Multiparametric Nonlinear Integer Programming and Explicit Quantized Optimal Control

Alberto Bemporad

Dip. Ingegneria dell'Informazione
University of Siena
Via Roma 56, 53100 Siena, Italy
Email: bemporad@unisi.it

August 27, 2003

Technical Report, Dept. Information Engineering, University of Siena

Abstract

This paper deals with multiparametric nonlinear integer problems where the optimization variables belong to a finite set and where the cost function and the constraints depend in an arbitrary nonlinear fashion on the optimization variables and in a linear fashion on the parameters. We examine the main theoretical properties of the optimizer and of the optimum as a function of the parameters, and propose a solution algorithm. The methodology is employed to investigate properties of quantized optimal control laws and optimal performance, and to obtain their explicit representation as a function of the state vector.

1 Introduction

In several control synthesis problems the number of possible control actions is finite, a situation usually referred to as *quantization* of the input signals. While in most applications the quantization introduced by analog-to-digital converters, finite precision arithmetic units, and digital to analog converters can be safely neglected by treating the control variables as continuous, in some problems this assumption may lead to an unacceptable deterioration of the closed-loop performance. Examples of control problems that must handle quantization range from more traditional mechanical problems (e.g., problems involving stepping motors) and hydraulic problems (e.g., with on/off valves), to new problems in communications, such as the one dealt with in [1], where quantized control is used to coordinate adaptation of multimedia applications and hardware resource, in order to provide user-preferable QoS requirements under resource contention and energy constraints.

It is therefore worthwhile to devise methods that take into account phenomena of quantization, either for the analysis of the effect of quantization of the input signal, or for the synthesis of quantized control laws. Both research topics are currently receiving a growing attention [2–9], especially in the field of hybrid systems because of the interactions between a continuous dynamical system and a discrete quantized controller.

Among other approaches, receding horizon optimal control ideas were proposed for synthesizing quantized control laws for linear systems with quantized inputs and quadratic optimality criteria. In [10], the authors ensure *practical stability* properties[1], by forcing the terminal state to belong to a special invariant set [6], they deal with state constraints, and propose on-line mixed-integer optimization for the implementation of the control law. In the absence of state-constraints, in [11] the authors show that the control law can be equivalently rewritten as a piecewise affine mapping.

Ideas for solving optimal control problems as an explicit function of the state vector were proposed earlier for linear systems [12–15], nonlinear systems [16], hybrid systems [17, 18], and uncertain linear systems [19]. These approaches rely on *multiparametric solvers*, namely solution algorithms that are able to express the optimizer vector (=the optimal input) as a function of a certain number of parameters (=the current states). The first method for solving multiparametric linear programs dates back to Gal and Nedoma [20]. The book [21] is an excellent reference for properties of generic nonlinear multiparametric problems.

Optimal control problems where all decision variables are quantized and where cost function and constraints depend on a real-valued state vector can be handled by *multiparametric integer programming* solvers. The first approaches to parametric integer programming were limited to scalar parameters [22], we refer the interested reader to the excellent annotated bibliographic survey [23] for more details.

A multiparametric integer solver for linear objectives and linear constraints was developed in [24, 25]. The algorithm finds the lexicographic minimum of the set of integer points which lie inside a convex polyhedron that depend linearly on one or more integral parameters, and is based on parameterized Gomory's cuts followed by a parameterized dual simplex method. An alternative method based on a contraction algorithm for multiparametric integer linear programming problems was proposed in [26]. Algorithms for solving a special class of multiparametric nonlinear integer programming problems were investigated in [27].

In this paper we propose a method for solving a quite general class of multiparametric nonlinear integer problems where: (1) the cost function and the constraints depend linearly on a vector of parameters, (2) they depend in an arbitrary nonlinear fashion on the optimization variables, and (3) these are restricted to belong to a finite set. Because of feature (2), the use of relaxation to non-quantized optimization variables and branching, which is the approach of most multiparametric mixed-integer solvers, would be inappropriate here.

The paper is organized as follows. After examining in Section 2 the main theoretical properties of the optimizer and optimum as a function of the parameters, we propose a solver in Section 3. Multiparametric integer programming is used in Section 4 in the context of quantized optimal control. Numerical results are finally reported in Section 5.

[1] As underlined in [2], the classical concept of stability must be replaced in a quantized context by "practical" stability

2 Multiparametric Nonlinear Integer Programming

We consider the following multiparametric optimization problem:

$$\begin{array}{rcll} V^*(\theta) & \triangleq & \min\limits_{x\in\mathcal{Q}} & f_1(x) + f_2'(x)\theta \\ & & \text{s.t.} & g_1(x) \le g_2'(x)\theta, \\ & & \theta \in \Theta & \end{array} \tag{1}$$

where: $x \in \mathbb{R}^n$ is the optimization vector, which is constrained to belong to the finite set of values $\mathcal{Q} = \{q_1, \ldots, q_N\}$, $q_i \in \mathbb{R}^n$, $\forall i = 1, \ldots, N$; $\theta \in \mathbb{R}^m$ is a vector of parameters, lying in the polyhedron $\Theta = \{\theta \in \mathbb{R}^m : \; T\theta \le S\} \subseteq \mathbb{R}^m$; $f_1 : \mathbb{R}^n \mapsto \mathbb{R}$, $f_2 : \mathbb{R}^n \mapsto \mathbb{R}^m$, $g_1 : \mathbb{R}^n \mapsto \mathbb{R}^p$, $g_2 : \mathbb{R}^n \mapsto \mathbb{R}^{m\times p}$ are generic nonlinear functions of the optimization variables.

A typical instance of $\mathcal{Q}$ is given when each component $x^{\{j\}}$ of x is restricted to a finite set $\Phi_j = \{\phi_{j1}, \ldots, \phi_{jN_j}\}$, $j = 1, \ldots, n$, so that $\mathcal{Q}$ is the Cartesian product $\Phi_1 \times \ldots \times \Phi_n$, and its cardinality $N = \prod_{j=1}^n N_j$.

A solution to the multiparametric program (1) is defined as follows:

Definition 1 *The* feasible parameter set Θ^* *is the set of all $\theta \in \Theta$ for which there is a vector $x \in \mathcal{Q}$ such that $g_1(x) \le g_2'(x)\theta$.*

Definition 2 *The* value function $V^* : \Theta^* \mapsto \mathbb{R}$ *is the function that associates to a parameter vector $\theta \in \Theta^*$ the corresponding optimum $V^*(\theta)$ of problem (1).*

Definition 3 *The* optimizer set function $X^* : \Theta^* \mapsto 2^{\mathcal{Q}}$ *is the function that associates to a parameter vector $\theta \in \Theta^*$ the corresponding set of optimizers $X^*(\theta) = \{x \in \mathcal{Q} : \; f_1(x) + f_2'(x)\theta = V^*(\theta)\}$ of problem (1).*

Definition 4 *The* optimizer function $x^* : \Theta^* \mapsto \mathcal{Q}$ *is the function that associates to a parameter vector $\theta \in \Theta^*$ the lexicographic[2] minimum $x^*(\theta)$ of $X^*(\theta)$.*

The following Lemma 1 and Theorem 1 establish the main properties of the multiparametric solution to problem (1).

Lemma 1 *Consider problem (1) without inequality constraints. Then $V^* : \Theta \mapsto \mathbb{R}$ is a concave piecewise affine function, and $x^* : \Theta \mapsto \mathbb{R}^n$ is a piecewise constant function.*

Proof. In the absence of inequality constraints, $V^*(\theta) = \min_{i=1,\ldots,N}\{f_1(q_i) + \theta' f_2(q_i)\}$ and by Schechter's result [28] it follows that V^* is a piecewise affine concave function over a polyhedral partition of Θ, where the hyperplanes defining the partition have either the form $(f_2(q_i) - f_2(q_j))'\theta \le f_1(q_j) - f_1(q_i)$ or $T^{\{h\}}\theta \le S^{\{h\}}$ ($^{\{h\}}$ denotes the hth row or component).

Example 2.1 For the parametric integer problem

$$V^*(\theta) \triangleq \min_{x\in\{0,1\}} (1-x)^3 + x\theta \tag{2}$$

[2] The lexicographic order is referred to the order of the elements of Q. For example, if $X^*(\theta) = \{q_i, q_j\} \subseteq Q$ and $i < j$, then $x^*(\theta) = q_i$.

($n = m = 1$, $\mathcal{Q} = \{0, 1\}$, $\Theta = \mathbb{R}$) we have

$$\begin{aligned} V^*(\theta) &= \min\{1, \theta\} = \begin{cases} \theta & \text{if } \theta \le 1 \\ 1 & \text{if } \theta > 1, \end{cases} \\ x^*(\theta) &= \begin{cases} 1 & \text{if } \theta \le 1 \\ 0 & \text{if } \theta > 1. \end{cases} \end{aligned}$$

The value function V^* is piecewise affine and concave over $\Theta^* = \mathbb{R}$, and is depicted in Figure 1(a). □

Next Theorem 1 establishes the main properties of the multiparametric solution of problem (1) with inequality constraints.

Theorem 1 *Let $\Theta^* \subseteq \Theta \subseteq \mathbb{R}^m$ be the feasible parameter set of (1), and let $V^* : \Theta \mapsto \mathbb{R}$, $x^* : \Theta \mapsto \mathcal{Q}$ the corresponding value function and optimizer function, respectively. Then Θ^* is the (possibly nonconvex[3]) union of at most N convex polyhedra, and V^*, x^* are a piecewise affine and a piecewise constant function, respectively, of the parameters over a partition of Θ^* in at most $2^N - 1$ (possibly nonconvex) polyhedra.*

Proof. For each $i \in \{1, \dots, N\}$ the linear inequality constraints $g_2'(q_i)\theta \ge g_1(q_i)$ and $T\theta \le S$ define a (possibly empty) polyhedron P_i in $\mathbb{R}^m$. Then, $\Theta^* = \bigcup_{i=1}^N P_i$. Consider now the set C of all combinations of indices $I = \{i_1, \dots, i_K\}$, $i_1 \ge 1$, $i_K \le N$, $K \le N$, $i_j < i_{j+1}$, $\forall j \in \{1, \dots, K-1\}$, without permutations and repetitions (e.g.: for $N = 3$ the combinations $\{1,2\}$, $\{2,1\}$, $\{1,1,2\}$, $\{1,2,1\}$, $\{1,2,2\}$, $\{2,1,1\}$, $\{2,1,2\}$, $\{2,2,1\}$ are only taken once as $\{1,2\}$). The number of elements of C is $\sum_{k=1}^N \binom{N}{k} = 2^N - 1$. Then, for $K = 1, \dots, N$ consider the (possibly nonconvex) polyhedral sets

$$R_{i_1 \dots i_K} = \{\theta \in \mathbb{R}^m : \ \theta \in P_j, \ \forall j \in \{i_1, \dots, i_K\} \quad \text{and} \quad \theta \notin P_h, \ \forall h \notin \{i_1, \dots, i_K\}\}$$

(for instance, for $N = 2$ we have $R_1 = P_1 \backslash (P_1 \cap P_2)$, $R_2 = P_2 \backslash (P_1 \cap P_2)$, $R_{12} = P_1 \cap P_2$; another example is reported in Figure 2, where it can be noticed that R_1, R_4, R_{14} are nonconvex polyhedral sets, and that R_1, R_4 are also disconnected).

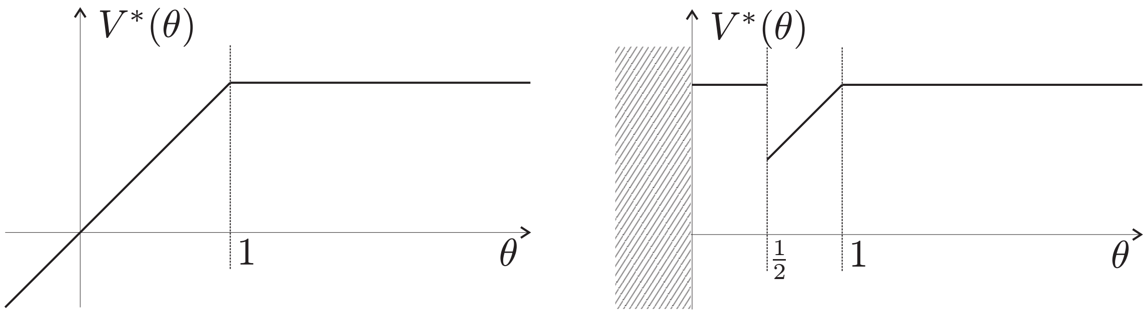
Define $\bar{C} \subseteq C$ as the subset of indices I for which R_I is nonempty (although R_I may not be full dimensional). As $\bigcup_{I \in \bar{C}} R_I = \Theta^*$, the sets R_I define a partition of Θ^* into a finite number of (possibly nonconvex) polyhedra.

On each set R_I, we have

$$V^*(\theta) = \min_{i \in I} \{f_1(q_i) + f_2'(q_i)\theta\}, \ \forall \theta \in R_I, \tag{3}$$

and by Lemma 1 we conclude that V^* is a concave piecewise affine function of θ over R_I. Hence, V^* is piecewise affine over Θ^*. For each given $\theta \in R_I$ the corresponding optimizer is defined as $x^*(\theta) = q_j$, where $j = \min\{i \in I : \ f_1(q_i) + f_2'(q_i)\theta = V^*(\theta)\}$, and where minimization is necessary to obtain the lexicographic minimum in case of multiple optima. □

[3] We use here the following definition of nonconvex polyhedral set: A set $\Omega \subseteq \mathbb{R}^m$ is a nonconvex polyhedral set if Ω is nonconvex and $\Omega = \bigcup_{i=1}^s \Omega_i$, where each set Ω_i is a convex polyhedron and $\Omega_i \cap \Omega_j$ is not full dimensional, $\forall i, j = 1, \dots, s$, $i \neq j$.



(a) Example 2.1

(b) Example 2.2

Figure 1: Value function for Examples 2.1 and 2.2

The proof of Theorem 1 is based on the enumeration of all possible subsets of $\mathcal{Q}$ that are feasible for problem (1), and provides a worst-case upper-bound to the complexity of the multiparametric solution. Fortunately, in general, the number of regions R_I that are useful for characterizing the solution is much smaller, for two reasons. First, emptiness of R_I for several combinations I; second, because of optimality considerations: if for some combination I and $j \notin I$ we have $f_1(q_i) + f_2'(q_i)\theta \leq f_1(q_j) + f_2'(q_j)\theta$ for all $i \in I$ and for all $\theta \in R_{I\cup\{j\}}$, then $R_{I\cup\{j\}}$ is not needed to characterize the solution (R_I is sufficient). The above feasibility and optimality considerations will be exploited in Section 3 to derive a solution algorithm for problem (1).

Example 2.2 If we add the linear constraint

$$x \leq 2\theta \tag{4}$$

to problem (2), the solution changes to

$$V^*(\theta) = \begin{cases} 1 & \text{if} \quad \theta < \frac{1}{2} \text{ or } \theta \geq 1 \\ \theta & \text{if} \quad \frac{1}{2} \leq \theta < 1 \end{cases} \tag{5a}$$

$$x^*(\theta) = \begin{cases} 0 & \text{if} \quad \theta < \frac{1}{2} \text{ or } \theta \geq 1 \\ 1 & \text{if} \quad \frac{1}{2} \leq \theta < 1. \end{cases} \tag{5b}$$

In this case the value function is piecewise affine over $\Theta^* = \{\theta \in \mathbb{R} : \ \theta \geq 0\}$ and has a discontinuity for $\theta = \frac{1}{2}$, as depicted in Figure 1(b). $\square$

Remark 2.1 If equality constraints of the form $h_1(x) + h_2'(x)\theta = 0$ are considered in problem (1), the set of feasible parameters Θ^* (or subsets of it) may not be full dimensional. In fact, as the optimizer function $x^*(\theta) \in \mathcal{Q}$ can only assume a finite number N of values, equality constraints $h_1(x^*(\theta)) + h_2'(x^*(\theta))\theta = 0$ would force θ to lie on a finite number of hyperplanes. More precisely, if $x^*(\theta) = q_i$ on some subset $\Theta_i^* \subseteq \Theta^*$, the dimension of Θ_i^* is $m - \text{rank}(h_2(q_i))$. In particular, when h_2 is an n-by-m full-rank matrix function on $\mathcal{Q}$, Θ^* reduces to a lattice. Note that, instead, in multiparametric mixed-integer problems the continuous components of the optimizer

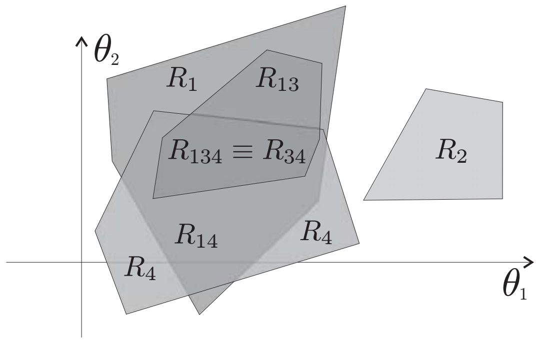


Figure 2: Example of a partition of Θ^* into (possibly nonconvex and disconnected) regions R_I, where R_I is the set of all $\theta \in \Theta$ such that $g_1(q_i) \le g_2'(q_i)\theta$ if and only if $i \in I$

may vary in a nonconstant fashion with the parameter θ, therefore allowing the satisfaction of equality constraints on full dimensional subsets of Θ. The above considerations have important implications when formulating finite-time optimal control problems with equality constraints on the terminal state, as discussed later in Section 4. $\square$

Remark 2.2 The piecewise linearity result for the value function of problem (1), in spite of the general nonlinear form of f_1, f_2, g_1, g_2, should not be surprising, as problem (1) can be reformulated as

$$\begin{array}{rcll} V^*(\theta) & \triangleq & \displaystyle\min_{i\in\{1,\ldots,N\}} & f_{1i} + f_{2i}'\theta \\ & & \text{s.t.} & g_{1i} \le g_{2i}'\theta, \\ & & \theta \in \Theta & \end{array} \tag{6}$$

where $f_{1i} \triangleq f_1(q_i)$, $f_{2i} \triangleq f_2(q_i)$, $g_{1i} \triangleq g_1(q_i)$, $g_{2i} \triangleq f_2(q_i)$ become constant data of the problem, for all $i = 1, \ldots, N$. $\square$

3 A Multiparametric Nonlinear Integer Programming Solver

Multiparametric programming solvers have been proposed for several classes of problems: linear [20,29,30], quadratic [12,15], mixed-integer linear (see [17] and references therein). A complete theory for general nonlinear multiparametric programming was developed in [21]. Most of the solvers rely upon the fact that the optimizer is a piecewise affine function of the parameters defined over *convex* polyhedra. On the other hand, Theorem 1 provides a characterization of the solution over a partition of *nonconvex* (in general) polyhedra. Although nonconvex polyhedra may be split into several convex components, this approach would largely increase the number of partitions. Moreover, mixed-integer solvers rely on the relaxation of integer constraints, an approach that cannot be followed in our context due to the arbitrary nonlinear dependence on the optimization variables.

Parametric programming solvers especially tailored to problems where all the variables are integer were proposed by several authors, as surveyed in [23], although most of them deal with scalar parameters. A multiparametric integer solver for linear objectives and linear constraints was developed in [24], which finds the lexicographic minimum of the set of integer points which lie inside a convex polyhedron that depend linearly on one or more integral parameters. An alternative method based on a contraction algorithm for multiparametric integer linear programming problems was proposed in [26]. In [27] the authors present algorithms for solving a special class of multiparametric nonlinear integer programs, where $f_1(x) = -\sum_{i=1}^n f_1^i(x^{\{i\}})$, $g_1^{\{j\}}(x) = \sum_{i=1}^n g_1^{ij}(x^{\{i\}}) + b$, f_1^i and g_1^{ij} are non-decreasing functions, $\forall i = 1, \dots, n$ and $\forall j = 1, \dots, p$, $f_2(x) \equiv 0$, $g_2(x) \equiv G_2$ is a constant diagonal matrix (i.e., the jth component of θ only perturbs the jth constraint, and therefore $p = m$), $\mathcal{Q} = \{x \in \mathbb{Z}^n : \; x^- \leq x \leq x^+\}$, and $\Theta = [0, 1]^m$.

In this paper we deal with a more general class of multiparametric nonlinear integer problems of the form (1), for which the aforementioned methods are not applicable. A direct application of the ideas used to prove Theorem 1 would lead to fully enumerating all $2^N - 1$ possible combinations of indices $I \in C$, test for nonemptiness of R_I, and characterize the value function and the optimizer on R_I according to (3). We provide here a more efficient solution method.

Before proceeding further, for any set of indices $I = \{i_1, \dots, i_K\} \subseteq \{1, \dots, N\}$, where N is the cardinality of $\mathcal{Q}$, let $P_I \triangleq \bigcap_{i \in I} P_i$, where $P_i = \{\theta \in \Theta : \; g_1(q_i) \leq g_2'(q_i)\theta\}$. Note that $R_I \subseteq P_I$. Moreover, denote by $V_i : \mathbb{R}^m \in \mathbb{R}$ the linear function that maps θ to $V_i(\theta) = f_1(q_i) + f_2'(q_i)\theta$, $i = 1, \dots, N$.

The method we propose here is based on two simple considerations. Let $I = \{i_1 \dots i_K\} \subseteq \{1, \dots, N\}$ and j any index such that $j \in \{i_K + 1, \dots, N\}$. The first consideration relates to *feasibility*: if P_I is empty, then $P_{I \cup \{j\}}$ is certainly empty. The second relates to *optimality*: we can avoid considering a polyhedral region $P_{I \cup \{j\}}$ if $V_j(\theta) \geq V_i(\theta)$ for all $i \in I$ and for all $\theta \in P_{I \cup \{j\}}$, or if $P_{I \cup \{j\}} \subset P_{I \cup \{h\}}$ and $V_j(\theta) \geq V_h(\theta)$ for all $\theta \in P_{I \cup \{h\}}$.

Based on the above considerations, a recursive algorithm for determining the feasible parameter set Θ^*, its subpartition, the value function V^*, and the optimizer function x^*, is summarized by Algorithm 3.1.

The algorithm builds an *optimality tree* $\mathcal{T}$, as depicted in Figure 3, where each node is characterized by a sequence $I = I_0 \cup \{j\}$ and a polyhedron $W_{I_0,j} = \{\theta \in \Theta : \; g_1(q_i) \leq g_2(q_i)'\theta, \; \forall i \in I, V_j(\theta) \leq V_i(\theta), \; \forall i \in I_0\}$, where I_0 is the sequence characterizing the father node.

The root node corresponds to $I = \emptyset$, $W_\emptyset = \Theta$. The maximum depth of the tree is $N = |\mathcal{Q}|$. The maximum number of nodes is 2^N. Clearly, $\mathcal{T}$ is always unbalanced by construction: a feasible combination $\{i_1, i_2, i_3\}$ will be always child of $\{i_1, i_2\}$ rather than $\{i_2, i_3\}$; in particular $\{N\}$ will always be a leaf node.

As the number of nodes in $\mathcal{T}$ depends not only on f_1, f_2, g_1, g_2, and on the number N of elements of $\mathcal{Q}$, but also on the order of the elements of $\mathcal{Q}$, at Step 2. the elements q_j that are infeasible for all $\theta \in \Theta$ (i.e., P_j is an empty convex polyhedron) are eliminated, and the remaining ones pre-ordered by increasing values of $f_1(q_j)$. An alternative is to consider the value $f_1(q_j) + \min_\theta\{f_2'(q_j)\theta$ subject to $g_1(q_j) \leq g_2'(q_j)\theta\}$ as an ordering criterion, which can be easily computed via linear programming for each feasible element $q_j \in Q$.

At step 5.2.1. the set $W_{I_0,j}$ represents the set of all vectors θ for which q_j is feasible, q_i is feasible for all $i \in I_0$, and that have a cost smaller than the cost at the father node (and, by induction, than the cost at all parent nodes). At step 5.2.2.,

---

1. $\mathcal{T} \leftarrow \{$`root_node`$\}$;
2. Remove the elements of $\mathcal{Q}$ that are infeasible for all $\theta \in \Theta$ and order the remaining elements by increasing cost f_1;
3. Execute `examine`($\mathcal{T}$,`root_node`,$\emptyset$);
4. End.

5. Function `examine`($\mathcal{T}$,`node`,I_0);
   - 5.1. If $I_0 \neq \emptyset$ then let $i \leftarrow$ largest element of I_0, otherwise let $i \leftarrow 0$;
   - 5.2. For $j \in \{i+1, \dots, N\}$:
     - 5.2.1. Let $W_{I_0,j} = \{\theta \in P_{I_0} : \; g_1(q_j) \leq g_2(q_j)'\theta, \; V_j(\theta) \leq V_i(\theta), \; \forall i \in I_0\}$;
     - 5.2.2. If $W_{I_0,j} \neq \emptyset$ and the set $\{h : i+1 \leq h < j, \; W_{I_0,j} \subseteq W_{I_0,h}$, and $V_j(\theta) \geq V_h(\theta)$, $\forall \theta \in W_{I_0,j}\} = \emptyset$:
       - 5.2.2.1 Append child node `node`$_j$ to `node` in $\mathcal{T}$;
       - 5.2.2.2 Execute `examine`($\mathcal{T}$,`node`$_j$,$I_0 \cup \{j\}$);
   - 5.3. End.

Algorithm 3.1: Multiparametric integer programming solver.

the algorithm determines if a child node must be generated. A node is not generated if $W_{I_0,j}$ is empty or if it is included in $W_{I_0,h}$ for some "brother" node labeled by $I_0 \cup \{h\}$ already considered so far, and if everywhere on $W_{I_0,j}$ the cost $V_j(\theta)$ is larger than $V_h(\theta)$.

After the execution of Algorithm 3.1 and the construction of the tree $\mathcal{T}$, the multiparametric solution can be simplified by removing branches from $\mathcal{T}$ according to a criterion similar to the one in Step 5.2.2.: for each node `node`$_j$ characterized by $I_0 \cup \{j\}$, we can check if there exists a "brother" node `node`$_h$, $j < h \leq N$, such that $W_{I_0,j} \subseteq W_{I_0,h}$ and $V_j(\theta) \geq V_h(\theta)$, $\forall \theta \in W_{I_0,j}$. If this happens, `node`$_j$ and its whole sub-tree can be safely removed, without affecting the multiparametric solution.

Remark 3.1 Complexity and suboptimality of the multiparametric solution can be traded off with minor modifications to Algorithm 3.1. In fact, given a suboptimality tolerance $\epsilon \geq 0$, we can modify the optimality requirement in Step 5.2.1. by imposing that $V_j(\theta) \leq V_i(\theta) - \epsilon$, so that a child node is added only if the cost improves at least by ϵ, and, similarly, in Step 5.2.2. by asking that $V_j(\theta) \geq V_h(\theta) - \epsilon$. □

3.1 Evaluation of the Solution

The tree structure $\mathcal{T}$ constructed by Algorithm 3.1 can be immediately used for storing the multiparametric solution in the form (7), and for evaluating the optimal value and the optimizer for a given $\theta \in \mathbb{R}^m$, as detailed in the recursive Algorithm 3.2.

During the execution of Algorithm 3.2, children nodes must be visited in lexicographic order, namely if $j < h$, the node corresponding to the sequence $I = \{i_1, \dots, i_k, j\}$ must be visited before the node corresponding to the sequence $I = \{i_1, \dots, i_k, h\}$. This ordering comes naturally by the way Algorithm 3.1 constructs tree $\mathcal{T}$. At Step 2.2., one can avoid evaluating the whole inclusion $\theta \in P_I$. Indeed, only checking $\theta \in P_{i_M}$, where $i_M \triangleq \max(I)$, is enough, as the remaining conditions

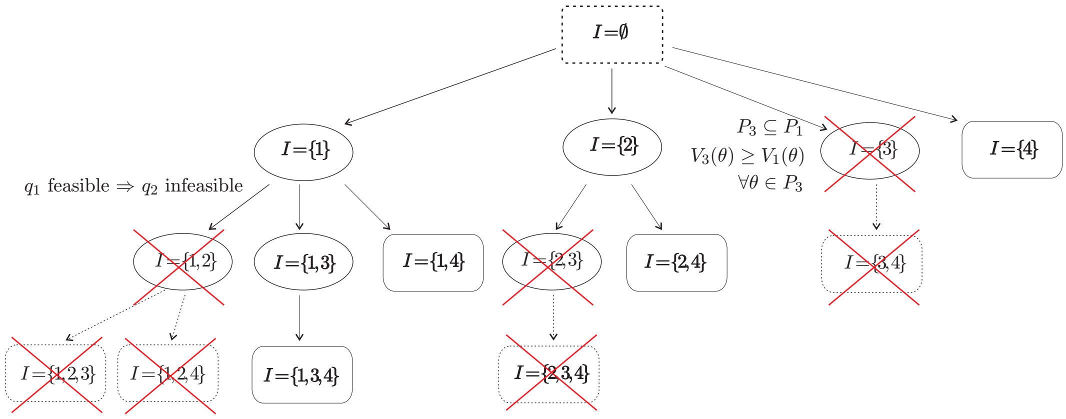


Figure 3: Optimality tree $\mathcal{T}$, related to the partition depicted in Figure 2

$\theta \in P_i$, by recursion, were already checked for all $i \in I \setminus \{i_M\}$. Moreover, only the inequalities of P_{i_M} which are not redundant on $P_{I\setminus\{i_M\}}$ need to be evaluated, which allows one to save memory space and computation time. In view of the above considerations, and since q_{i_M} is always the optimizer, only $i_M = \max(I)$ needs to be stored in the node, rather than the whole sequence I. Moreover, all the constraints defining P_{i_M} belong to a finite constraint store, so that rather than memorizing copies of the same constraints one can memorize pointers to entries of such a constraint store.

Remark 3.2 Multiparametric linear, quadratic, and mixed-integer linear solvers partition the set of feasible parameters Θ^* into the union of convex polyhedral sets with nonoverlapping interiors [12,13,15,17,20]. A potential drawback of this type of representation is that when the number of polyhedra is large the evaluation of the solution may be computationally expensive, because to determine which polyhedron contains a given parameter vector θ may require (in the worst case) evaluating the defining inequalities of all polyhedra. A useful approach to overcome this problem was taken in [31], where the authors suggested to organize a given polyhedral partition on a search tree, so that the amount of computation for evaluating the solution is on average logarithmic in the number of polyhedra.

Here we have taken a different approach by expressing the solution as a multi-level conditional expression (i.e., as a tree of nested conditionals), similarly to what is done in [25] for multiparametric integer linear programming[4]. In fact, the multi-

[4]In [25] the authors denominate a multi-level conditional expression a *quast*.

1. $[V^*(\theta), x^*(\theta)]$ ←eval($\mathcal{T}$,`root_node`,θ);

2. Function $[V^*, x^*]$ ←eval($\mathcal{T}$,`node`,θ);
   2.1. Let $V^* \leftarrow +\infty$, $x^* \leftarrow \emptyset$;
   2.2. If $\theta \in P_I$:
      2.2.1. Let $I \leftarrow$ combination associated with `node`;
      2.2.2. Let $c \leftarrow$ number of children of `node`; Let $i \leftarrow 0$;
      2.2.3. While $i < c$ and $V^* = +\infty$:
         2.2.3.1 $i \leftarrow i + 1$;
         2.2.3.2 Let $\texttt{node}_i \leftarrow i$-th child of `node`;
         2.2.3.3 $[V^*, x^*]$ ←eval($\mathcal{T}$,$\texttt{node}_i$,θ);
      2.2.4. If $V^* = +\infty$ and $I \neq \emptyset$:
         2.2.4.1 Let $i^* \leftarrow$ largest element of I;
         2.2.4.2 Let $x^* \leftarrow q_{i^*}$, $V^* \leftarrow f_1(q_{i^*}) + f_2'(q_{i^*})\theta$;
   2.3. Return $[V^*, x^*]$;
   2.4. End.

Algorithm 3.2: Evaluation of the optimal value $V^*(\theta)$ and of the lexicographic minimum $x^*(\theta)$

parametric solution can be written as:

$$
\begin{array}{l}
\text{if } \theta \in \Theta \text{ then} \\
\quad \text{if } H_1\theta \leq K_1 \text{ then} \\
\quad\quad \vdots \\
\quad\quad \text{if } H_i\theta \leq K_i \text{ then} \\
\quad\quad\quad x^*(\theta) = q_i \\
\quad\quad \vdots \\
\quad \text{elseif } H_k\theta \leq K_k \text{ then} \\
\quad\quad \vdots \\
\quad \text{else} \\
\quad\quad \text{problem is infeasible} \\
\quad \text{end} \\
\text{else} \\
\quad \text{solution is undefined } (\theta \notin \Theta) \\
\text{end}
\end{array}
\tag{7}
$$

where $H_{()}$, $K_{()}$, are (possibly empty) matrices/vectors of suitable dimensions. □

Example 3.1 The solution reported in (5) can be obtained by running Algorithm 3.1 for $\Theta = \{\theta : \|\theta\|_\infty \leq 10\}$, with $\mathcal{Q} = \{1, 0\}$ (the elements q_1, q_2 of $\mathcal{Q}$ are ordered by increasing cost $f_1(q_i)$), and arranged as follows:

$$
\begin{array}{l}
\text{if } \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] \theta \leq \left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}\right] \text{ then} \\
\quad \text{if } -2\theta \leq -1 \text{ then} \\
\quad\quad \text{if } -\theta \leq -1 \text{ then} \\
\quad\quad\quad x^*(\theta) = q_2,\ V^*(\theta) = (1-q_2)^3 + q_2\theta, \text{ where } q_2 = 0 \\
\quad\quad \text{else} \\
\quad\quad\quad x^*(\theta) = q_1,\ V^*(\theta) = (1-q_1)^3 + q_1\theta, \text{ where } q_1 = 1 \\
\quad\quad \text{end} \\
\quad \text{elseif } -2\theta \leq 0 \text{ then} \\
\quad\quad x^*(\theta) = q_2,\ V^*(\theta) = (1-q_2)^3 + q_2\theta, \text{ where } q_2 = 0 \\
\quad \text{else} \\
\quad\quad \text{problem is infeasible} \\
\quad \text{end} \\
\text{else} \\
\quad \text{solution is undefined (because } \|\theta\|_\infty > 10) \\
\text{end}
\end{array}
\tag{8}
$$

Note that (8) is not a minimal multi-level conditional expression. Determining ways for ensuring the minimality of the multilevel conditional solution is a topic that remains to be investigated. □

4 Explicit Quantized Optimal Control

Consider the following linear discrete time invariant system

$$x(t+1) = Ax(t) + Bu(t) \tag{9}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathcal{U} \triangleq \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_L\}$, $\bar{u}_i \in \mathbb{R}^{n_u}$ are the levels of quantization, and (A, B) is a stabilizable pair. Starting from the initial state $x(0)$, we wish to control the final state $x(T)$ to a target set Ω while satisfying the constraints

$$\bar{A}x(t) + \bar{B}u(t) \leq \bar{C},\ \ t = 0, \ldots, T-1. \tag{10}$$

Constraints (10) are generic linear constraints on input and state variables. A typical instance are box constraints of the form $x_{\min} \leq x_k \leq x_{\max}$ (constraints of the form $u_{\min} \leq u_k \leq u_{\max}$ can be immediately taken into account by simply excluding from $\mathcal{U}$ those values $\bar{u}_i$ outside the bounds). We assume that the set Ω is a full-dimensional polyhedral terminal set for the state vector, as in case of non full-dimensional sets Ω, the set Θ of initial states $x(0)$ for which (10) are feasible may be lower-dimensional, for instance if $\Omega = \{0\}$, corresponding to the constraint $x(T) = 0$, Θ would be a lattice. In the following subsections we show how the multiparametric integer solver developed earlier can be used to derive explicit optimal control laws based on the minimization of a quadratic or linear performance index.

4.1 Quadratic Quantized Optimal Control

Consider the following optimal control problem:

$$\min_U \left\{ J(U,\theta) = x_T' P x_T + \sum_{k=0}^{T-1} \left(x_k' Q x_k + u_k' R u_k \right) \right\} \tag{11a}$$

$$\text{s.t.} \begin{cases} x_0 = \theta \\ x_{k+1} = Ax_k + Bu_k,\ k = 0,\dots,T-1, \\ \bar{A}x_k + \bar{B}u_k \le \bar{C},\ k = 0,\dots,T-1, \\ x_T \in \Omega \\ u_k \in \mathcal{U} \triangleq \{\bar{u}_1,\dots,\bar{u}_L\} \end{cases} \tag{11b}$$

where $R = R' > 0$, $Q = Q' \ge 0$, $P \ge 0$ are matrices of suitable dimensions, θ represents a generic initial condition, $U \triangleq [u_0'\ u_1'\ \dots\ u_{T-1}']' \in \mathbb{R}^{mT}$ is the set of free control moves, $U \in \mathcal{Q}$, where $\mathcal{Q} \triangleq \mathcal{U}^T = \mathcal{U} \times \dots \times \mathcal{U}$, and $U^*(\theta) \triangleq [u_0^{*'}\ u_1^{*'}\ \dots\ u_{T-1}^{*'}](\theta)'$ is the minimizer (or, in case of multiple optima, the lexicographic minimum of the set of optimizers).

It is immediate to cast problem (11) as an integer quadratic program (IQP). Indeed, by substituting $x_k = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$, Eq. (11a) can be rewritten as

$$\begin{array}{ll} \min\limits_U & \left\{ \frac{1}{2} U'HU + U'F'\theta + \frac{1}{2}\theta' Y \theta \right\} \\ \text{subj. to} & GU \le W + E\theta \\ & U \in \mathcal{Q} \end{array} \tag{12}$$

where the column vector $U \triangleq [u_0',\dots,u_{T-1}']' \in \mathbb{R}^{mT}$ is the optimization vector, $H = H' > 0$, and H, F, Y, G, W, E are easily obtained from Q, R, and (11a).

The optimization problem (11) is an IQP which depends on the initial state θ. The multiparametric nonlinear integer programming algorithm developed earlier can be conveniently used to compute the piecewise constant solution $U^*(x_0)$ to the optimal control problem (11). In fact, after taking apart the quadratic term $\frac{1}{2}\theta' Y \theta$ that does not affect the optimizer $U^*(\theta)$, problem (12) can be recast in the form (1) by setting $f_1(U) = \frac{1}{2}U'HU$, $f_2(U) = FU$, $g_1(U) = GU - W$, $g_2(U) = E'$.

The following result immediately follows by Theorem 1.

Corollary 1 *Consider the optimal control problem (11), parameterized by the initial condition $x_0 = \theta$. Then*

(i) The set of parameters $\Theta^* \subseteq \mathbb{R}^{n_x}$ for which a solution to (11) exists is the union of at most L^T convex polyhedra.

(ii) The value function $V^* : \mathbb{R}^{n_x} \in \mathbb{R}$ is a piecewise quadratic function of x_0 (more exactly, the sum of a convex quadratic and a piecewise affine function) over a partition of Θ^* in at most $2^{L^T} - 1$ (possibly nonconvex) polyhedra.

(iii) The optimizer function $U^* : \mathbb{R}^{n_x} \in \mathcal{U}^T$ is a piecewise constant function of x_0 defined over the same partition of Θ^*.

Moreover, in the absence of inequality constraints $\bar{A}x_k + \bar{B}u_k \le \bar{C}$, $k = 0,\dots,T-1$, and $x_T \in \Omega$, V^* is the sum of a convex quadratic and a piecewise affine concave function of x_0.

4.2 Linear Quantized Optimal Control

Consider the following optimal control problem:

$$\min_{U} \left\{ J(U,\theta) = \|Px_T\|_\infty + \sum_{k=0}^{T-1} (\|Qx_k\|_\infty + \|Ru_k\|_\infty) \right\} \tag{13a}$$

$$\text{s.t.(11b)} \tag{13b}$$

where R, Q, P are full-rank matrices with n_R, n_Q, and n_P rows, respectively, and a suitable number of columns, and θ, U, $\mathcal{U}$, $\mathcal{Q}$, and $U^*(\theta)$ are defined as above.

Similarly to [17], problem (13) can be cast as an integer linear program (ILP). To this end, we introduce the following constraints

$$\begin{array}{rcl} \epsilon_t^u & \geq & \pm R^{\{j\}} u_t,\ \forall j = 1,\ldots,n_R,\ \forall t = 0,\ldots,T-1 \\ \epsilon_t^x & \geq & \pm Q^{\{j\}} x_t,\ \forall j = 1,\ldots,n_Q,\ \forall t = 0,\ldots,T-1 \\ \epsilon_T^x & \geq & \pm P^{\{j\}} x_T,\ \forall j = 1,\ldots,n_P, \end{array} \tag{14}$$

where ${}^{\{j\}}$ denotes the jth row. By substituting again $x_k = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$, and by letting $\mathcal{E} = [\epsilon_0^x \epsilon_0^u\ \ldots\ \epsilon_{T-1}^x \epsilon_{T-1}^u \epsilon_T^x]$, problem (13) can be rewritten as

$$\begin{array}{ll} \min\limits_{U,\mathcal{E}} & \left\{ \epsilon_T^x + \sum_{t=1}^{T-1} \epsilon_t^x + \epsilon_t^u \right\} \\ \text{subj. to} & G \left[\begin{smallmatrix} U \\ \mathcal{E} \end{smallmatrix} \right] \leq W + E\theta \\ & \mathcal{E} \geq 0 \\ & \left[\begin{smallmatrix} U \\ \mathcal{E} \end{smallmatrix} \right] \in \mathcal{Q} \times \mathcal{Q}_{\mathcal{E}}, \end{array} \tag{15}$$

where $\mathcal{Q}_{\mathcal{E}} \subset \mathbb{R}^{2T+1}$ is also a finite set, as, at optimality, for each optimal component of $\mathcal{E}$ at least one of the constraints (14) is active, so that also the components of $\mathcal{E}$ are indeed quantized. Therefore, the optimization problem (13) is an ILP which depends on the initial state θ, and the multiparametric integer programming algorithm developed earlier may be used to compute the explicit piecewise constant solution $\left[\begin{smallmatrix} U^* \\ \mathcal{E}^* \end{smallmatrix} \right] (x_0)$. A corollary of Theorem 1 similar to Corollary 1 may be easily stated.

On the other hand, the approaches of [26] and of [24], that are specialized for multiparametric integer linear problems may be more suitable here. Alternatively, since continuous relaxations of (15) are multiparametric linear programs, it may be more convenient to treat $\mathcal{E}$ as continuous variables and solve (15) by using multiparametric mixed-integer linear solvers, as done in [17] for the generic case of hybrid systems with continuous and discrete inputs.

4.3 Explicit Quantized Receding Horizon Control

A useful way for transforming the $U^*(\theta)$ into a closed-loop control law is to adopt the so called *receding horizon* philosophy. The receding horizon controller is defined as

$$u(t) = u_0^*(x(t)), \tag{16}$$

where $u_0^*(x(t))$ is the first element of the minimizer $U^*(x(t))$ of the finite-time quantized optimal control problem, initialized at the current state $\theta = x(t)$.

An immediate corollary of Corollary 1 is that the control law (16) is a piecewise constant law defined over a polyhedral partition. Criteria for selecting the terminal set Ω in order to guarantee practical stability properties of the quantized control law (16) were analyzed in [10].

Remark 4.1 As only the first part $u_0^*(x(t))$ of the minimizer $U^*(x(t))$ is of interest, after the execution of Algorithm 3.1 the multiparametric solution can be simplified by removing subtrees of $\mathcal{T}$ where the first optimal move u_0^* is the same in all nodes (in depth search of such subtrees would just serve to determine $u_1^*, \ldots, u_{N-1}^*$). $\square$

4.4 Logic-Based Constraints

A *Boolean expression* is inductively defined by the grammar

$$\begin{aligned} \phi \ \ ::= \ \ & X|\neg\phi_1|\phi_1 \vee \phi_2|\phi_1 \oplus \phi_2|\phi_1 \wedge \phi_2| \\ & \phi_1 \leftarrow \phi_2|\phi_1 \rightarrow \phi_2|\phi_1 \leftrightarrow \phi_2|(\phi_1), \end{aligned} \tag{17}$$

where $X \in \{0,1\}$ is a Boolean variable, and the logic operators $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\leftarrow$ (implied by), $\rightarrow$ (implies), $\leftrightarrow$ (iff) have the usual semantics. Every Boolean expression can be rewritten in *conjunctive normal form* (CNF), which is defined by the following grammar:

$$\phi \ \ ::= \ \ \psi|\phi \wedge \psi, \tag{18}$$

$$\psi \ \ ::= \ \ \psi_1 \vee \psi_2|\neg X|X. \tag{19}$$

By generalizing results of [32–34], in [35] the authors illustrated techniques for equivalently expressing arbitrary Boolean functions by a set of linear inequalities over 0-1 variables. In particular, by first converting a Boolean formula into CNF (a task that can be performed automatically by using one of the several available techniques), by letting the CNF have the form $\bigwedge_{j=1}^{m_\ell}\left(\bigvee_{i\in P_j} X_i \bigvee \bigvee_{i\in N_j} \neg X_i\right)$, $N_j, P_j \subseteq \{1,\ldots,\ell\}$, $\forall j = 1,\ldots,m_\ell$, the corresponding set of integer linear inequalities is

$$\begin{cases} 1 \le \sum_{i\in P_1} X_i + \sum_{i\in N_1}(1-X_i), \\ \vdots \\ 1 \le \sum_{i\in P_m} X_i + \sum_{i\in N_m}(1-X_i), \end{cases} \tag{20}$$

that define a polyhedron $P_{\mathrm{CNF}} \subset \mathbb{R}^\ell$.

As a consequence, Boolean constraints involving 0-1 variables $x = [X_1 \ \ldots \ X_n]$ and 0-1 parameters $\theta = [X_{n+1} \ \ldots \ X_{n+m}]$, $\ell = n+m$, having the form

$$g_B(x,\theta) = 0 \tag{21}$$

where g_B is an arbitrary Boolean function, can be translated into linear inequalities

$$A_g x \le c_g + B_g\theta. \tag{22}$$

Clearly, (22) fits the general multiparametric integer programming framework (1) with $\mathcal{Q} = \{0,1\}^n$, and $\Theta = \{0,1\}^m \subset [0,1]^m$.

N	25	36	49	64	81	100	121
Nodes in $\mathcal{T}$	14	14	20	23	30	32	38
Algorithm 3.1	1.42 s	2.06 s	3.15 s	5.11 s	7.93 s	10.22 s	13.79 s
Algorithm 3.2	2.13 ms	2.37 ms	2.45 ms	2.89 ms	3.40 ms	3.78 ms	4.23 ms
Enumeration	14.27 ms	20.05 ms	26.98 ms	35.17 ms	44.63 ms	59.94 ms	65.65 ms

Table 1: Computational experience for Problem (23): number N of elements of $\mathcal{Q}$, number of nodes in $\mathcal{T}$ generated by Algorithm 3.1 and required CPU time, CPU time for evaluating the solution using Algorithm 3.2 and using enumeration (CPU times are averaged over 1089 values of θ uniformly distributed over Θ)

5 Examples

Example 5.1 Consider the multiparametric integer problem

$$\begin{aligned}
V^*(\theta) \;\triangleq\; & \min_{x\in\mathcal{Q}} \; x_1^3 + |x_2| \\
& \text{s.t.} \; \begin{bmatrix} 1 & 0 \\ 0.3090|x_2| & 0.9511|x_1| \\ -0.8090 & 0.5878 \\ -0.8090|\sin(x_1/5)| & -0.5878|x_2| \\ 0.3090 & -0.9511 \\ -1 & 0 \end{bmatrix} (1-\tfrac{1}{2}\sqrt{x_1^2+x_2^2}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
& \quad - \begin{bmatrix} 6.7506 \\ 3.1557 \\ 5.0342 \\ 4.4299 \\ 6.4565 \\ 0 \end{bmatrix} \le \begin{bmatrix} 2.6210 & 1.1543 \\ -0.4353 & 2.9194 \\ -4.8150 & 4.2181 \\ 3.2141 & 2.3821 \\ -0.5530 & -3.2373 \\ 0 & 0 \end{bmatrix} (1-\tfrac{1}{2}\sqrt{x_1^2+x_2^2}) \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \\
& \quad -10 \le \theta_1, \theta_2 \le 10
\end{aligned} \tag{23}$$

where $\mathcal{Q}$ is obtained by gridding the square $[0,2]\times[0,2]$ with a grid-step of $\frac{2}{n}$, $n = 4,5,\ldots,10$, leading to $N = (n+1)^2$ elements in $\mathcal{Q}$. In Table 1 we report the computation time required by Algorithms 3.1 and 3.2, and compare the latter with the time required for computing the optimal solution by enumeration. In Figure 4 we show the value function when the grid step is 0.2 ($\mathcal{Q}$ contains $N = 121$ elements). □

Example 5.2 Consider the following optimal reliability design problem proposed in [27, Example 4]:

$$\begin{aligned}
\max \quad & z = (1-(1-0.6)^{x^{\{1\}}})(1-(1-0.9)^{x^{\{2\}}})(1-(1-0.55)^{x^{\{3\}}})(1-(1-0.75)^{x^{\{4\}}}) \\
\text{s.t.} \quad & \begin{bmatrix} 6.2 & 3.8 & 6.5 & 5.3 \\ 9.5 & 5.5 & 3.8 & 4.0 \end{bmatrix} x \le \begin{bmatrix} 50 + 15\theta^{\{1\}} \\ 50 + 20\theta^{\{2\}} \end{bmatrix} \\
& x \in \mathcal{Q} = \{x \in \mathbb{Z}^4 : \; 1 \le x \le \bar{x}\} \\
& \Theta = \{\theta \in \mathbb{R}^2 : \; 0 \le \theta \le 1\}.
\end{aligned} \tag{24}$$

In (24) we have added the upper bound $\bar{x}$, obtained via linear programming by maximizing $x^{\{j\}}$ with respect to (x, θ) subject to the linear inequality constraints

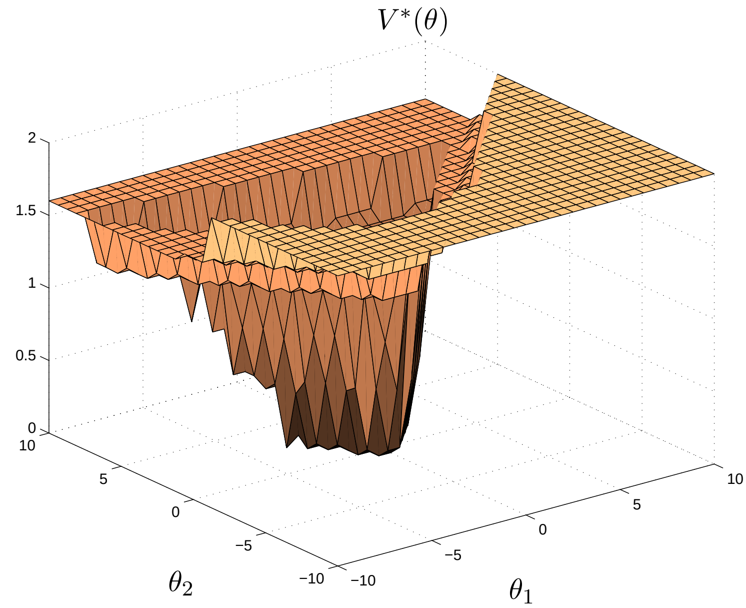


Figure 4: Value function for problem (1) when $\mathcal{Q}$ contains $N = 121$ elements

in (24) and to $\theta \in \Theta$, for $j = 1, 2, 3, 4$, and by rounding off the result to the nearest smaller or equal integer, which provides $\bar{x} = [5\ 9\ 7\ 9]'$.

The set $\mathcal{Q}$ contains 2835 elements, of which 2484 are infeasible. Algorithm 3.1 is executed in 24.95 s, and provides the solution depicted in Figure 5. The solution coincides with the one reported in [27, Figure 3]. The corresponding optimality tree consists of 18 nodes, where all nodes are leaf nodes, except the root node. $\square$

Example 5.3 Consider an extremely simplified version of the problem of landing a spacecraft on a planet, where we consider only the vertical motion described by the equations

$$\left\{ \begin{array}{rcl} m\frac{dv}{dt} & = & -\beta v + u \\ \frac{dh}{dt} & = & v, \end{array} \right. \tag{25}$$

where h is the height from ground, v the vertical velocity, and the overall force u acting on the spacecraft is given by

$$u = \left\{ \begin{array}{ll} -mg & \text{thruster off} \\ 0 & \text{thruster on (gravity compensation)} \\ mg & \text{double thruster on.} \end{array} \right. \tag{26}$$

By choosing the parameters $\beta = 1$, $m = 1$, $g = 1$ (units are omitted here, as the parameters have no particular meaning in this example), and by discretizing the dynamics with a sampling time $T_s = 1$, we obtain the discrete-time linear model

$$x(t+1) = \left[\begin{array}{cc} 1 & 0.6321 \\ 0 & 0.3679 \end{array} \right] x(t) + \left[\begin{array}{c} 0.7358 \\ 1.2642 \end{array} \right] u(t), \tag{27}$$

where $u(t) \in \mathcal{U} \triangleq \{-1, 0, 1\}$, and $x = [\begin{smallmatrix} h \\ v \end{smallmatrix}]$. We wish to design a controller that brings the height of the spacecraft and its velocity to zero while satisfying the constraints

$$\begin{array}{rcl} h & \geq & 0 \\ v & \geq & -\bar{v}, \end{array} \tag{28}$$

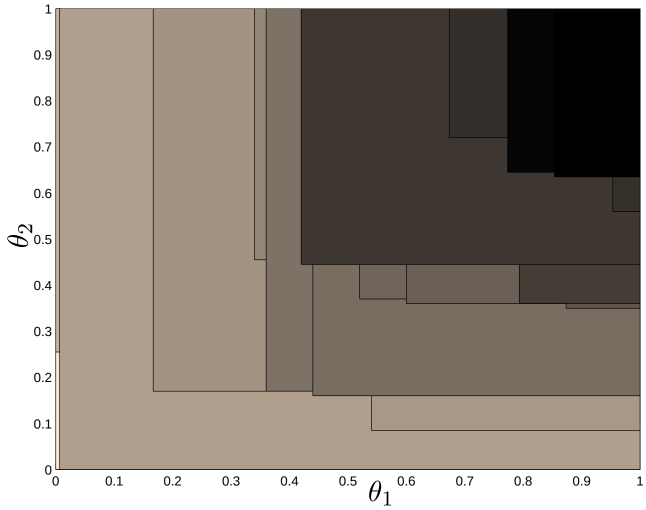


Figure 5: Partition associated with the solution to Problem (24) (the level of darkness is proportional to the value of $V^*(\theta)$, which is piecewise constant over each region). The exact definition of the inequalities defining the partition and the corresponding optimal values are reported in [27, p. 250]

where $\bar{v} = 1.5$. To this end, we consider the finite-time optimal control problem

$$\begin{array}{ll} \min\limits_{u_0,u_1} & x_2' P x_2 + \sum\limits_{k=0}^{1} \left(x_k' Q x_k + u_k' R u_k \right) \\ \text{s.t.} & x_1 \geq \left[\begin{smallmatrix} 0 \\ -v \end{smallmatrix} \right] \\ & u_0, u_1 \in \{-1, 0, 1\}, \end{array}$$

where $R = 10$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right]$, and $P \approx \left[\begin{smallmatrix} 3.1240 & 1.5677 \\ 1.5677 & 2.3241 \end{smallmatrix} \right]$ solves the Riccati equation $P = (A + BK_{LQ})' P (A + BK_{LQ}) + K_{LQ}' R K_{LQ} + Q$, $K_{LQ} = -(R + B'PB)^{-1} B'PA$.

The mp-IQP problem associated with the optimal control law (29) has the form (12) with $\theta = x_0$ and

$$\begin{aligned} H &= \begin{bmatrix} 0.7675 & 0.2924 \\ 0.2924 & 0.6323 \end{bmatrix}, \quad F = \begin{bmatrix} 0.2160 & 0.2132 \\ 0.1477 & 0.1468 \end{bmatrix} \\ G &= \begin{bmatrix} -0.7358 & 0 \\ -1.2642 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ 1.5 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0.6321 \\ 0 & 0.3679 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where we have neglected the constant term $\frac{1}{2}\theta' Y \theta$. By running Algorithm 3.1 on $\Theta = \{\theta : \|\theta\|_\infty \leq 15\}$, the multiparametric solution is computed in 0.85 s on a Pentium III 800 Mhz running Matlab 5.3, and the associated tree $\mathcal{T}$ consists of 24 nodes and has a depth of 5 levels, as depicted in Figure 6. The number of inequalities associated with each node varies between one and four. An evaluation of the value function V^* takes an average of 1.36 ms (this value is obtained by averaging over

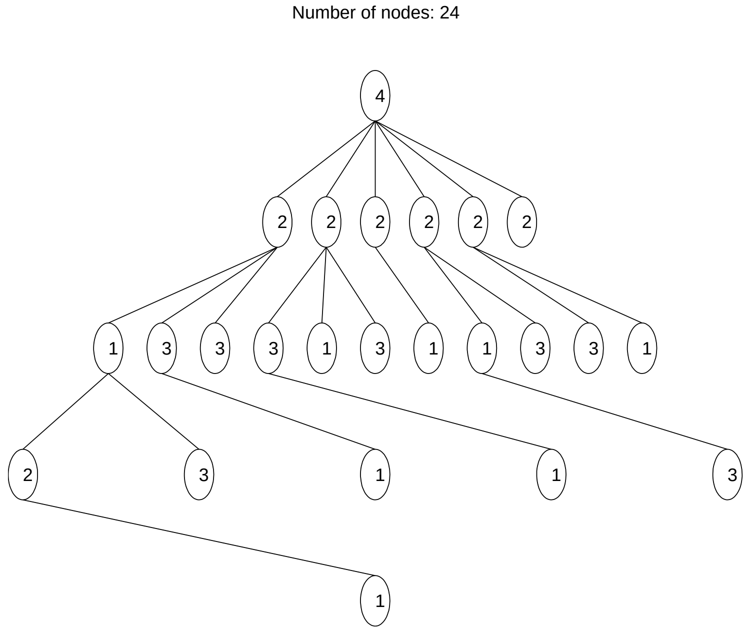


Figure 6: Optimality tree associated with the optimal control problem (29). For each node is reported the number of linear inequalities that must be checked at that node during the on-line evaluation of the solution for a given x_0

a grid of 4225 samples of Θ), against about 6.01 ms needed to compute V^* by enumeration. Even from this simple problem where the number of elements of $\mathcal{Q}$ is only $N = 9$, it is clear the advantage of having an explicit representation of V^*.

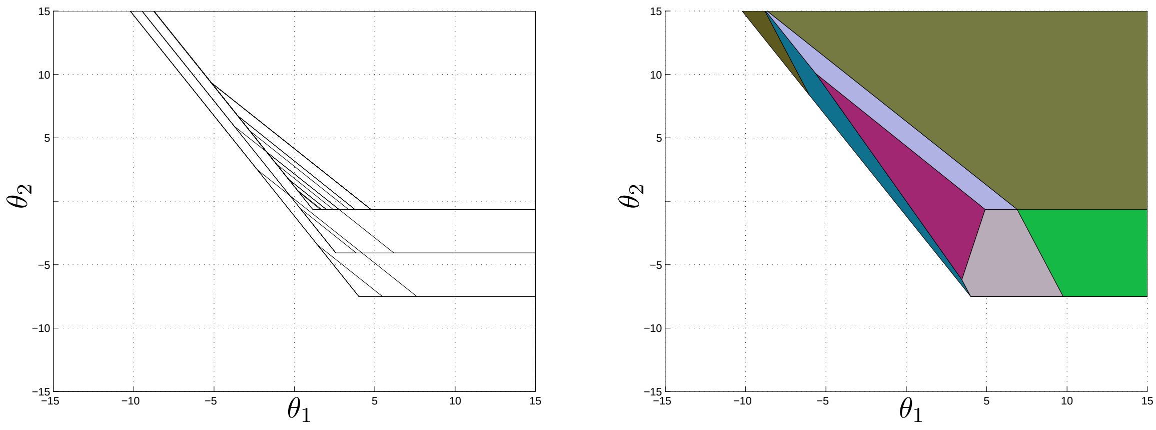
We compare now the solution $U^*(\theta)$ of the integer quadratic problem with the quantization $\hat{U}(\theta)$ to the nearest (in Euclidean norm) feasible point in $\mathcal{Q}$ of the solution $U^*_{\text{QP}}(\theta)$ of the continuous quadratic program $\min_U \in \mathbb{R}^{mT}\{\frac{1}{2}U'HU + \theta' F'U$ subject to $GU \leq W + E\theta\}$. The partition associated with $U^*_{\text{QP}}(\theta)$, obtained in 0.22 s using the algorithm reported in [15], is depicted in Figure 7(b), while the partition associated with $U^*(\theta)$ is depicted in Figure 7(a). In Figure 8, we report the difference $\hat{V}(\theta) - V^*(\theta)$, where $\hat{V}(\theta) = \frac{1}{2}\hat{U}'(\theta)H\hat{U}(\theta) + \theta' F'\hat{U}(\theta)$, and $V^*(\theta)$ is the optimal value function for the integer quadratic program; clearly $V^*(\theta) \leq \hat{V}(\theta)$, for all $\theta \in \Theta^*$.

By implementing the multiparametric solution in a receding horizon fashion, we obtain the trajectories plotted in Figure 9, that show the closed-loop behavior of the system for the initial condition $x(0) = \left[\begin{smallmatrix}10\\0\end{smallmatrix}\right]$.

$\square$

6 Conclusions

For multiparametric nonlinear integer problems where the cost function and the constraints depend in an arbitrary nonlinear fashion on the optimization variables and in a linear fashion on the parameters, and where the optimization variables only belong to a finite set, we have characterized the main theoretical properties of the solution and proposed a solution algorithm. The methodology was employed to investigate properties of quantized optimal control laws and to obtain their explicit



(a) Partition associated with the solution $U^*(\theta)$ of the multiparametric integer program

(b) Partition associated with the solution $U^*_{\mathrm{QP}}(\theta)$ of the continuous multiparametric quadratic program

Figure 7: Comparison between the solutions of the continuous and of the integer multiparametric quadratic program

representation as a function of the state vector. A potential benefit to the presented methodology may Techniques based on the integration of multiparametric and dynamic programming are currently under investigation for solving quantized optimal control problems

An interesting topic for further research is the problem of obtaining minimal representations of the multiparametric solution, and the application of the multiparametric nonlinear integer programming algorithm to other classes of quantized optimal control problems. .

7 Acknowledgements

This work was partially supported by the European Project "Computation and Control". We thank Eric Triki for pointing out a few typos in the original manuscript.

References

[1] W. Yuan, K. Nahrstedt, and X. Gu. Coordinating energy-aware adaptation of multimedia applications and hardware resource. In *Proc. 9th ACM Multimedia Conference*, Ottawa, Canada, October 2001.

[2] D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Trans. Automatic Control*, 35(8):916–924, 1990.

[3] N. Elia and S. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. Automatic Control*, 46(9):1384–1400, 2001.

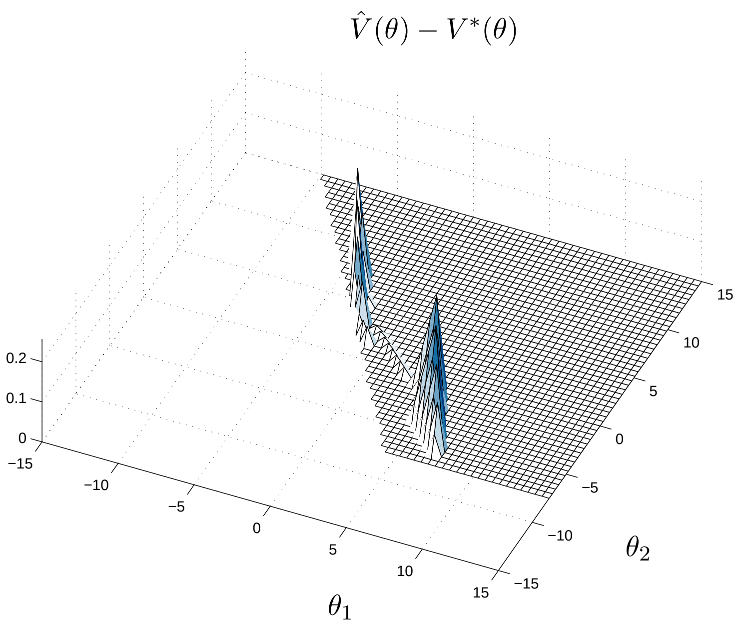


Figure 8: Difference $\hat{V}(\theta) - V^*(\theta)$, where $\hat{V}(\theta) = \frac{1}{2}\hat{U}'(\theta)H\hat{U}(\theta) + \theta' F' \hat{U}(\theta)$, and $\hat{U}(\theta)$ is obtained by quantizing the solution of the continuous quadratic program to the nearest feasible point in $\mathcal{Q}$

[4] F. Fagnani and S. Zampieri. Stability analysis and synthesis for scalar linear systems with a quantized feedback. Technical Report 20, Dipartimento di Matematica, Politecnico di Torino, Italy, May 2001.

[5] A. Bicchi, A. Marigo, and B. Piccoli. On the reachability of quantized control systems. *IEEE Trans. Automatic Control*, 47(4):546–563, 2002.

[6] B. Picasso, F. Gouaisbaut, and A. Bicchi. Construction of invariant and attractive sets for quantizedinput linear systems. In *Proc. 41th IEEE Conf. on Decision and Control*, pages 824–829. 2002.

[7] S. Tatikonda, A. Sahai, and S. Mitter. Control of LQG systems under communication constraints. In *Proc. 37th IEEE Conf. on Decision and Control*, 1998.

[8] W. Wong and R. Brockett. Systems with finite communication bandwidth constraints - Part I: State estimation problems. *IEEE Trans. Automatic Control*, 42:1294–1299, 1997.

[9] W. Wong and R. Brockett. Systems with finite communication bandwidth constraints - Part II: Stabilization with limited information feedback. *IEEE Trans. Automatic Control*, 44:1049–1053, 1999.

[10] B. Picasso, S. Pancanti, A. Bemporad, and A. Bicchi. Receding–horizon control of LTI systems with quantized inputs. In *IFAC Conf. on Analysis and Design of Hybrid Systems*, Saint Malo, France, June 2002.

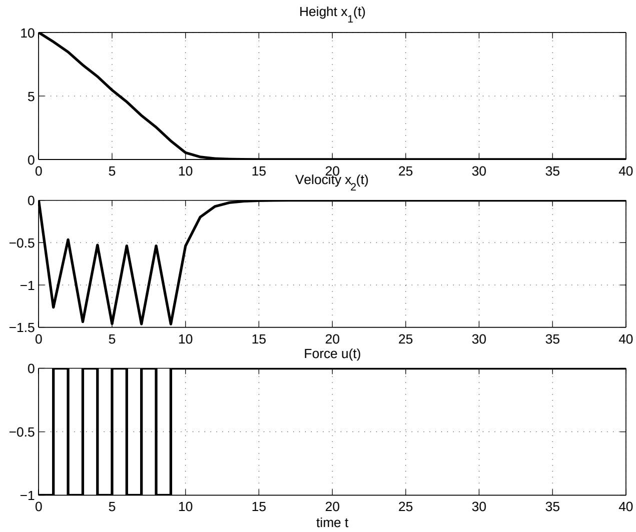


Figure 9: Receding horizon optimal control with quantized inputs

[11] D.E. Quevedo, J.A. De Doná, and G.C. Goodwin. Receding horizon linear quadratic control with finite input constraint sets. In *Proc. 15th IFAC Triennal World Congress*, Barcelona, Spain, July 2002.

[12] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, 2002.

[13] A. Bemporad, F. Borrelli, and M. Morari. Model predictive control based on linear programming — The explicit solution. *IEEE Trans. Automatic Control*, 47(12):1974–1985, December 2002.

[14] M.M. Seron, J.A. DeDoná, and G.C. Goodwin. Global analytical model predictive control with input constraints. In *Proc. 39th IEEE Conf. on Decision and Control*, pages 154–159, 2000.

[15] P. Tøndel, T.A. Johansen, and A. Bemporad. An algorithm for multi-parametric quadratic programming and explicit MPC solutions. *Automatica*, 39(3):489–497, March 2003.

[16] T.A. Johansen. On multi-parametric nonlinear programming and explicit non-linear model predictive control. In *Proc. 41th IEEE Conf. on Decision and Control*, pages 2768–2773, Las Vegas, Nevada, USA, December 2002.

[17] A. Bemporad, F. Borrelli, and M. Morari. Piecewise linear optimal controllers for hybrid systems. In *American Control Conference*, pages 1190–1194, Chicago, IL, June 2000.

[18] E.C. Kerrigan and D.Q. Mayne. Optimal control of constrained, piecewise affine systems with bounded disturbances. In *Proc. 41th IEEE Conf. on Decision and Control*, pages 1552–1557, Las Vegas, Nevada, USA, December 2002.

[19] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Trans. Automatic Control*, 2003. In press.

[20] T. Gal and J. Nedoma. Multiparametric linear programming. *Management Science*, 18:406–442, 1972.

[21] A.V. Fiacco. *Introduction to sensitivity and stability analysis in nonlinear programming.* Academic Press, London, U.K., 1983.

[22] R.A. Marsten and T. Morin. Parametric integer programming: the right hand side case. Technical Report Working Papers n. 0106, National Bureau of Economic Research, Inc., Cambridge, MA, October 1975.

[23] H.J. Greenberg. An annotated bibliography for post-solution analysis in mixed integer programming and combinatorial optimization. In D.L. Woodruff, editor, *Advances in Computational and Stochastic Optimization, Logic Programming, and Heuristic Search*, pages 97–148. Kluwer Academic Publishers, Boston, MA, 1998. (Note: BibTex bibliography maintained online at `http://www.cudenver.edu/ hgreenbe/aboutme/pubrec.html`).

[24] P. Feautrier. Parametric integer programming. *RAIRO Recherche Opérationnelle*, 22:243–268, September 1988.

[25] P. Feautier, J.-F. Collard, and C. Bastoul. Solving systems of affine (in)equalities: PIP's user's guide. `http://www.prism.uvsq.fr/~cedb/bastools/piplib.html`, January 2003.

[26] A. Crema. A contraction algorithm for the multiparametric integer linear programming problem. *European Journal of Operational Research*, 101(1):130–139, 1997.

[27] M.S. Chern, R.H. Jan, and R.J. Chern. Parametric nonlinear integer programming: The right-hand side case. *European Journal of Operational Research*, 54(2):237–255, 1991.

[28] M. Schechter. Polyhedral functions and multiparametric linear programming. *Journal of Optimization Theory and Applications*, 53(2):269–280, May 1987.

[29] T. Gal. *Postoptimal Analyses, Parametric Programming, and Related Topics.* de Gruyter, Berlin, 2nd edition, 1995.

[30] F. Borrelli, A. Bemporad, and M. Morari. A geometric algorithm for multiparametric linear programming. *Journal of Optimization Theory and Applications*, 118(3):515–540, September 2003.

[31] P. Tøndel, T.A. Johansen, and A. Bemporad. Evaluation of piecewise affine control via binary search tree. *Automatica*, 39(5):945–950, May 2003.

[32] R. Raman and I.E. Grossmann. Relation between MILP modeling and logical inference for chemical process synthesis. *Computers Chem. Engng.*, 15(2):73–84, 1991.

[33] H.P. Williams. *Model Building in Mathematical Programming.* John Wiley & Sons, Third Edition, 1993.

[34] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, March 1999.

[35] F.D. Torrisi and A. Bemporad. HYSDEL — A tool for generating computational hybrid models. *IEEE Transactions on Control Systems Technology*, 2002. Accepted for publication. `http://control.ethz.ch/~hybrid/hysdel`.