader can find the proof of Theorem 1 in [20]. Next, we state sufficient conditions for ISS (in the sense of Definition 2) of discrete-time non-smooth nonlinear MPC.

Theorem 2. (ISS of Non-smooth Nonlinear MPC) *Let $\mathbb{W}$ be a compact subset of $\mathbb{R}^l$ that contains the origin and let $\mathbb{X}$ be a robustly positively invariant (RPI) set* [20] *for the MPC closed-loop system* (2b)-(4) *and disturbances in $\mathbb{W}$, with $0 \in \operatorname{int}(\mathbb{X})$. Let $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ for some positive constants a, b, c, λ and let $\sigma \in \mathcal{K}$. Suppose $L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$, $V_{MPC}(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{X}$ and that:*

$$V_{MPC}(\tilde{g}(x, u^{MPC}(x), w)) - V_{MPC}(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \quad \forall x \in \mathbb{X},\ \forall w \in \mathbb{W}. \tag{6}$$

Then, the perturbed system (2b) *in closed-loop with the MPC control* (4) *obtained by solving Problem 1 at each sampling-instant is ISS for initial conditions in $\mathbb{X}$ and disturbance inputs in $\mathbb{W}$. Moreover, the ISS property of Definition 2 holds for $\beta(s,k) \triangleq \alpha_1^{-1}(2\rho^k \alpha_2(s))$ and $\gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right)$, where $\rho \triangleq 1 - \frac{c}{b} \in [0,1)$.*

For a proof of Theorem 2 we refer the reader to [20]. Note that the hypotheses of Theorem 1 and Theorem 2 allow $g(\cdot,\cdot)$, $\tilde{g}(\cdot,\cdot,\cdot)$ and $V_{\text{MPC}}(\cdot)$ to be discontinuous when $x \neq 0$. They *only* imply continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$.

5 A robust MPC scheme for discontinuous PWA systems

In this section we consider the class of discrete-time piecewise affine systems, i.e.

$$x_{k+1} = g(x_k, u_k) \triangleq A_j x_k + B_j u_k + f_j \qquad \text{if} \quad x_k \in \Omega_j, \tag{7a}$$

$$\tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, u_k, w_k) \triangleq A_j \tilde{x}_k + B_j u_k + f_j + w_k \quad \text{if} \quad \tilde{x}_k \in \Omega_j, \tag{7b}$$

where $w_k \in \mathbb{W} \subset \mathbb{R}^n$, $k \in \mathbb{Z}_+$, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $j \in \mathcal{S}$ with $\mathcal{S} \triangleq \{1, 2, \dots, s\}$ a *finite set* of indices. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\operatorname{int}(\Omega_i) \cap \operatorname{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 \triangleq \{j \in \mathcal{S} \mid 0 \in \operatorname{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 \triangleq \{j \in \mathcal{S} \mid 0 \notin \operatorname{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (7a) with $u = 0$. Therefore, we require that $f_j = 0$ for all $j \in \mathcal{S}_0$. Note that this does not exclude PWA systems which are *discontinuous over the boundaries*. Next, let $\|\cdot\|$ denote the ∞-norm and consider the case when the ∞-norm is used to define the MPC cost function, i.e. $F(x) \triangleq \|Px\|$ and $L(x,u) \triangleq \|Qx\| + \|Ru\|$. Here $P \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{r\times m}$ are assumed to be known matrices that have full-column rank. In the PWA setting we take the auxiliary controller $h(x) \triangleq K_j x$ when $x \in \Omega_j$, where $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$.

In [17] the authors developed ways to compute (off-line) the terminal weight P and the feedbacks $\{K_j \mid j \in \mathcal{S}\}$ such that inequality (5) holds and $\mathbb{X}_T$ is a positively invariant set for the PWA system (7a) in closed-loop with the piecewise linear (PWL) state-feedback $h(\cdot)$. Then, it can be shown that PWA systems in closed-loop with MPC controllers calculated as in (4) and using an ∞-norm based cost in Problem 1 satisfy the hypothesis of Theorem 1, thereby establishing Lyapunov stability for the origin of the closed-loop system. A similar result for quadratic cost based MPC and PWA prediction models can be found in [20]. However, since both the system (7) and the hybrid MPC value function will be discontinuous in general, it follows, as pointed out in [12], that the closed-loop system may not be robust (ISS) to *arbitrarily small* disturbances, despite the fact that nominal asymptotic stability is guaranteed.

In this section we present a new design method based on tightened constraints for setting up ISS MPC schemes for a class of *discontinuous* PWA systems. One of the advantages of the proposed approach is that the resulting MPC optimization problem can still be formulated as a *mixed integer linear programming* (MILP) problem, which is a standard problem in hybrid MPC. Note that in this case the assumption of Section 3 on the existence of an optimal sequence of controls is satisfied, see, for example, [14,20].

Let $\eta \triangleq \max_{j\in\mathcal{S}} \|A_j\|$, $\xi \triangleq \|P\|$ and define, for any $\mu > 0$ and $i \in \mathbb{Z}_{\geq 1}$,

$$\mathcal{L}_\mu^i \triangleq \left\{ x \in \mathbb{R}^n \mid \|x\| \leq \mu \sum_{p=0}^{i-1} \eta^p \right\}.$$

Consider now the following (tightened) set of admissible input sequences:

$$\widetilde{\mathcal{U}}_N(x_k) \triangleq \left\{ \mathbf{u}_k \in \mathbb{U}^N \mid x_{i|k} \in \mathbb{X}_i,\ i = 1, \dots, N-1,\ x_{N|k} \in \mathbb{X}_T \right\},\ k \in \mathbb{Z}_+, \quad (8)$$

where $\mathbb{X}_i \triangleq \cup_{j\in\mathcal{S}}\{\Omega_j \sim \mathcal{L}_\mu^i\} \subseteq \mathbb{X}$ for all $i = 1, \dots, N-1$ and $(x_{1|k}, \dots, x_{N|k})$ is the state sequence generated from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k$ to the PWA model (7a). Let $\widetilde{\mathbb{X}}_f(N)$ denote the set of feasible states for Problem 1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$, and let $\widetilde{V}_{\mathrm{MPC}}(\cdot)$ denote the corresponding MPC value function. For any $\mu > 0$, define $\mathcal{B}_\mu \triangleq \{w \in \mathbb{R}^n \mid \|w\| \leq \mu\}$ and recall that $\mathbb{X}_{\mathbb{U}} = \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$.

Theorem 3. *Assume that* $0 \in \mathrm{int}(\Omega_{j^*})$ *for some* $j^* \in \mathcal{S}$. *Take* $N \in \mathbb{Z}_{\geq 1}$, $\theta > \theta_1 > 0$ *and* $\mu > 0$ *such that* $\mu \leq \frac{\theta - \theta_1}{\xi \eta^{N-1}}$,

$$\mathbb{F}_\theta \triangleq \{x \in \mathbb{R}^n \mid F(x) \leq \theta\} \subseteq (\Omega_{j^*} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_{\mathbb{U}}$$

and $g(x, h(x)) \in \mathbb{F}_{\theta_1}$ *for all* $x \in \mathbb{F}_\theta$. *Set* $\mathbb{X}_T = \mathbb{F}_{\theta_1}$. *Furthermore, suppose that Assumption 1 holds and inequality* (5) *is satisfied for all* $x \in \mathbb{F}_\theta$. *Then:*

(i) *If* $\tilde{x}_k \in \widetilde{\mathbb{X}}_f(N)$, *then* $\tilde{x}_{k+1} \in \widetilde{\mathbb{X}}_f(N)$ *for all* $w_k \in \mathcal{B}_\mu$, *where* $\tilde{x}_{k+1} = A_j \tilde{x}_k + B_j u^{MPC}(\tilde{x}_k) + f_j + w_k$. *Moreover,* $\mathbb{X}_T \subseteq \widetilde{\mathbb{X}}_f(N)$.

(ii) *The perturbed PWA system* (7b) *in closed-loop with the MPC control* (4) *obtained by solving Problem 1 (with* $\widetilde{\mathcal{U}}_N(x_k)$ *instead of* $\mathcal{U}_N(x_k)$ *and* (7a) *as prediction model) at each sampling instant is ISS for initial conditions in* $\widetilde{\mathbb{X}}_f(N)$ *and disturbance inputs in* $\mathcal{B}_\mu$.

The proof of Theorem 3 is given in the appendix. The tightened set of admissible input sequences (8) may become very conservative as the prediction horizon increases, since it requires that the state trajectory must be kept farther and farther away from the boundaries. The conservativeness can be reduced by introducing a pre-compensating state-feedback, which is a common solution in robust MPC.

6 Illustrative example

To illustrate the application of Theorem 3 and how to construct the parameters θ, θ_1 and μ for a given $N \in \mathbb{Z}_{\geq 1}$, we present an example. Consider the following *discontinuous* PWA system:

$$x_{k+1} = \tilde{g}(x_k, u_k, w_k) \triangleq g(x_k, u_k) + w_k \triangleq A_j x_k + B_j u_k + w_k \text{ if } x_k \in \Omega_j,\ j \in \mathcal{S}, \tag{9}$$

where $\mathcal{S} = \{1, \ldots, 5\}$, $A_1 = \left[\begin{smallmatrix} -0.0400 & -0.4610 \\ -0.1390 & 0.3410 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0.6552 & 0.2261 \\ 0.5516 & -0.0343 \end{smallmatrix}\right]$, $A_3 = \left[\begin{smallmatrix} -0.7713 & 0.7335 \\ 0.4419 & 0.5580 \end{smallmatrix}\right]$, $A_4 = \left[\begin{smallmatrix} -0.0176 & 0.5152 \\ 0.6064 & 0.2168 \end{smallmatrix}\right]$, $A_5 = \left[\begin{smallmatrix} -0.0400 & -0.4610 \\ -0.0990 & 0.6910 \end{smallmatrix}\right]$, $B_1 = B_2 = B_3 = B_4 = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]^\top$ and $B_5 = \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]^\top$. The state and the input of system (9) are constrained at all times in the sets $\mathbb{X} = [-3, 3] \times [-3, 3]$ and $\mathbb{U} = [-0.2, 0.2]$, respectively. The state-space partition is plotted in Figure 1. The method presented in [17] was employed to compute the terminal weight matrix $P = \left[\begin{smallmatrix} 2.3200 & 0.3500 \\ -0.2100 & 2.4400 \end{smallmatrix}\right]$ and the feedback $K = \left[\begin{smallmatrix} -0.04 & -0.35 \end{smallmatrix}\right]$ such that inequality (5) of Assumption 1 holds for all $x \in \mathbb{R}^2$, the ∞-norm MPC cost with $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.01$ and $h(x) = Kx$. Based on inequality (5), it can be shown that the sublevel sets of the terminal cost $F(\cdot)$, i.e. also $\mathbb{F}_\theta$, are λ-contractive sets [20] for the dynamics $g(x, h(x))$, with $\lambda = 0.6292$. Then, for any θ_1 with $\theta > \theta_1 \geq \lambda\theta$ it holds that $g(x, h(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$. This yields $\mu \leq \frac{(1-\lambda)\theta}{\xi \eta^{N-1}}$. However, μ and θ must also be such that $\mathbb{F}_\theta \subseteq (\Omega_5 \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_{\mathbb{U}}$. Hence, a trade-off must be made in choosing θ and μ. A large θ implies a large μ, which is desirable since μ is an upper bound on $\|w\|$, but θ must also be small enough to ensure the above inclusion. We chose $\theta = 0.96$ and $\theta_1 = \lambda\theta = 0.6040$. Then, with $\eta = 1.5048$, $\xi = 2.67$ and a prediction horizon

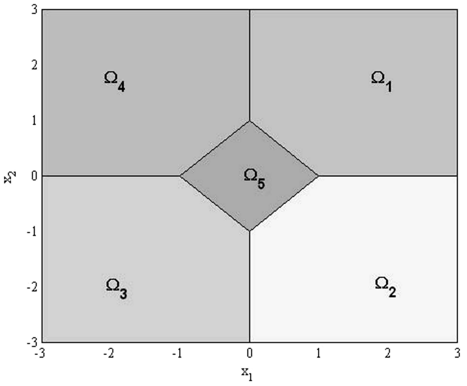

Fig. 1. State-space partition for system (9).

$N = 2$ one obtains that any μ with $0 \leq \mu \leq 0.0886$ is an admissible upper bound on $\|w\|$. For $\mu = 0.0886$ it holds that $\mathbb{F}_\theta \subseteq (\Omega_5 \sim \mathcal{L}_\mu^1) \cap \mathbb{X}_\mathbb{U}$ (see Figure 2 for an illustrative plot). Hence, the hypothesis of Theorem 3 is satisfied for any $w \in \mathcal{B}_\mu = \{w \in \mathbb{R}^2 \mid \|w\| \leq 0.0886\}$.

Then, we used the multi parametric toolbox (MPT) [21] to calculate the MPC control law (4) as an explicit PWA state-feedback, and to simulate the resulting MPC closed-loop system (9)-(4) for randomly generated disturbances in $\mathcal{B}_\mu$. The explicit MPC controller is defined over 132 state-space regions. The set of feasible states $\widetilde{\mathbb{X}}_f(2)$ is plotted in Figure 2 together with the partition corresponding to the explicit MPC control law.

Note that, by Theorem 3, ISS is ensured for the closed-loop system for initial conditions in $\widetilde{\mathbb{X}}_f(2)$ and disturbances in $\mathcal{B}_\mu$, without employing a *continuous MPC value function*. Indeed, for example, $\widetilde{V}_{\text{MPC}}(\cdot)$ and the closed-loop PWA dynamics (9)-(4) are discontinuous at $x = [0 \;\; 1]^\top \in \text{int}(\widetilde{\mathbb{X}}_f(2))$.

7 Conclusion

In this paper we have presented an overview of stability and robustness theory for discrete-time nonlinear MPC while focusing on the application and the extension of the classical results to *discontinuous* nonlinear systems. A stability theorem has been developed, which unifies many of the previous results. An ISS result for discrete-time discontinuous nonlinear MPC has also been presented. A new MPC scheme with an ISS guarantee has been developed for a particular class of discontinuous PWA systems.

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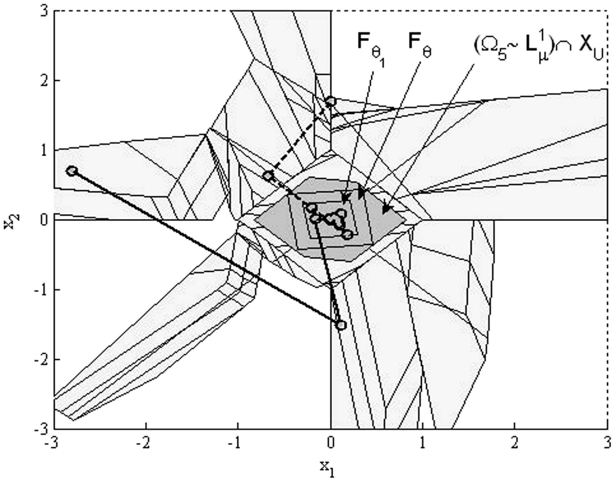

Fig. 2. State trajectories for the MPC closed-loop system (9)-(4) with $x_0 = [0.003\ 1.7]^\top$ - dashed line and $x_0 = [-2.8\ 0.7]^\top$ - solid line.

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A Proof of Theorem 3

Let $(x^*_{1|k}, \ldots, x^*_{N|k})$ denote the state sequence obtained from initial state $x_{0|k} \triangleq \tilde{x}_k$ and by applying the input sequence $\mathbf{u}^*_k$ to (7a). Let $(x_{1|k+1}, \ldots, x_{N|k+1})$ denote the state sequence obtained from the initial state $x_{0|k+1} \triangleq \tilde{x}_{k+1} = x_{k+1} + w_k = x^*_{1|k} + w_k$ and by applying the input sequence $\mathbf{u}_{k+1} \triangleq (u^*_{1|k}, \ldots, u^*_{N-1|k}, h(x_{N-1|k+1}))$ to (7a).

(i) The constraints in (8) are such that: (P1) $(x_{i|k+1}, x^*_{i+1|k}) \in \mathit{\Omega}_{j_{i+1}} \times \mathit{\Omega}_{j_{i+1}}$, $j_{i+1} \in \mathcal{S}$, for all $i = 0, \ldots, N-2$ and, $\|x_{i|k+1} - x^*_{i+1|k}\| \leq \eta^i \mu$ for $i = 0, \ldots, N-1$. This is due to the fact that $x_{0|k+1} = x^*_{1|k} + w_k$, $x_{i|k+1} = x^*_{i+1|k} + \prod_{p=1}^{i} A_{j_p} w_k$ for $i = 1, \ldots, N-1$ and $\|\prod_{p=1}^{i} A_{j_p} w_k\| \leq \eta^i \mu$, which yields $\prod_{p=1}^{i} A_{j_p} w_k \in \mathcal{L}^{i+1}_\mu$. Pick the indices $j_{i+1} \in \mathcal{S}$ such that $x^*_{i+1|k} \in \mathit{\Omega}_{j_{i+1}}$ for all $i = 1, \ldots, N-2$. Then, due to $x^*_{i+1|k} \in \mathit{\Omega}_{j_{i+1}} \sim \mathcal{L}^{i+1}_\mu$, it follows by Lemma 2 of [9] that $x_{i|k+1} \in \mathit{\Omega}_{j_{i+1}} \sim \mathcal{L}^i_\mu \subset \mathbb{X}_i$ for $i = 1, \ldots, N-2$. From $x_{N-1|k+1} = x^*_{N|k} + \prod_{p=1}^{N-1} A_{j_p} w_k$ it follows that $F(x_{N-1|k+1}) - F(x^*_{N|k}) \leq \xi \eta^{N-1} \mu$, which implies that $F(x_{N-1|k+1}) \leq \theta_1 + \xi \eta^{N-1} \mu \leq \theta$ due to $x^*_{N|k} \in \mathbb{X}_T = \mathbb{F}_{\theta_1}$ and $\mu \leq \frac{\theta - \theta_1}{\xi \eta^{N-1}}$. Hence, $x_{N-1|k+1} \in \mathbb{F}_\theta \subset \mathbb{X}_\mathbb{U} \cap (\mathit{\Omega}_{j^*} \sim \mathcal{L}^{N-1}_\mu) \subset \mathbb{X}_\mathbb{U} \cap \mathbb{X}_{N-1}$ so that $h(x_{N-1|k+1}) \in \mathbb{U}$ and $x_{N|k+1} \in \mathbb{F}_{\theta_1} = \mathbb{X}_T$. Thus, the sequence of inputs $\mathbf{u}_{k+1}$ is feasible at time $k+1$ and Problem 1 with

$\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ remains feasible. Moreover, from $g(x, h(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$ and $\mathbb{F}_{\theta_1} \subset \mathbb{F}_\theta$ it follows that $\mathbb{F}_{\theta_1}$ is a positively invariant set for system (7a) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$. Then, since

$$\mathbb{F}_{\theta_1} \subset \mathbb{F}_\theta \subseteq (\Omega_{j^*} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_\mathbb{U} \subset \mathbb{X}_i \cap \mathbb{X}_\mathbb{U} \quad \text{for all} \quad i = 1, \ldots, N-1$$

and $\mathbb{X}_T = \mathbb{F}_{\theta_1}$, the sequence of control inputs $(h(x_{0|k}), \ldots, h(x_{N-1|k}))$ is feasible with respect to Problem 1 (with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$) for all $x_{0|k} \triangleq \tilde{x}_k \in \mathbb{F}_{\theta_1}$. Therefore, $\mathbb{X}_T = \mathbb{F}_{\theta_1} \subseteq \widetilde{\mathbb{X}}_f(N)$.

(ii) The result of part (i) implies that $\widetilde{\mathbb{X}}_f(N)$ is a RPI set for system (7b) in closed-loop with the MPC control (4) and disturbances in $\mathcal{B}_\mu$. Moreover, since $0 \in \operatorname{int}(\mathbb{X}_T)$, we have that $0 \in \operatorname{int}(\widetilde{\mathbb{X}}_f(N))$. The choice of the terminal cost and of the stage cost ensures that there exist $a, b > 0$, $\alpha_1(s) \triangleq as$ and $\alpha_2(s) \triangleq bs$ such that $\alpha_1(\|x\|) \leq \widetilde{V}_{\text{MPC}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \widetilde{\mathbb{X}}_f(N)$. Let $\tilde{x}_{k+1}$ denote the solution of (7b) in closed-loop with $u^{\text{MPC}}(\cdot)$ obtained as indicated in part (i) of the proof and let $x^*_{0|k} \triangleq \tilde{x}_k$. Due to full-column rank of Q there exists $\gamma > 0$ such that $\|Qx\| \geq \gamma\|x\|$ for all x. Then, by optimality, property (P1), $x_{N-1|k+1} \in \mathbb{F}_\theta$ and from inequality (5) it follows that:

$$\begin{aligned}
&\widetilde{V}(\tilde{x}_{k+1}) - \widetilde{V}(\tilde{x}_k) \leq J(\tilde{x}_{k+1}, \mathbf{u}_{k+1}) - J(\tilde{x}_k, \mathbf{u}_k^*) = -L(x^*_{0|k}, u^*_{0|k}) + F(x_{N|k+1}) \\
&\ + [-F(x_{N-1|k+1}) + F(x_{N-1|k+1})] - F(x^*_{N|k}) + L(x_{N-1|k+1}, h(x_{N-1|k+1})) \\
&\ + \sum_{i=0}^{N-2} \Big[L(x_{i|k+1}, \mathbf{u}_{k+1}(i+1)) - L(x^*_{i+1|k}, u^*_{i+1|k})\Big] \\
&\ \leq -L(x^*_{0|k}, u^*_{0|k}) + F(x_{N|k+1}) - F(x_{N-1|k+1}) + L(x_{N-1|k+1}, h(x_{N-1|k+1})) \\
&\ + \left(\xi\eta^{N-1} + \|Q\| \sum_{p=0}^{N-2} \eta^p\right) \|w_k\| \\
&\ \overset{(5)}{\leq} -\|Qx^*_{0|k}\| + \sigma(\|w_k\|) \leq -\alpha_3(\|\tilde{x}_k\|) + \sigma(\|w_k\|),
\end{aligned}$$

with $\sigma(s) \triangleq (\xi\eta^{N-1} + \|Q\| \sum_{p=0}^{N-2} \eta^p)s$ and $\alpha_3(s) \triangleq \gamma s$. Thus, it follows that $\widetilde{V}_{\text{MPC}}(\cdot)$ satisfies the hypothesis of Theorem 3. Hence, the closed-loop system (7b)-(4) is ISS for initial conditions in $\widetilde{\mathbb{X}}_f(N)$ and disturbance inputs in $\mathcal{B}_\mu$. □

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