JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 117, No. 1, pp. 9–38, April 2003 (© 2003)

Suboptimal Explicit Receding Horizon Control via Approximate Multiparametric Quadratic Programming[1]

A. Bemporad[2] and C. Filippi[3]

Communicated by D. Q. Mayne

Abstract. Algorithms for solving multiparametric quadratic programming (MPQP) were recently proposed in Refs. 1–2 for computing explicit receding horizon control (RHC) laws for linear systems subject to linear constraints on input and state variables. The reason for this interest is that the solution to MPQP is a piecewise affine function of the state vector and thus it is easily implementable online. The main drawback of solving MPQP exactly is that, whenever the number of linear constraints involved in the optimization problem increases, the number of polyhedral cells in the piecewise affine partition of the parameter space may increase exponentially. In this paper, we address the problem of finding approximate solutions to MPQP, where the degree of approximation is arbitrary and allows to tradeoff between optimality and a smaller number of cells in the piecewise affine solution. We provide analytic formulas for bounding the errors on the optimal value and the optimizer, and for guaranteeing that the resulting suboptimal RHC law provides closed-loop stability and constraint fulfillment.

Key Words. Receding horizon control, model predictive control, multiparametric programming, convex quadratic programming, error bounds, piecewise linear control.

[1]We thank Domenico Mignone for providing the LMI-based routine for computing quadratic and piecewise-quadratic Lyapunov functions. We also thank the anonymous referees and Professor David Q. Mayne, whose comments greatly helped to improve the quality of the paper.
[2]Assistant Professor, Dipartimento di Ingegneria dell'Informazione, Università di Siena, Siena, Italy.
[3]Assistant Professor, Dipartimento di Matematica Pura e Applicata, Università di Padova, Padova, Italy.

1. Introduction

In industrial applications, the design of feedback controllers must often cope with the presence of constraints over actuators and other process components. Such constraints must be adequately handled by the control design. Receding horizon control (RHC), also referred to as model predictive control (MPC) especially in industry, has become the accepted standard for complex constrained multivariable control problems in the process industries (Ref. 3). Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. At the next time step, the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy. The solution relies on a linear dynamic model, satisfies all the input and output constraints, and optimizes a quadratic performance index.

Although RHC has long been recognized as the winning alternative for constrained systems, due to the considerable online computation effort, its applicability has been limited to relatively slow systems.

For RHC based on linear prediction models and a quadratic performance index, in Ref. 1 the authors proposed a new approach to move offline all the computations necessary for the implementation of RHC while preserving all its other characteristics. The approach consists of solving offline the optimization problem associated with RHC for all the expected measurement values by using multiparametric quadratic programming (MPQP) solvers. The resulting feedback controller inherits all the stability and performance properties of the linear RHC and is piecewise affine. For this reason, the online computation associated with explicit RHC controllers reduces to the function evaluation of a piecewise affine mapping. Therefore, the approach is extremely promising, as it broadens the scope of applicability of linear RHC to small-size, fast-sampling applications. Alternative approaches for obtaining explicit RHC solutions were investigated in Refs. 4–5.

The problem of reducing online computation, although addressed by explicit RHC techniques, is not yet solved. In fact, whenever the number of constraints involved in the optimization problem increases, the number of linear gains associated with the piecewise affine control algorithm may increase exponentially, which still makes the online implementation of the piecewise affine controller prohibitive on low-cost control hardware.

The technique proposed in Ref. 4 attempts to reduce complexity by reducing a priori the allowed combinations of active constraints, based on engineering insight on the control problem.

In this paper, we propose a new algorithm for reducing the complexity of the explicit RHC controller, by computing suboptimal solutions to the multiparametric quadratic problem. The idea is based on relaxing the first-order Karush-Kuhn-Tucker (KKT) optimality conditions (except primal feasibility, so that the computed move is feasible) by some arbitrary degree ϵ, which serves as a design knob for tuning the complexity of the controller. We show that, for $\epsilon \to \infty$, the complexity of the controller is reduced to an affine control law, highly suboptimal, while for $\epsilon \to 0$ the controller converges to the explicit RHC controller (Ref. 1), fully optimal with respect to the chosen performance index. We analyze a general relaxation scheme, where all the KKT conditions (except primal feasibility) may be relaxed; also, we analyze a particular relaxation scheme, where only dual feasibility is relaxed. For the general perturbation scheme, we show how to compute a posteriori the maximum error between the optimizer and the suboptimizer. For the particular perturbation scheme, we provide also a criterion for choosing ϵ so that the distances between the optimal value and the suboptimal value and between the exact solution and the approximate solution are bounded a priori, and so that the resulting suboptimal RHC law provides closed-loop stability and constraint fulfillment.

2. Receding Horizon Control

We start by reviewing briefly basic facts on RHC and MPQP; see Ref. 1 for details. Consider the discrete-time linear time-invariant system

$$x(t+1) = \mathscr{A}x(t) + \mathscr{B}u(t), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^{n_u}$ are the state and input vectors, respectively, and where the pair $(\mathscr{A}, \mathscr{B})$ is stabilizable. Consider the problem of regulating the state $x(t)$ to the origin while fulfilling the constraints

$$D_1 x(t) + D_2 u(t) \leq d, \tag{2}$$

at all time instants $t \geq 0$, where d has strictly positive components. Assume that a full measurement of the state $x(t)$ is available at the current time t. Then, the following optimization problem:

$$\min_U \quad x_T' P x_T + \sum_{k=0}^{T-1} [x_k' Q x_k + u_k' R u_k], \tag{3a}$$

$$\text{s.t.} \quad D_1 x_k + D_2 u_k \leq d, \quad k = 0, \ldots, T-1, \tag{3b}$$

$$x_T \in \Omega, \tag{3c}$$

$$x_{k+1} = \mathscr{A} x_k + \mathscr{B} u_k, \qquad k = 0, \ldots, T-1, \tag{3d}$$

$$x_0 = x(t), \tag{3e}$$

is solved with respect to the column vector

$$U \triangleq [u'_0 \ldots u'_{T-1}]' \in \mathbb{R}^r, \qquad r \triangleq n_u T,$$

at each time t, where x_k denotes the predicted state vector at time $t+k$, obtained by applying the input sequence $u_0, \ldots, u_{k-1}$ to the model (1) starting from the state $x(t)$. In (3), we assume that Q and R are symmetric and positive-definite matrices, that P is symmetric and nonnegative definite, and that the set of terminal states Ω is polyhedral and contains the origin.

The MPC control law is based on the following idea. At time t, compute the optimizer

$$U^*(x(t)) = [(u_0^*)', \ldots, (u_{T-1}^*)']'$$

to problem (3), apply

$$u(t) = u_0^* = I^1 U^*(x(t)), \qquad I^1 \triangleq [I_{n_u}, 0, \ldots, 0], \tag{4}$$

as input to the system (1), and repeat the optimization (3) at the next time step $t+1$, based on the new measured or estimated state $x(t+1)$. By substituting

$$x_k = \mathscr{A}^k x(t) + \sum_{j=0}^{k-1} \mathscr{A}^j \mathscr{B} u_{k-1-j}$$

in (3), this can be written as

$$\min_U \quad (1/2)U'HU + x'(t)\, C'U + (1/2)x'(t)\, Yx(t), \tag{5a}$$

$$\text{s.t.} \quad AU \leq b + Fx(t), \tag{5b}$$

where $H = H'$ is positive definite and H, C, Y, A, b, F are easily obtained from (3), with

$$H \in \mathbb{R}^{r \times r}, \quad C \in \mathbb{R}^{r \times n}, \quad Y \in \mathbb{R}^{n \times n}, \quad A \in \mathbb{R}^{q \times r}, \quad b \in \mathbb{R}^q, \quad F \in \mathbb{R}^{q \times n}.$$

The optimization problem (5) is a quadratic program (QP), which depends on the current state $x(t)$, and therefore is a multiparametric quadratic program (MPQP).

3. Multiparametric Quadratic Programming

Consider the optimization problem

$$(\text{QP}_x) \quad \min_U \quad (1/2)U'HU + x'C'U + (1/2)x'Yx, \tag{6a}$$

$$\text{s.t.} \quad AU \leq b + Fx, \tag{6b}$$

where $U \in \mathbb{R}^r$ is the vector of decision variables, $x \in \mathbb{R}^n$ is the parameter vector (for ease of notation, we have removed the explicit reference to time), and all data are defined as in Section 2. As only the optimizer U^* is needed, the term involving Y is usually removed from (6). Here, we retain such term since it is essential to the arguments of Section 5.2. We say that a parameter vector x is feasible if the corresponding problem (QP_x) admits a solution, i.e., there exists a vector U satisfying the constraints of (QP_x); we denote by $X_f \subseteq \mathbb{R}^n$ the set of feasible parameter vectors. As a consequence of the definition, X_f is the orthogonal projection onto the x-space of the polyhedron $\{(U, x): AU - Fx \leq b\}$, i.e.,

$$X_f = \{x: \exists U \text{ s.t. } AU - Fx \leq b\}.$$

Thus, X_f is a convex polyhedron. Let $\phi^*: X_f \mapsto \mathbb{R}$ denote the value function, which associates with every $x \in X_f$ the optimal value of (QP_x). As H is positive definite, for every $x \in X_f$ the corresponding quadratic program has a unique optimal solution.

Multiparametric quadratic programming (MPQP) amounts to determining the optimal solution U^* and the value function ϕ^* as explicit functions of x, for all $x \in X_f$.

Let the rows of A be indexed by $M \triangleq \{1, 2, \ldots, q\}$. For any $N \subseteq M$, we denote by A_N the submatrix of A consisting of the rows indexed by N. Analogously, if $s \in \mathbb{R}^q$, we denote by s_N the subvector of s consisting of the entries indexed by N. Finally, we recall that a constraint of (QP_x) is active at a vector U if it is satisfied as an equality by U.

Definition 3.1. Let $U^*(x)$ be the optimal solution of (QP_x). The optimal partition associated with x is the partition $(B(x), N(x))$ of M, where $N(x)$ is the index set of the active constraints at $U^*(x)$ and $B(x) = M \setminus N(x)$.

Definition 3.2. Let $(B, N) = (B(x), N(x))$ for some $x \in X_f$. We call critical region associated with (B, N) the set of parameters

$$CR^* \triangleq \{x \in X_f: N(x) = N\}.$$

The following result can be found in Ref. 1, Theorem 2. In view of the following developments, we restate its proof here.

Theorem 3.1. Let H be positive definite. Let (B, N) be an optimal partition, and let CR^* be the associated critical region. Assume that the rows of A_N are linearly independent. Then, the optimizer U^* and the associated vector of Lagrange multipliers λ^* are the following uniquely defined, affine functions of x over CR^*:

$$U^*(x) = Z_U x + \zeta_U,$$

$$\lambda_N^*(x) = Z_\lambda x + \zeta_\lambda,$$

$$\lambda_B^*(x) = 0,$$

where

$$Z_\lambda = -(A_N H^{-1} A_N')^{-1}(F_N + A_N H^{-1} C),$$

$$\zeta_\lambda = -(A_N H^{-1} A_N')^{-1} b_N,$$

$$Z_U = -H^{-1} A_N' Z_\lambda - H^{-1} C,$$

$$\zeta_U = -H^{-1} A_N' \zeta_\lambda.$$

Proof. Once an optimal partition (B, N) is fixed, the first-order KKT conditions for problem (QP_x) may be written as follows (Ref. 6, p. 504):

$$HU + Cx + A'\lambda = 0, \tag{7a}$$

$$A_B U + s_B = b_B + F_B x, \qquad s_B \geq 0, \tag{7b}$$

$$A_N U + s_N = b_N + F_N x, \qquad s_N = 0, \tag{7c}$$

$$\lambda_B = 0, \tag{7d}$$

$$\lambda_N \geq 0, \tag{7e}$$

where $\lambda \in \mathbb{R}^q$ is the vector of Lagrange multipliers and s_B, s_N are a partition of the vector of primal slack variables $s \in \mathbb{R}^q$. We solve (7a) for U,

$$U = -H^{-1}(A_N' \lambda_N + Cx), \tag{8}$$

and substitute the result into (7c), getting

$$-A_N H^{-1}(A_N' \lambda_N + Cx) - b_N - F_N x = 0.$$

Assuming that A_N is of full row rank, $(A_N H^{-1} A_N')^{-1}$ exists and therefore we obtain

$$\lambda_N = -(A_N H^{-1} A_N')^{-1}[b_N + (F_N + A_N H^{-1} C)x]. \tag{9}$$

Thus, λ is an affine function of x. We can substitute λ_N from (9) into (8) to obtain

$$U = H^{-1} A_N' (A_N H^{-1} A_N')^{-1}[b_N + (F_N + A_N H^{-1} C)x] - H^{-1} Cx, \tag{10}$$

and note that U is also an affine function of x. Relations (9) and (10) lead to the assertion. □

Theorem 3.1 characterizes the solution only locally in the neighborhood of a specific x_0, as it does not provide the construction of the set CR* where this characterization remains valid. On the other hand, this region can be characterized immediately. By construction, conditions (7a), (7c), (7d) are satisfied as identities by $U^*(x)$ and $\lambda^*(x)$. By substituting the expressions of $U^*(x)$ and $\lambda^*(x)$ in (7b) and (7e), we get

$$(A_B Z_U - F_B)x \leq b_B - A_B \zeta_U, \tag{11a}$$

$$-Z_\lambda x \leq \zeta_\lambda. \tag{11b}$$

The representation (11) may be improved by removing possible redundant inequalities. Obviously, CR* is a polyhedron in the x-space and represents the largest set of $x \in X_f$ such that the combination of the active constraints at the minimizer corresponds to the chosen index set N.

3.1. Degeneracy. So far, we have assumed that the rows of A_N are linearly independent. It can happen that, by solving (QP_x), one determines a set of active constraints for which this assumption is violated. For instance, this happens when more than r constraints are active at the optimizer $U^*(x) \in \mathbb{R}^r$, i.e., in the case of primal degeneracy. In this case, the vector of Lagrange multipliers λ^* might not be uniquely defined, as the dual problem of (QP_x) is not strictly convex (instead, dual degeneracy cannot occur, because we assumed H positive definite, which implies that the minimizer is always unique). In Ref. 1, the authors suggest a simple way to handle degeneracy by extracting from A_N an arbitrary maximal subset of linearly independent rows and then proceed with the corresponding reduced set of active constraints.

3.2. Continuity and Convexity Properties. The result stated below makes use of the following definition.

Definition 3.3. A function $z: X \mapsto \mathbb{R}^m$, where $X \subseteq \mathbb{R}^n$ is a polyhedral set, is piecewise affine [resp. piecewise quadratic] if the following conditions hold:

(a) it is possible to partition X into finitely many convex polyhedral regions CR_i, $i = 1, \ldots, p$;
(b) inside CR_i, z is an affine [resp. quadratic] function, for all $i = 1, \ldots, p$.

Continuity of the value function ϕ^* and the solution U^* can be shown as simple corollaries of the linearity result of Theorem 3.1. Together with the convexity of the set of feasible parameters X_f and of the value function ϕ^*, this fact is proved in the next theorem (Ref. 1, Theorem 4).

Theorem 3.2. Consider the multiparametric quadratic program (QP_x), and let H be positive definite. Then, the optimizer $U^*: X_f \mapsto \mathbb{R}^r$ is continuous and piecewise affine, and the value function $\phi^*: X_f \mapsto \mathbb{R}$ is continuous, convex, and piecewise quadratic.

We note that the same continuity and convexity results can be obtained as special cases of the general nonlinear results in Ref. 7, Chapter 2.

4. Approximate MPQP

Let the parameter vector $x \in X_f$ be arbitrarily chosen[4], and let (B, N) be the corresponding optimal partition. In order to obtain a suboptimal solution to (QP_x), we relax the KKT conditions (7) as

$$-\epsilon_1 \leq HU + Cx + A'\lambda \leq \epsilon_1, \tag{12a}$$

$$A_B U + s_B = b_B + F_B x, \qquad s_B \geq 0, \tag{12b}$$

$$A_N U + s_N = b_N + F_N x, \qquad 0 \leq s_N \leq \epsilon_2, \tag{12c}$$

$$-\epsilon_4 \leq \lambda_B \leq \epsilon_4, \qquad \lambda_N \geq -\epsilon_3, \tag{12d}$$

where

$$\epsilon_1 \in \mathbb{R}^r, \qquad \epsilon_2, \epsilon_3 \in \mathbb{R}^{|N|}, \qquad \epsilon_4 \in \mathbb{R}^{|B|}$$

are the relaxation vectors that determine the degree of approximation, with $\epsilon_k \geq 0$ (componentwise) for $k = 1, \ldots, 4$. The relaxed KKT conditions (12) define a polyhedron in the (U, x, λ, s)-space. The approximate critical region is defined as the projection onto the x-space of such a polyhedron, and it is denoted by $\mathrm{CR}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, or in short, by CR_ϵ.

Assume for the time being that CR_ϵ has been computed (this issue will be discussed in Section 4.1). Then, the rest of the space $X_f \backslash \mathrm{CR}_\epsilon$ has to be explored and new critical regions have to be generated. An effective approach for partitioning $X_f \backslash \mathrm{CR}_\epsilon$ by polyhedral sets is based on the following theorem; see Ref. 1, Theorem 3.

[4]A vector $x \in X_f$ can be computed by finding a pair (U, x) satisfying $AU - Fx \leq b$, e.g., via linear programming.

Theorem 4.1. Let $X \subseteq \mathbb{R}^n$ be a polyhedron, and let $\mathrm{CR}_\epsilon = \{\theta \in X: Gx \leq g\}$ be a nonempty polyhedral subset of X, where $G \in \mathbb{R}^{h \times n}$. Also, let

$$R_i = \{x \in X: G_{\{i\}}x > g_{\{i\}}, G_{\{j\}}x \leq g_{\{j\}}, \forall j < i\}, \qquad i = 1, \ldots, h,$$

where $G_{\{i\}}$ denotes the ith row of G and $g_{\{i\}}$ denotes the ith entry of g. Then:

(i) $X = \left(\bigcup_{i=1}^{h} R_i \right) \cup \mathrm{CR}_\epsilon$;

(ii) $\mathrm{CR}_\epsilon \cap R_i = \emptyset$, for all i, and $R_i \cap R_j = \emptyset$, for all $i \neq j$; i.e. $\{\mathrm{CR}_\epsilon, R_1, \ldots, R_h\}$ is a partition of X.

After partitioning the rest of the space, we proceed recursively: for each region R_i, we choose a new vector x_0, compute the approximate critical region CR_ϵ, compute the rest of the space $R_i \backslash \mathrm{CR}_\epsilon$, and so on. Clearly, in order to minimize the number of regions R_i generated at each recursion, before applying Theorem 4.1 it is convenient to remove all redundant inequalities from the representation of CR_ϵ.

4.1. Orthogonal Projections. Before proceeding further, it is useful to rewrite the approximate KKT conditions (12a) in the form

$$HU + Cx + A'_N\lambda_N + A'_B\lambda_B + v = 0, \qquad -\epsilon_1 \leq v \leq \epsilon_1, \tag{13}$$

where $v \in \mathbb{R}^r$ represents the violation of the first KKT condition (7a). From (13), we obtain

$$U = -H^{-1}(A'_N\lambda_N + A'_B\lambda_B + Cx + v);$$

thus, by substitution into (12c) and under the assumption that $A_N H^{-1} A'_N$ is invertible,

$$\lambda_N = E_v v + E_s s_N + E_\lambda \lambda_B + Z_\lambda x + \zeta_\lambda, \tag{14}$$

where

$$E_s \triangleq (A_N H^{-1} A'_N)^{-1}, \qquad E_v \triangleq -E_s A_N H^{-1}, \qquad E_\lambda \triangleq E_v A'_B.$$

The approximated critical region CR_ϵ is now the projection onto the x-space of the polyhedron in the (v, s_N, λ_B, x)-space described by the inequalities

$$-\epsilon_1 \leq v \leq \epsilon_1, \tag{15a}$$

$$-A_B H^{-1}[A'_N(E_v v + E_s s_N + E_\lambda \lambda_B + Z_\lambda x + \zeta_\lambda) + A'_B\lambda_B + Cx + v]$$
$$\leq b_B + F_B x, \tag{15b}$$

$$0 \le s_N \le \epsilon_2, \tag{15c}$$

$$-\epsilon_4 \le \lambda_B \le \epsilon_4, \tag{15d}$$

$$E_v v + E_s s_N + E_\lambda \lambda_B + Z_\lambda x + \zeta_\lambda \ge -\epsilon_3. \tag{15e}$$

Rather than projecting with respect to the whole set of variables v, s_N, λ_B, we can restrict the amount of relaxations and accordingly consider the following three cases.

Case A: $\epsilon_2 = 0$. This special case implies $s_N = 0$, and therefore amounts to fixing the index set N of constraints which are active at the optimizer of (QP_x). The projection is performed only with respect to v, λ_B.

Case B: $\epsilon_2 = 0, \epsilon_4 = 0$. This special case implies $\lambda_B = 0, s_N = 0$, and corresponds to avoiding the relaxation of the second KKT condition (7b). Equivalently, it implies that the given optimal partition (B, N) is maintained. The simplification of the projection procedure is obvious: we need only to project with respect to v.

Case C: $\epsilon_1 = 0, \epsilon_2 = 0, \epsilon_4 = 0$. In this final special case, we relax only the nonnegativity condition on the Lagrange multipliers corresponding to nonactive constraints of (QP_x). Hence, we need no projection, as similarly to (11) for the exact case, the approximated critical region reduces to

$$(A_B Z_U - F_B)x \le b_B - A_B \zeta_U, \tag{16a}$$

$$-Z_\lambda x \le \epsilon_3 + \zeta_\lambda. \tag{16b}$$

4.2. Properties of Approximated Critical Regions. Since the primal feasibility of the optimizer is never relaxed, the approximate critical region is always contained in X_f. It is of interest to study its behavior as a function of the amount of relaxation.

Lemma 4.1. Let $(B, N) = (B(x), N(x))$ for some $x \in X_f$, and let $CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ be the associated approximate critical region. Then, the following statements hold:

(i) if $\epsilon_k \le \epsilon'_k, \forall k = 1, \ldots, 4$, then

$$CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \subseteq CR(\epsilon'_1, \epsilon'_2, \epsilon'_3, \epsilon'_4);$$

(ii) $$\bigcap_{\delta_1, \delta_2, \delta_3, \delta_4 \ge 0} CR(\delta_1, \delta_2, \delta_3, \delta_4) = CR^*,$$

where $\delta_1 \in \mathbb{R}^r$, $\delta_2, \delta_3 \in \mathbb{R}^{|N|}$, $\delta_4 \in \mathbb{R}^{|B|}$, and CR^* is the exact critical region represented by (11);

(iii) $\forall \epsilon_3, \epsilon_4 \geq 0, \bigcup_{\delta_1 \geq 0} \mathrm{CR}(\delta_1, 0, \epsilon_3, \epsilon_4) = X_N,$

where X_N is the projection onto the x-space of

$\{(U, x): A_B U - F_B x \leq b_B, A_N U - F_N x = b_N\};$

(iv) $\forall \epsilon_3, \epsilon_4 \geq 0, \bigcup_{\delta_1, \delta_2 \geq 0} \mathrm{CR}(\delta_1, \delta_2, \epsilon_3, \epsilon_4) = X_f.$

Proof. In order to prove the lemma, we refer to the relaxed KKT conditions (12). Statements (i) and (ii) follow immediately from (12).

To prove statement (iii), we show that

(a) $\bigcup_{\delta_1 \geq 0} \mathrm{CR}(\delta_1, 0, \epsilon_3, \epsilon_4) \subseteq X_N,$

(b) $\bigcup_{\delta_1 \geq 0} \mathrm{CR}(\delta_1, 0, \epsilon_3, \epsilon_4) \supseteq X_N.$

Take a vector

$$\bar{x} \in \bigcup_{\delta_1 \geq 0} \mathrm{CR}(\delta_1, 0, \epsilon_3, \epsilon_4).$$

Then,

$$\bar{x} \in \mathrm{CR}(\bar{\delta}_1, 0, \epsilon_3, \epsilon_4), \qquad \text{for some } \bar{\delta}_1 \geq 0,$$

and hence there exists a vector $\bar{U}$ such that $(\bar{U}, \bar{x})$ satisfies (12b), (12c), implying that $\bar{x} \in X_N$ and proving (a).

Viceversa, take $\bar{x} \in X_N$. Then, there exists a vector $\bar{U}$ such that $(\bar{U}, \bar{x})$ satisfies (12b), (12c). (12d) is satisfied for instance by taking $\bar{\lambda} = 0$, while (12a) is satisfied, e.g., for $\epsilon_1 = \bar{\delta}_1 = |H\bar{U} + C\bar{x}|$, where $|\cdot|$ is intended componentwise, implying $\bar{x} \in \mathrm{CR}(\bar{\delta}_1, 0, \epsilon_3, \epsilon_4)$ and therefore proving (b).

The proof of statement (iv) is similar to the proof of (iii) and thus is omitted. □

4.3. Approximate Optimizer. So far, we have described a suboptimal method for partitioning the parameter set X_f, but contrarily to the exact case described in Theorem 3.1, we have not specified yet an approximate optimizer, which will be denoted by $\hat{U}(x)$. Similarly to the exact case, we wish to have $\hat{U}(x)$ to be a piecewise affine function of x, defined over the partition into approximate critical regions given by the recursive method mentioned above, such that $\hat{U}(x)$ is primal feasible for all $x \in \mathrm{CR}_\epsilon$, for each approximate critical region CR_ϵ. Moreover, we wish $\hat{U}(x)$ to be as close as possible to the exact solution $U^*(x)$.

For Case C, it turns out that, for any given index set N, a good choice is to set $\hat{U}(x)$ as in (10), because it provides primal feasibility for all $x \in \mathrm{CR}_\epsilon$ and optimality for $x \in \mathrm{CR}^* \subseteq \mathrm{CR}_\epsilon$, i.e.,

$$\hat{U}(x) = U^*(x), \qquad \forall x \in \mathrm{CR}^*.$$

For Cases A and B, primal feasibility should be enforced explicitly. To this end, the following lemma may be useful.

Lemma 4.2. Let $V = \{V_1, \ldots, V_h\}$ be a set of vectors of $\mathbb{R}^n$ such that $\mathrm{CR}_\epsilon \subseteq \mathrm{conv}(V)$. Let $\hat{U}(x)$ be an affine function of x. Then,

$$A\hat{U}(V_i) \leq b + FV_i, \qquad \text{for all } V_i \in V,$$

implies

$$A\hat{U}(x) \leq b + Fx, \qquad \text{for all } x \in CR_\epsilon.$$

Proof. It is straightforward, by convexity. □

A natural choice for V is the set of vertices of CR_ϵ. Although good packages exist for determining the set of vertices of CR_ϵ (see Ref. 8), for high-dimensional x-spaces this might be computationally too expensive. Alternatively, the set V can be obtained by determining a union of hyper-rectangles which outer approximates CR_ϵ (Ref. 9). After a set V fulfilling Lemma 4.2 is chosen, we compute the affine suboptimal solution

$$\hat{U}(x) = \hat{Z}x + \hat{\zeta},$$

where $\hat{Z}$ and $\hat{\zeta}$ are obtained by solving the following constrained quadratic least squares problem

$$\min_{Z \in \mathbb{R}^{r \times n}, \zeta \in \mathbb{R}^r} \quad \sum_{i=1}^{h} \|\mathscr{W}[U^*(V_i) - (ZV_i + \zeta)]\|^2, \tag{17a}$$

$$\text{s.t.} \quad A(ZV_i + \zeta) \leq b + FV_i, \quad i = 1, 2, \ldots, h, \tag{17b}$$

which provides the best fit to the optimal solutions $U^*(V_i)$ under the constraint of primal feasibility over $\mathrm{conv}(V) \supseteq CR_\epsilon$, where $\mathscr{W}$ is a weighting matrix. When the approximate MPQP is used to solve an RHC problem, a sensible choice for $\mathscr{W}$ is

$$\mathscr{W} = \begin{bmatrix} I_{n_u} & 0 \\ 0 & 0 \end{bmatrix},$$

as only the first n_u components of the solution are used to build the suboptimal explicit RHC law. Moreover, for the approximate region which contains the origin (i.e., the region corresponding to the empty combination of the active constraints), in (17) we impose $\zeta = 0$, so that it is possible to achieve asymptotic convergence to the origin.

Remark 4.1. Consider the original RHC problem of Section 2. If the constraints (2) have the popular form

$$u_{\min} \leq u(t) \leq u_{\max} \qquad \text{(hard constraints)},$$

$$y_{\min} - \sigma \leq y(t) \leq y_{\max} + \sigma \qquad \text{(soft constraints)},$$

with

$$u_{\min} \leq 0 \leq u_{\max}, \qquad y_{\min} \leq 0 \leq y_{\max},$$

and if $\sigma \geq 0$ is an additional slack optimization variable, then problem (17) is always solvable. Indeed, at least the input sequence $U = 0$ is feasible for all $x \in \operatorname{conv}(V)$ for some sufficiently large σ. In general, unless some other particular hypotheses on A, b, F are assumed, problem (17) may be infeasible, especially for large ϵ. In this case, a possibility is to reduce iteratively (e.g., halve) the entries of ϵ until feasibility of (17) is reached.

Remark 4.2. Contrarily to the exact case, the overall piecewise affine function may not be continuous. Note that the approximate critical regions that we obtain are always closed sets, whereas by Theorem 4.1 we should apply our method to a partition of X_f formed by sets R_i which are defined by both strict and nonstrict inequalities. Actually, we propose to apply the search for an approximate critical region in the closure of such sets R_i. The resulting approximate descriptions of the optimizer is then redundant, in the sense that it may be defined more than once for some $x \in X_f$. For such vectors x, we define arbitrarily $\hat{U}(x)$ as one of the possible values.

4.4. Approximate Value Function. Because of the property of primal feasibility given by (17b) (see Cases A and B) or (16a) (see Case C), the following proposition follows immediately.

Proposition 4.1. Let $\hat{\phi}(x) \triangleq (1/2)\hat{U}(x)'H\hat{U}(x) + x'C'\hat{U}(x) + (1/2)x'Yx$ be the approximate value function, and let ϕ^* be the (exact) value function of problem (QP_x). Then, $\hat{\phi}(x) \geq \phi^*(x)$ for all $x \in X_f$; i.e., $\hat{\phi}(x)$ is an upper bound for $\phi^*(x)$.

In Lemma 4.3, we will give a bound on the gap between $\hat{\phi}(x)$ and $\phi^*(x)$, valid for Case C.

4.5. Suboptimality Figures. Once the suboptimal solution to the MPQP problem has been determined, it is interesting to compute (a posteriori) the degree of suboptimality of the resulting approximate explicit RHC controller with respect to the original RHC problem; in other words, it is of interest to compute the difference between the first n_u components of $\hat{U}(x)$ and $U^*(x)$. To this end, we define the absolute error

$$e_{\text{abs}} \triangleq \max_{x \in X_f \cap \bar{X}} \|I^1[\hat{U}(x) - U^*(x)]\|_\infty$$

and the relative error

$$e_{\text{rel}} \triangleq \max_{x \in X_f \cap \bar{X}} \{\|I^1[\hat{U}(x) - U^*(x)]\|_\infty / \|x\|_1\},$$

where $\bar{X} \subset \mathbb{R}^n$ is a bounded polyhedron, containing the state vector of interest. Typically, $\bar{X}$ is a box,

$$\bar{X} = \{x: \underline{x} \leq x \leq \bar{x}\}.$$

The following proposition shows that such errors can be computed numerically. A constructive proof can be found in Ref. 10.

Proposition 4.2. Let the exact optimizer $U^*(x)$ and the approximate optimizer $\hat{U}(x)$ be given. The absolute error e_{abs} can be computed by solving $2n_u$ mixed integer linear programs (MILPs). Furthermore, the relative error e_{rel} can be computed by solving $2n_u$ monoparametric mixed integer linear programs (MPMILPs) and one maximization of a piecewise hyperbolic scalar function.

4.6. A Priori Error Bounds. Analytic forms for expressing the error between the optimizer and a feasible vector can be found in Refs. 11–12 for linear complementarity problems. Although in principle these results may be applied to our MPQP context, they rely on the existence of constants whose determination is not given constructively. Therefore, in this paper, we follow a different route and develop a direct approach to analyze the error between the optimal solution and the suboptimal solution.

Consider the special case $\epsilon_1 = 0, \epsilon_2 = 0, \epsilon_4 = 0$ (Case C). Inside CR_ϵ, defined by (16), we take as approximate optimizer

$$\hat{U}(x) \triangleq Z_U x + \zeta_U,$$

and we take as approximate vector of Lagrange multipliers

$$\hat{\lambda}_N(x) \triangleq Z_\lambda x + \zeta_\lambda, \quad \hat{\lambda}_B(x) \triangleq 0,$$

where $Z_U, \zeta_U, Z_\lambda, \zeta_\lambda$ are defined as in Theorem 3.1. As mentioned already, this choice provides primal teasibility for all $x \in \mathrm{CR}_\epsilon$ and optimality for all $x \in \mathrm{CR}^* \subseteq \mathrm{CR}_\epsilon$. Accordingly, we take as approximate value function

$$\hat{\phi}(x) \triangleq (1/2)\hat{U}'(x)\, H\hat{U}(x) + x'C'\hat{U}(x) + (1/2)x'Yx.$$

Our goal is to impose an a priori bound ρ on the absolute error,

$$\max_{x \in X_f} \|I^1[\hat{U}(x) - U^*(x)]\|_\infty \leq \rho.$$

Lemma 4.3. Let $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$. Then, for all $x \in \mathrm{CR}_\epsilon$,

$$\hat{\phi}(x) - \phi^*(x) \leq (1/2)\epsilon_3' A_N H^{-1} A_N' \epsilon_3. \tag{18}$$

Proof. Since

$$U = -H^{-1}(Cx + A'\lambda),$$

the Dorn dual (QD_x) of problem (QP_x) may be written as (Ref. 6, pp. 232–233)

$$\max_\lambda \{-(1/2)\lambda' A H^{-1} A'\lambda - [b + (F + AH^{-1}C)x]'\lambda$$
$$- (1/2)x'C'H^{-1}Cx : \lambda \geq 0\} + (1/2)x'Yx.$$

By convexity, (QP_x) and (QD_x) have the same optimum $\phi^*(x)$. Since

$$\hat{\lambda}_N(x) + \epsilon_3 \geq 0, \qquad \hat{\lambda}_B(x) = 0$$

is feasible for (QD_x) for all $x \in \mathrm{CR}_\epsilon$, and by noting that

$$b_N + (F_N + A_N H^{-1} C)x = -A_N H^{-1} A_N' \hat{\lambda}_N(x),$$

we have

$$\begin{aligned}
&\phi^*(x) - (1/2)x'Yx \\
&\geq -(1/2)[\hat{\lambda}_N(x) + \epsilon_3]' A_N H^{-1} A_N' [\hat{\lambda}_N(x) + \epsilon_3] \\
&\quad - [b_N + (F_N + A_N H^{-1} C)x]'[\hat{\lambda}_N(x) + \epsilon_3] - (1/2)x'C'H^{-1}Cx \\
&= (1/2)\hat{\lambda}_N(x)' A_N H^{-1} A_N' \hat{\lambda}_N(x) - (1/2)\epsilon_3' A_N H^{-1} A_N' \epsilon_3 \\
&\quad - (1/2)x'C'H^{-1}Cx.
\end{aligned}$$

Furthermore, by noting that

$$\hat{U}(x) = -H^{-1}(A_N' \hat{\lambda}_N(x) + Cx),$$

we get

$$\hat{\phi}(x) - (1/2)x'Yx$$
$$= (1/2)\hat{U}'(x)H\hat{U}(x) + x'C'\hat{U}(x)$$
$$= (1/2)\hat{\lambda}_N(x)'A_N H^{-1} A'_N \hat{\lambda}_N(x) - (1/2)x'C'H^{-1}Cx.$$

Clearly, inequality (18) follows.

Lemma 4.4. Let $\epsilon_1 = 0, \epsilon_2 = 0, \epsilon_4 = 0$, and let $\Delta U(x) \triangleq \hat{U}(x) - U^*(x)$. Then, for all $x \in \mathrm{CR}_\epsilon$,

$$\Delta U'(x)\, H \Delta U(x) \leq \epsilon'_3 A_N H^{-1} A'_N \epsilon_3. \tag{19}$$

Proof. We have

$$\hat{\phi}(x) - \phi^*(x)$$
$$= (1/2)\hat{U}'(x)H\hat{U}(x) + x'C'\hat{U}(x) - (1/2)U^{*\prime}(x)HU^*(x) + x'C'U^*(x),$$

and so,

$$\hat{\phi}(x) - \phi^*(x)$$
$$= -(1/2)\Delta U'(x)H\Delta U(x) + \Delta U'(x)[H\hat{U}(x) + Cx]. \tag{20}$$

Define the function

$$f(t) \triangleq (1/2)[\hat{U}(x) - t\Delta U(x)]'H[\hat{U}(x) - t\Delta U(x)]$$
$$+ x'C'[\hat{U}(x) - t\Delta U(x)] + (1/2)x'Yx.$$

Note that $f(t)$ is the objective value of (QP_x) associated with $\hat{U}(x) - t\Delta U(x)$, which is feasible for (QP_x) for all $t \in [0, 1]$, as $\hat{U}(x)$ and $U^*(x)$ are both feasible. Since $f(1) = \phi^*(x)$, then $f(t)$ must be decreasing on a left neighbor of $t = 1$. Hence,

$$f'(t) = \Delta U'(x)\, H\Delta U(x)\, t - \Delta U'(x)[H\hat{U}(x) + Cx] \leq 0, \quad \text{if } t = 1,$$

and so,

$$\Delta U'(x)[H\hat{U}(x) + Cx] \geq \Delta U'(x)\, H\Delta U(x). \tag{21}$$

From (20), we then obtain

$$\hat{\phi}(x) - \phi^*(x) \geq (1/2)\, \Delta U'(x)\, H\Delta U(x), \tag{22}$$

which in addition to (18) implies the thesis. □

Lemma 4.5. Let $z \in \mathbb{R}^r$, and consider the following optimization problem:

$$V^* = \max_z \quad \|I^1 z\|_\infty \tag{23a}$$

$$\text{s.t.} \quad z' H z \leq \alpha, \tag{23b}$$

where H is positive definite. Then,

$$V^* = \max_\alpha \{ \sqrt{\alpha [H^{-1}]_{ii}}, \quad i = 1, \dots, n_u \},$$

where $[\cdot]_{ij}$ denotes the (i, j)th entry of $[\cdot]$.

Proof. Consider the optimization problem

$$\max_z \{c' z : z' H z \leq \alpha\}.$$

Since the quadratic constraint must be active at the optimizer z^*, the optimization problem is equivalent to

$$\max_z \{c' z : z' H z = \alpha\}.$$

For the latter problem, denoting by β the Lagrange multiplier associated with the quadratic constraint, the necessary optimality conditions are

$$c + 2H z \beta = 0, \qquad z' H z - \alpha = 0,$$

from which we obtain

$$z = -(H^{-1} c)/2\beta, \qquad \beta = \pm (\sqrt{c' H^{-1} c})/2\sqrt{\alpha},$$

and finally the KKT points

$$z = \mp (H^{-1} c \sqrt{\alpha}) / \sqrt{c' H^{-1} c}.$$

The maximum is therefore $\sqrt{\alpha c' H^{-1} c}$. By letting

$$c = \pm I^1_{\{i\}}, \qquad i = 1, \dots, n_u,$$

where $I^1_{\{i\}}$ denotes the ith row of I^1, we prove the lemma. □

Theorem 4.2. Let $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$. Assume that, for each optimal partition (B, N), the corresponding approximated critical region CR_ϵ is generated by setting $\epsilon_3 = \epsilon(N) 1$, where $1 \triangleq [1, 1, \dots, 1]'$ and

$$\epsilon(N) \leq [\rho / \sqrt{1' A_N H^{-1} A_N' 1}]. \min_{i=1,\dots,n_u} [1/\sqrt{[H^{-1}]_{ii}}]. \tag{24}$$

Then,

$$\max_{x \in X_f} \|I^1[\hat{U}(x) - U^*(x)]\|_\infty \leq \rho.$$

Proof. As a consequence of Lemma 4.4, $z = \hat{U}(x) - U^*(x)$ satisfies the ellipsoidal constraint (19) for all $x \in CR_\epsilon$. By setting

$$\alpha = \epsilon^2(N)\underline{1}' A_N H^{-1} A_N' \underline{1},$$

Lemma 4.5 guarantees that

$$\|I^1[\hat{U}(x) - U^*(x)]\|_\infty \leq \max_{i=1,\ldots,n_u} \{\sqrt{\epsilon^2(N)\underline{1}' A_N H^{-1} A_N' \underline{1}\,[H^{-1}]_{ii}}\} \leq \rho,$$

for all $x \in \mathrm{CR}_\epsilon$, and for all approximated critical regions CR_ϵ. □

5. Suboptimal Receding Horizon Control

This section discusses the two main issues regarding RHC policies, namely, the feasibility of the optimization problem (3) at each time step $t \geq 0$ and the stability of the resulting closed-loop system.

5.1. Feasibility. As stressed in the previous section, primal feasibility is maintained in the approximate MPQP solution. Note that, when the RHC setup of Section 2 is augmented by additional constraints aimed at guaranteeing feasibility at each time step t (Ref. 13), such constraints will be fulfilled also by the suboptimal RHC solution. For instance, if constraints which enforce the predicted terminal state x_T to lie in a polyhedral invariant set are included (Ref. 14), feasibility at each time step t is guaranteed. This point is clarified below in the proof of Theorem 5.2, where we indeed show that X_f is an invariant set. In conclusion, the feasibility of the RHC problem at each time t does not depend on optimality.

5.2. Stability. The suboptimal controller proposed in this paper does not inherit directly intrinsic nominal stability properties of its optimal RHC counterpart based on the exact minimization of (3).

As the closed-loop suboptimal RHC system, composed by a linear plant in feedback with the suboptimal explicit RHC controller, is a piecewise affine system, *a posteriori* stability criteria based on piecewise or common quadratic Lyapunov functions (Ref. 15–17) or on reachability analysis (Ref. 16) can be applied to analyze if a certain suboptimal RHC controller is stabilizing; this will be exemplified in Section 6.

On the other hand, we are interested in synthesizing suboptimal RHC controllers that, by construction and independently on the particular value of the tuning parameters, are stabilizing; or in other words, we are interested in providing *a priori* stability guarantees.

Lemma 5.1. Let $d > 0$ in (2); i.e., the interior of the polyhedron given by (2) contains the origin. Then, the critical region $\mathrm{CR}_{\varnothing}$ corresponding to the empty combination of active constraints is a full-dimensional subset of $\mathbb{R}^n$.

Proof. We prove that there exists a scalar $\alpha > 0$ such that

$$\Phi_\alpha \triangleq \{x \in \mathbb{R}^n: \|x\| \leq \alpha\} \subset \mathrm{CR}_{\varnothing}.$$

This is equivalent to showing that, for each state $x_0 \in \Phi_\alpha$, the unconstrained optimal control sequence $U = -H^{-1}Cx_0$ satisfies the constraints in (3) and therefore is optimal also for the constrained problem (3). To this end, it is enough to find a small enough positive scalar α such that x_0 and $U = [u_0', \ldots, u_{T-1}']'$ satisfy the constraints in (3). Let

$$\Theta \triangleq -H^{-1}C,$$

and let

$$u_k = \Theta_{\{k\}} x_0$$

denote the kth control move, $k = 0, \ldots, T-1$. Since

$$x_k = \mathscr{A}^k x_0 + \sum_{j=0}^{k-1} \mathscr{A}^j \mathscr{B}\, u_{k-1-j},$$

if $\alpha > 0$ is sufficiently small, we obtain

$$D_1 x_k + D_2 u_k = \left[D_1 \mathscr{A}^k + D_1 \sum_{j=0}^{k-1} \mathscr{A}^j \mathscr{B}\, \Theta_{\{k-1-j\}} + D_2 \Theta_{\{k\}} \right] x_0 \leq d,$$

for all $k = 0, \ldots, T-1$ and for all $x_0 \in \Phi_\alpha$. □

Before proceeding further, we recall the following from Ref. 18.

Definition 5.1. Consider the linear autonomous system $x(t+1) = \bar{\mathscr{A}} x(t)$ and the polyhedron $\mathscr{P} \triangleq \{x: Bx \leq c\}$. The set $\Omega \triangleq \{x: B\bar{\mathscr{A}}^t x \leq c, \forall t \geq 0\}$ is called the maximum output admissible set (MOAS) contained in $\mathscr{P}$.

Theorem 5.1. Let $\bar{\mathscr{A}}$ be a strictly Hurwitz matrix (all eigenvalues contained in the interior of the unit disk), let $\mathscr{P}$ be bounded, and let $0 \in \mathrm{int}(\mathscr{P})$, where $\mathrm{int}(\mathscr{P})$ denotes the interior of P. Then, the MOAS Ω contained in $\mathscr{P}$ is determined by a finite number of facet inequalities.

Proof. See Ref. 18, Theorem 4.1.

We concentrate now on the following special class of suboptimal RHC laws, that will be referred to as SRHC:

(a) P is the solution of the Riccati equation

$$P = (\mathscr{A} + \mathscr{B}\mathscr{K})' P (\mathscr{A} + \mathscr{B}\mathscr{K}) + Q + \mathscr{K}' R \mathscr{K},$$

where

$$\mathscr{K} \triangleq -(R + \mathscr{B}' P \mathscr{B})^{-1} \mathscr{B}' P \mathscr{A};$$

(b) the set

$$\mathscr{P} \triangleq \{x: (D_1 + D_2 \mathscr{K}) x \leq d\}$$

is bounded and contains the origin in its interior[5];

(c) the terminal set Ω is the MOAS contained in $\mathscr{P}$;

(d) only dual feasibility is relaxed ($\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$) and $\epsilon_3 \triangleq \epsilon \underline{1}$, $\epsilon \geq 0$;

(e) the first critical region generated by the suboptimal multiparametric solver is

$$\mathrm{CR}_\varnothing \triangleq \{x: Wx \leq w\}, \qquad w \in \mathbb{R}^{n_w},$$

associated with the void combination of active constraints $(B_1, N_1) \triangleq (M, \varnothing)$.

Note that assumption (e) implies that the critical region associated with $(M, \varnothing)$ is not approximated.

Definition 5.2. The function $f: \mathscr{X} \mapsto \mathbb{R}$ is said to be positive definite if $f(x) > 0$ for all $x \in \mathscr{X}$ and $f(x) = 0$ if and only if $x = 0$. The function f is said to be negative definite if $-f$ is positive definite.

Theorem 5.2. Consider the SRHC controller defined above, and let γ be the maximum positive number for which the ellipsoid $\mathscr{E} \triangleq \{x: x'Qx \leq \gamma\}$ is contained in $\mathrm{CR}_i\varnothing$. Let (B_h, N_h), $h = 2, \ldots, \hat{l}$, be the optimal partitions of the approximate solution to the MPQP problem (6), and let $\mathrm{CR}_{\epsilon(N_h)}$ denote the associated approximate critical regions. If $\epsilon(N_h)$ is chosen satisfying

$$\epsilon(N_h) \leq \sqrt{2\gamma / (\underline{1}' A_{N_h} H^{-1} A'_{N_h} \underline{1})}, \tag{25}$$

[5]This hypothesis is satisfied for instance when the constraints (2) have the popular form $u_{\min} \leq u(t) \leq u_{\max}$, $x_{\min} \leq x(t) \leq x_{\max}$, with $u_{\min} < 0 < u_{\max}$, $x_{\min} < 0 < x_{\max}$.

for all $h = 2, \ldots, \hat{l}$, then SRHC asymptotically stabilizes the system (1) while fulfilling the constraints (2) at each time $t \geq 0$, for all $x(0) \in X_f$.

Proof. In order to prove the theorem, we show that the exact value function ϕ^* is a Lyapunov function for the system (1) in closed-loop with the suboptimal controller SRHC. Let

$$\hat{U} \triangleq [\hat{u}_0', \ldots, \hat{u}_{T-1}']'$$

be the suboptimizer at time t, $t \geq 0$. At time $t+1$, consider the vector of inputs

$$\tilde{U} \triangleq [\hat{u}_1', \ldots, \hat{u}_{T-1}'(\mathscr{K}\hat{x}_T)']',$$

where

$$\hat{x}_k \triangleq \mathscr{A}^k x(t) + \sum_{j=0}^{k-1} \mathscr{A}^j \mathscr{B} \hat{u}_{k-1-j}, \qquad k = 0, \ldots, T.$$

By the definition of Ω, the condition $\hat{x}_T \in \Omega$ implies that

$$\hat{x}_{T+1} \triangleq (\mathscr{A} + \mathscr{B}\mathscr{K})\hat{x}_T \in \Omega,$$

which together with the feasibility of $\hat{U}$ at time t implies the feasibility of $\tilde{U}$ at time $t+1$, which also proves that

$$x(t+1) \in X_f.$$

Then,

$$\begin{aligned}
&\phi^*(x(t+1)) - \phi^*(x(t)) \\
&\leq (1/2)\tilde{U}'H\tilde{U} + x'(t+1)C'\tilde{U} + (1/2)x'(t+1)Yx(t+1) - \phi^*(x(t)) \\
&= \hat{x}_{T+1}'P\hat{x}_{T+1} + \sum_{k=1}^{T} (\hat{x}_k'Q\hat{x}_k + \hat{u}_k'R\hat{u}_k) - \phi^*(x(t)) \\
&= \hat{x}_T'(\mathscr{A} + \mathscr{B}\mathscr{K})'P(\mathscr{A} + \mathscr{B}\mathscr{K})\hat{x}_T + \sum_{k=0}^{T-1} (\hat{x}_k'Q\hat{x}_k + \hat{u}_k'R\hat{u}_k) \\
&\quad + \hat{x}_T'Q\hat{x}_T + (\mathscr{K}\hat{x}_T)'R(\mathscr{K}\hat{x}_T) - \hat{x}_0'Q\hat{x}_0 - \hat{u}_0'R\hat{u}_0 - \phi^*(x(t)) \\
&= \hat{x}_T'[(\mathscr{A} + \mathscr{B}\mathscr{K})'P(\mathscr{A} + \mathscr{B}\mathscr{K}) + Q + \mathscr{K}'\mathscr{R}\mathscr{K}]\hat{x}_T \\
&\quad + \sum_{k=0}^{T-1} (\hat{x}_k'Q\hat{x}_k + \hat{u}_k'R\hat{u}_k) - x(t)'Qx(t) - u(t)'Ru(t) - \phi^*(x(t)) \\
&= \hat{x}_T'P\hat{x}_T + \sum_{k=0}^{T-1} (\hat{x}_k'Q\hat{x}_k + \hat{u}_k'R\hat{u}_k) - x(t)'Qx(t) - u(t)'Ru(t) - \phi^*(x(t)) \\
&= \hat{\phi}(x(t)) - x(t)'Qx(t) - u(t)'Ru(t) - \phi^*(x(t)).
\end{aligned}$$

For all $x(t) \in CR_{\varnothing}$, we have

$$\hat{\phi}(x(t)) = \phi^*(x(t)),$$

and therefore,

$$\phi^*(x(t+1)) - \phi^*(x(t)) < 0, \qquad \text{for all } x(t) \in CR_{\varnothing} \setminus \{0\}.$$

Consider now

$$x(t) \in CR_{\epsilon(N_h)}, \qquad h = 2, \ldots, \hat{l}.$$

Since

$$x(t)'Qx(t) > \gamma, \qquad \text{for all } x(t) \notin CR_{\varnothing},$$

if $\epsilon(N_h)$ satisfies (25), then

$$\begin{aligned} &\phi^*(x(t+1)) - \phi^*(x(t)) \\ &\leq \hat{\phi}(x(t)) - \phi^*(x(t)) - x(t)'Qx(t) - u(t)'Ru(t) \\ &\leq (1/2)\epsilon(N_h)^2 \underline{1}' A_{N_h} H^{-1} A'_{N_h} \underline{1} - x(t)'Qx(t) - u(t)'Ru(t) \\ &< \gamma - \gamma - u(t)'Ru(t) \leq 0. \end{aligned} \tag{26}$$

By letting

$$\Delta\phi^*(x) \triangleq \phi^*(\mathscr{A}x + \mathscr{B}I^1 \hat{U}(x)) - \phi^*(x),$$

Eq. (26) proves that $\Delta\phi^*$ is a negative-definite function. Since

$$\phi^*(x) \geq x'Qx,$$

and since Q is positive definite, it follows that ϕ^* is positive definite and radially unbounded $[\phi^*(x) \to \infty, \text{for } \|x\| \to \infty]$. Therefore, we can apply LaSalle invariance principle for discrete-time systems (Ref. 19, Theorem 4.2) on the level sets of ϕ^* to conclude that the origin is asymptotically stable with domain of attraction X_f. □

In conclusion, whenever a new optimal partition (B_h, N_h) is generated by the recursive algorithm, Lemma 4.3, Theorem 4.2, and Theorem 5.2 provide constructive criteria for choosing the relaxation $\epsilon(N_h)$, so that error bounds on the value function and the optimizer as well as stability can be guaranteed a priori.

5.3. Complexity. The suboptimal RHC control law is

$$\hat{u}_0(x) = I_1 \hat{U}(x).$$

As approximate critical regions where the first n_u components of $\hat{U}(x)$ are the same and whose union is a convex set can be joined during a post-processing phase (Ref.20), similarly to the exact explicit solution of RHC (Ref. 1), $\hat{u}_0(x)$ has the following piecewise affine form:

$$\hat{u}_0(x) = \bar{F}^i x + \bar{g}^i, \qquad \text{if } \bar{H}^i x \leq \bar{k}^i, i = 1, \ldots, \hat{l}_{\text{rhc}}, \tag{27}$$

where the polyhedral sets

$$\{x: \bar{H}^i x \leq \bar{k}^i\}, \qquad i = 1, \ldots, \hat{l}_{\text{rhc}},$$

partition X_f; clearly, $\hat{l}_{\text{rhc}} \leq \hat{l}$.

While offline complexity of the suboptimal RHC algorithm can be investigated similarly to (Ref. 1), online complexity of (27) is more interesting, especially from an application point of view. The simplest way to implement the piecewise affine feedback law (27) is to store the polyhedral cells $\{x: \bar{H}^i x \leq \bar{k}^i\}$, perform an online linear search through them to locate the one which contains the current state $x(t)$, look up the corresponding $\bar{F}^i, \bar{g}^i$, and evaluate $\bar{F}^i x(t) + \bar{g}^i$. This search procedure can be easily parallelized or more efficiently organized according to a balanced search tree, a research topic currently under investigation.

6. Examples

Example 6.1. A second-order nonminimum phase system with transfer function $2(s-1)/(s^2+2s+5)$ is sampled with $T_s = 0.1$ s to obtain the discrete time state-space representation

$$x(t+1) = \begin{bmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{bmatrix} x(t) + \begin{bmatrix} 0.1271 \\ 0.0132 \end{bmatrix} u(t), \tag{28a}$$

$$y(t) = [1.4142 \quad -0.7071]\, x(t). \tag{28b}$$

The task is to regulate the system to the origin while fulfilling the input constraint

$$-1 \leq u(t) \leq 1.$$

To this aim, we design an RHC controller based on the optimization problem (3) with

$$T = 6, \qquad R = 0.1, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_1 = 0, \qquad D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \Omega = \mathbb{R}^2,$$

and P solves the Lyapunov equation

$$P = \mathscr{A}' P \mathscr{A} + Q.$$

Note that this choice of P corresponds to setting

$$u_k = 0, \qquad \text{for } k \geq 6,$$

and to minimize $\sum_{k=0}^{\infty} x_k' x_k + 0.1 u_k^2$. The MPQP problem associated with the RHC law has the form (QP_x) with $r = 6$, $q = 12$, and $n = 2$; see Ref. 10 for details. The explicit RHC controller was computed using the exact MPQP algorithm of (Ref. 1), and the corresponding polyhedral partition of the state-space is depicted in Fig. 1(a).

In order to reduce the number of regions, we apply the approximate MPQP algorithm. By setting $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$, and choosing a constant ϵ_3, we get the solutions shown in Figs. 1(b)–1(e) (for simplicity, from now on, we let all the components of ϵ_k to be equal, and denote by ϵ_k the single component). Each approximate MPQP solution was computed in less than 15 s of CPU time on a Pentium III 650 MHz running Matlab 5.3. Note that, despite the relaxation of dual feasibility ($\epsilon_3 > 0$), the region containing the origin does not change with respect to the exact solution. This is justified by the fact that, since $N = \varnothing$, the constraints defining the critical region are all of the form

$$AU^*(x) \leq b + Fx,$$

and therefore are not affected by the relaxation. For all the suboptimal RHC laws, we computed the maximum absolute errors e_{abs} according to Proposition 4.2 running CPLEX 7.0 (Ref. 21) on the same machine (about 10 s of CPU time per computation) and verified that the closed-loop system is quadratically stable with common quadratic Lyapunov function $U'LU$ (Ref. 17):

$$\text{for } \epsilon_3 = 0.03, \qquad L = \begin{bmatrix} 8.0052 & 3.8681 \\ 3.8681 & 17.3940 \end{bmatrix}, \qquad e_{\text{abs}} = 0.09530;$$

$$\text{for } \epsilon_3 = 0.05, \qquad L = \begin{bmatrix} 9.3106 & 4.3013 \\ 4.3013 & 17.9287 \end{bmatrix}, \qquad e_{\text{abs}} = 0.15884;$$

$$\text{for } \epsilon_3 = 0.15, \qquad L = \begin{bmatrix} 10.2754 & 4.7303 \\ 4.7303 & 19.0681 \end{bmatrix}, \qquad e_{\text{abs}} = 0.28158;$$

$$\text{for } \epsilon_3 = 0.2, \qquad L = \begin{bmatrix} 11.0924 & 5.1412 \\ 5.1412 & 20.2768 \end{bmatrix}, \qquad e_{\text{abs}} = 0.28158.$$

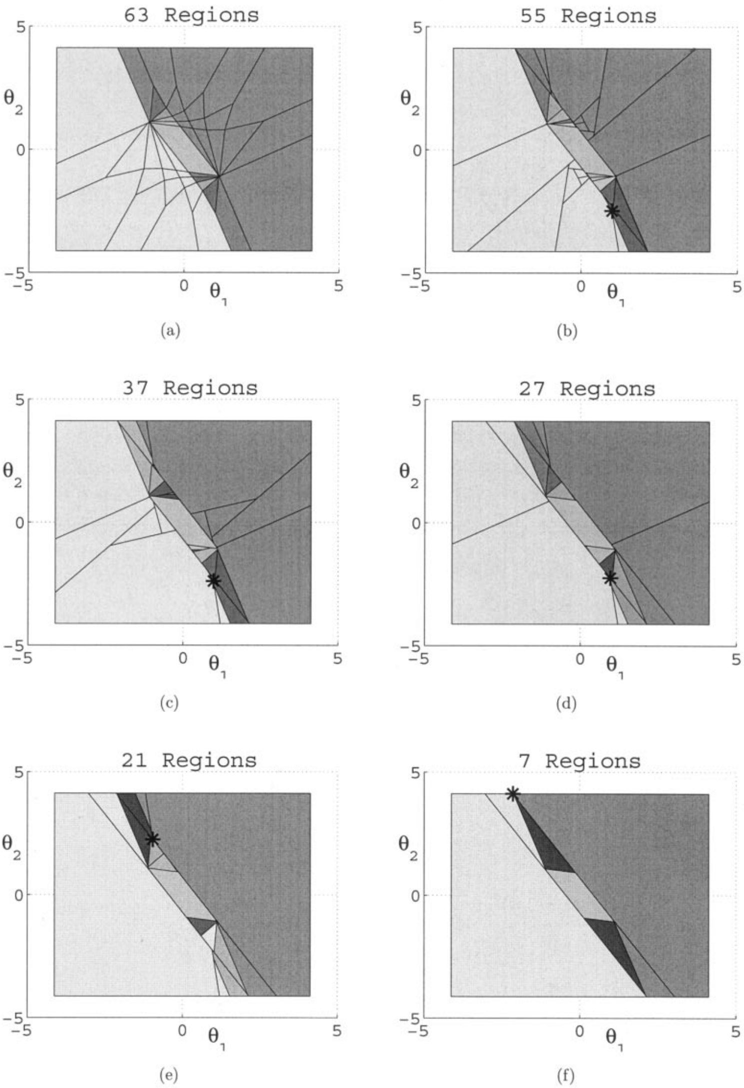

Figure 1. MPQP solutions of Example 6.1 for different degrees of approximation when the critical region containing the origin is maintained exact:

(a) $\epsilon_i = 0, i = 1, \ldots, 4$ (exact solution);
(b) $\epsilon_3 = 0.03$ and $\epsilon_1 = \epsilon_2 = \epsilon_4 = 0$ (Case C);
(c) $\epsilon_3 = 0.05$ and $\epsilon_1 = \epsilon_2 = \epsilon_4 = 0$ (Case C);
(d) $\epsilon_3 = 0.15$ and $\epsilon_1 = \epsilon_2 = \epsilon_4 = 0$ (Case C);
(e) $\epsilon_3 = 0.20$ and $\epsilon_1 = \epsilon_2 = \epsilon_4 = 0$ (Case C);
(f) $\epsilon_i = 10000, i = 1, \ldots, 4$.

The states where the maximum errors are achieved are marked by asterisks in Figs. 1(b)–1(e).

By choosing ϵ_3 adaptively in accordance with Theorem 4.2, we obtained:

for $\rho = 0.03$, $e_{abs} = 0.01927$, 53 regions ;

for $\rho = 0.05$, $e_{abs} = 0.03211$, 47 regions;

for $\rho = 0.1$, $e_{abs} = 0.05794$, 43 regions;

for $\rho = 0.15$, $e_{abs} = 0.08690$, 39 regions;

for $\rho = 0.25$, $e_{abs} = 0.12639$, 37 regions;

for $\rho = 0.5$, $e_{abs} = 0.25278$, 25 regions;

for $\rho = 1$, $e_{abs} = 0.28158$, 21 regions;

for $\rho = 5$, $e_{abs} = 0.28158$, 21 regions.

It is apparent that the a posteriori error bound e_{abs} is always smaller than the prespecified a priori error bound ρ. This is not surprising, as the choice for ϵ_3 suggested by Theorem 4.2 is based on the conservative overestimate (19). Moreover, for $x \in CR_\epsilon$, the piecewise affine function $\hat{U}(x) - U^*(x)$ does not span the whole ellipsoidal set described by the constraint in (23), so that further conservativeness is introduced. The fact that the intrinsic polyhedral structure of the partition may not allow it to reach the a priori error bound ρ is further testified by the fact that, as ρ increases, e_{abs} saturates at 0.28158.

Next, we vary all ϵ_i. To maintain the solution $\hat{U}(x)$ exact in the region where no constraint is active, we do not relax the KKT conditions for such a region and set

$$\epsilon_i = 10000, \qquad i = 1, \ldots, 4.$$

The suboptimal RHC control law was computed in 13.4 s and is depicted in Figure 1(f); its analytical expression can be found in Ref. 10. The control law is stabilizing, as

$$L = \begin{bmatrix} 19.5936 & 5.7937 \\ 5.7937 & 19.6299 \end{bmatrix}$$

provides a common Lyapunov function for the closed-loop piecewise affine system. The maximum absolute error is $e_{abs} = 1.9369$, while the maximum relative error is $e_{rel} = 0.97431$, attained at $\left[\begin{smallmatrix} -0.1351 \\ 0.9249 \end{smallmatrix}\right]$. Note that, by construction, the control law is exact in the central region. If we relax also the region corresponding to the unconstrained case, then we obtain different approximate stabilizing explicit RHC laws, which are reported in Ref. 10.

Example 6.2. We synthesize the suboptimal RHC law with a priori stability guarantees for the double integrator $1/s^2$, according to Theorem 5.2. We sample the dynamics with $T_s = 1$ s to obtain the discrete-time state-space model with state transition matrices

$$\mathscr{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

We constrain the inputs and states within the range

$$-1 \le u(t) \le 2, \qquad \|x(t)\|_\infty \le 100,$$

and choose the MPC parameters

$$T = 6, \qquad Q = I, \qquad R = 0.01.$$

By choosing P as the solution to the resulting Riccati equation, using the algorithm of Ref. 18 and in accordance with Theorem 5.2, we obtain that

$$\Omega = \left\{ x: \begin{bmatrix} -0.6630 & -0.3304 \\ 1.3261 & 0.6609 \\ 0.1048 & 0.2169 \\ -0.2097 & -0.4338 \end{bmatrix}; x \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is the corresponding maximum output admissible set. The first critical region generated by the suboptimal multiparametric solver, associated with the void combination of active constraints, is $CR_\varnothing = \Omega$, for which $\gamma = 0.4555$ is the maximum positive number such that the ellipsoid

$$\epsilon \triangleq \{x: x'Qx \le \gamma\}$$

is contained in $CR_\varnothing$. The exact explicit RHC controllers and the suboptimal SRHC controllers consist of 53 regions and 39 regions, respectively, which are depicted in Fig. 2. For SRHC, the relaxation parameter $\epsilon(N_h)$ ranges between 0.0304 and 64.9584. The closed-loop trajectories for the exact and the suboptimal controllers are indistinguishable.

7. Conclusions

In this paper, we addressed the problem of reducing the number of polyhedral cells associated with explicit solutions to RHC problems. Such number tends to increase exponentially with the number of constraints involved in the optimization problem. Our solution consists of finding an approximate solution to MPQP by relaxing the KKT conditions for optimality, except primal feasibility. Bounds for the errors on the optimal value

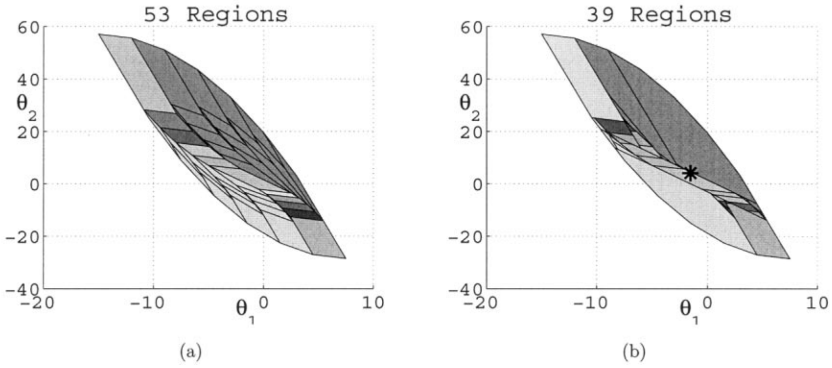

Figure 2. Exact and approximate RHC controllers with stability guarantees for Example 6.2: (a) exact RHC law; (b) RHC law.

and the optimizer are provided, and constraint fulfillment and closed-loop stability of the resulting suboptimal RHC law are guaranteed by explicit formulas. In principle, the degree of approximation is arbitrary and allows to tradeoff between optimality and a comparatively small number of cells in the piecewise affine solution. Clearly, the choice of the degree of relaxation depends also on stability requirements, although it may be more a reflection of the particular Lyapunov function chosen to prove the closed-loop stability properties than fundamental limitations on the proposed procedure. This is a point which is worthy further study. Future work will be also devoted to extend the approach of this paper to multiparametric linear programming (MPLP) and to multiparametric mixed integer linear programming (MPMILP).

References

1. Bemporad, A., Morari, M., Dua, V., and Pistikopoulos, E., *The Explicit Linear-Quadratic Regulator for Constrained Systems*, Automatica, Vol. 38, No. 1, pp. 3–20, 2002.
2. Tøndel, P., Johansen, T. A., and Bemporad, A., *An Algorithm for Multiparametric Quadratic Programming and Explicit MPC Solutions*, Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, Florida, pp. 1199–1204, 2001.
3. Meadows, E., and Rawlings, J., *Nonlinear Process Control*, Prentice Hall, Upper Saddle River, New Jersey, 1997.

4. Johansen, T. A., Petersen, I., and Slupphaug, O., *On Explicit Suboptimal LQR with State and Input Constraints*, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000.
5. Seron, M. M., De Doná, J. A., and Goodwin, G. C., *Global Analytical Model Predictive Control with Input Constraints*, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000.
6. Bazaraa, M. S., Sherali, H. D., and Shetty, C. M., *Nonlinear Programming—Theory and Algorithms*, 2nd Edition, John Wiley and Sons, New York, NY, 1993.
7. Fiacco, A. V., *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, Academic Press, London, UK, 1983.
8. Fukuda, K., *cdd/cdd+ Reference Manual*, Institute for Operations Research, ETH-Zentrum, Zurich, Switzerland, 1997.
9. Bemporad, A., Filippi, C., and Torrisi, F. D., *Inner and Outer Approximations of Polytopes Using Boxes*, IFA Technical Report AUT 02-06, Automatic Control Laboratory, ETH, Zürich, Switzerland, 2002.
10. Bemporad, A., and Filippi, C., *Suboptimal Explicit RHC via Approximate Multiparametric Quadratic Programming*, IFA Technical Report AUT 02–07, Automatic Control Laboratory, ETH, Zürich, Switzerland, 2002.
11. Luo, Z. Q., Mangasarian, O. L., Ren, J., and Solodov, M. V., *New Error Bounds for the Linear Complementarity Problem*, Mathematics of Operations Research, Vol. 19, No. 4, pp. 880–892, 1994.
12. Mangasarian, O. L., and Ren, J., *New Improved Error Bounds for the Linear Complementarity Problem*, Mathematical Programming, Vol. 66, pp. 241–255, 1994.
13. Mayne, D. Q., Rawlings, J. B., Rao, C. V., and Scokaert, P. O. M., *Constrained Model Predictive Control: Stability and Optimality*, Automatica, Vol. 36, No. 6, pp. 789–814, 2000.
14. Kerrigan, E. C., and Maciejowski, J. M., *Invariant Sets for Constrained Nonlinear Discrete-Time Systems with Application to Feasibility in Model Predictive Control*, Proceedings of the 39th IEEE Conference on Decision and Control, Sidney, Australia, 2000.
15. Johannson, M., and Rantzer, A., *Computation of Piecewise Quadratic Lyapunov Functions for Hybrid Systems*, IEEE Transactions on Automatic Control, Vol. 43, No. 4, pp. 555–559, 1998.
16. Bemporad, A., Torrisi, F. D., and Morari, M., *Performance Analysis of Piecewise Linear Systems and Model Predictive Control Systems*, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000.
17. Mignone, D., Ferrari-trecate, G., and Morari, M., *Stability and Stabilization of Piecewise Affine and Hybrid Systems: An LMI Approach*, Proceedings of the 39th IEEE Conference on Decision and Control, Sidney, Australia, 2000.
18. Gilbert, E. G., and Tan, K. T., *Linear Systems with State and Control Constraints: The Theory and Applications of Maximal Output Admissible Sets*, IEEE Transactions on Automatic Control, Vol. 36, No. 9, pp. 1008–1020, 1991.
19. LaSalle, J. P., *Stability Theory for Difference Equations*, Studies in Ordinary Differential Equations, Edited by J. Hale, Mathematical Association of America Studies in Mathematics, Vol. 14, pp. 1–31, 1997.

20. Bemporad, A., Fukuda, K., and Torrisi, F. D., *Convexity Recognition of the Union of Polyhedra*, Computational Geometry, Vol. 18, pp. 141–154, 2001.
21. ILOG, *CPLEX* 7.0 *User Manual*, Gentilly, France, 2000.
22. Bemporad, A., and Morari, M., *Control of Systems Integrating Logic, Dynamics, and Constraints*, Automatica, Vol. 35, No. 3, pp. 407–427, 1999.
23. Bemporad, A., Ferrari-trecate, G., and Morari, M., *Observability and Controllability of Piecewise Affine and Hybrid Systems*, IEEE Transactions on Automatic Control, Vol. 45, No. 10, pp. 1864–1876, 2000.
24. Dua, V., and Pistikopoulos, E. N., *An Algorithm for the Solution of Multiparametric Mixed Integer Linear Programming Problems*, Annals of Operations Research, Vol. 99, pp. 123–139, 2000.