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A Posteriori Multistage Optimal Trading under Transaction Costs and a Diversification Constraint

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Algorithmic assistance to traders and portfolio managers has become standard practice. This assistance can be categorized as either *algorithmic trading*—that is, fully automated high- or low-frequency trading—or *algorithmic screening*—semi-automated high- or low-frequency trading with computer programs providing recommendations to the human trader. Both algorithmic trading and screening are fundamentally based on predictions of future developments. Predictions may be made based on, for example, financial accountancy, technical chart analysis, global macroeconomic analysis, news, sentiments, and combinations thereof. There exists a plethora of literature on *financial times-series forecasting*. For methods based on support vector machines, see, for example, Tay and Cao (2001), Kim (2003), Van Gestel et al. (2001), and Chowdhury et al. (2018). In general, influential factors on trading decisions are trading frequency, targeted time horizons, performance expectations, asset choices, foreign exchange rates, transaction costs, and risk management, such as investment diversification. Our study belongs to the class of technical chart analysis. The data on which the analysis is based are daily adjusted closing prices of various currencies and assets. Short selling, borrowing of money, and the trading of derivatives are not treated, even though the presented methodologies can be extended to include them.

The motivation and contribution of this article is threefold: 1) the development of a simple algorithm for *a posteriori* (historical) multivariate, multistage optimal trading under transaction costs and a diversification constraint, including the discussion of unconstrained trading frequency, a fixed number of total admissible trades, and the waiting of a specific time period after every executed trade until the next trade; 2) the quantification of the effects of transaction costs on *a posteriori* optimal trading evaluated on real-world data; and 3) the preparatory labeling of financial time-series data for supervised machine learning.

This article is related most closely to the work of Boyarshinov and Magdon-Ismail (2010), who discussed a dynamic programming solution to the optimal investment in either one stock or one bond under consideration of unconstrained trading frequency and a bound on the admissible number of trades. Additionally, a method for optimization of Sterling and Sharpe ratios is presented. No real-world data analysis is conducted, however. Additional differences include our discussion of a diversification constraint, the constraint of introducing a waiting period after every executed trade until the next trade, and a synchronous trading constraint. Furthermore, we introduce a heuristic for each of the constrained optimal investment problems (with a bound on the admissible number of trades

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and a waiting period constraint), thereby reducing the computational complexity of the methods while not compromising optimality. For an overview of measures to reduce risk by the introduction of various constraints (e.g., on the drawdown probability or short selling), see Lobo et al. (2007), who conducted *one-step-ahead* optimization, importantly, based on estimates of one-step-ahead returns and covariance matrices of a set of risky assets. In contrast, this article is concerned about *multistage* optimization and historical optimal trading with hindsight. The mathematical approaches therefore differ significantly (convex optimization versus graph search). Optimal trading based on stochastic models, usually stochastic differential equations (SDEs) and the consideration of fixed and/or proportional transaction costs is treated in, for example, Altarovici et al. (2015), Lo et al. (2001), Morton and Pliska (1995), and Korn (1998). In contrast, this study is data based only, without consideration of any mathematical model explaining the generation of this data. For a discussion about the existence of trends in financial time series, see Fliess and Join (2009). For the general discussion of *transaction cost analysis* (TCA), see Gomes and Waelbroeck (2010) and, further, Kissell (2006, 2008, 2013) for discussion of how TCA can be used by portfolio managers to improve performance and the development of a framework for pre-, intra- and post-trade analysis.

ONE-STAGE MODELING OF TRANSITION DYNAMICS

Notation

Let time index $t \in Z_+$ be associated with the trading period T_s, such that trading instants are described as tT_s, whereby T_s may typically be, for example, one week, one day, or less (for intraday trading). The system state z_t at time t is defined as an eight-dimensional vector of mixed integer and real-valued quantities,

$$z_t = [i_t \quad k_t \quad j_t \quad m_t^{c_t} \quad n_t \quad w_t^0 \quad d_t \quad c_t], \tag{1}$$

where $i_t \in \mathcal{I} = (\mathcal{I}_{N_c} \cup \mathcal{I}_{N_a})$ denotes investment identification numbers partitioned into N_c currencies and N_a different risky noncurrency assets, such that $\mathcal{I}_{N_c} = \{0, 1, \ldots, N_c - 1\}$ and $\mathcal{I}_{N_a} = \{N_c, \ldots, N_c + N_a - 1\}$. For ease of reference, in the following, we lump currencies and noncurrency assets in the term asset and only distinguish when context-necessary. The integer number of conducted trades along an *investment trajectory* shall be denoted by $k_t \in \mathrm{Z}$, whereby an investment trajectory is here defined as a sequence of states z_t, $t = 0, 1, \ldots, N_t$, where N_t is the time horizon length. Let j_t denote the investment identification number preceding i_t at time $t - 1$ (parent node), i.e., $j_t = i_{t-1}$. We define $m_t^{c_t} \in \mathrm{R}_+$ as the real-valued and positive *cash position* (liquidity) held in the currency identified by $c_t \in \mathcal{I}_{N_c}$. The number $n_t \in \mathrm{Z}_+$ indicates the number of noncurrency assets held. The current wealth, composed of cash position and noncurrency asset, is denoted by w_t^0 and shall always be in monetary units EUR. Euro is considered as our reference currency and shall throughout this paper be identified by $i_t = 0$. The integer number of time samples since the last trade is defined by $d_t \in \mathrm{Z}_+$. The (unitless) foreign exchange (FX) rate $x_t^{c_1,c_2}$ for two currencies $c_1 \in \mathcal{I}_{N_c}$ and $c_2 \in \mathcal{I}_{N_c}$ is defined as $x_t^{c_1,c_2}$ such that $m_t^{c_2} = m_t^{c_1} x_t^{c_1,c_2}$. Thus, $m_t^{c_1}$ and $m_t^{c_2}$ have numerical values, however, with units identified by $c_1 \in \mathcal{I}_{N_c}$ and $c_2 \in \mathcal{I}_{N_c}$, respectively. Noncurrency asset prices are denoted by $p_t^{c_t,a}$, whereby c_t identifies the price unit and $a \in \mathcal{I}_{N_a}$ the asset. We treat foreign exchange rates and asset prices as time-varying parameters obtained from data. In the sequel, various sets of admissible system states are defined. For brevity, we use a shorthand notation. For example, we define a set as $\mathcal{Z}_t = \{z_t \colon i_t = 10\}$, implying $\mathcal{Z}_t = \{z_t \colon i_t = 10$, associated with z_t according to Equation 1$\}$.

Transaction Costs

We follow the notation of Lobo et al. (2007), modeling transaction costs as nonconvex with a fixed charge for any nonzero trade (fixed transaction costs) and a linear term scaling with the quantity traded (proportional transaction costs). Thus, for a foreign exchange at time $t - 1$, we model $m_t^{c_t} = m_{t-1}^{c_{t-1}} x_{t-1}^{c_{t-1},c_t} (1 - \epsilon_{\text{fx}}^{c_{t-1},c_t}) - \beta_{\text{fx}}^{c_{t-1},c_t}$, where $\epsilon_{\text{fx}}^{c_{t-1},c_t}$ and $\beta_{\text{fx}}^{c_{t-1},c_t}$ denote the linear term and the fixed charge, respectively. Similarly, transaction costs for transactions from currency to noncurrency assets, between assets of different currencies and the like, can be defined. We can further differentiate between linear terms for buying and selling. To fully introduce notation for transaction costs $(\epsilon_{\text{buy}}^{i_t}, \beta_{\text{buy}}^{i_t} \geq 0)$, we state the transaction from a cash position toward an asset investment and vice

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versa. For a transaction of buying n_{t-1} of asset i_{t-1} at time $t-1$, we obtain

$$\begin{aligned} m_t^{c_t} &= m_{t-1}^{c_{t-1}} x_{t-1}^{c_{t-1},c_t} (1-\epsilon_{\text{fx}}^{c_{t-1},c_t}) \\ &\quad -\beta_{\text{fx}}^{c_{t-1},c_t} - n_{t-1} p_{t-1}^{c_t,i_t} (1+\epsilon_{\text{buy}}^{i_t}) - \beta_{\text{buy}}^{i_t}. \end{aligned}$$

For a transaction of selling n_{t-1} of asset i_{t-1} and transforming to currency c_t, we obtain

$$\begin{aligned} m_t^{c_t} &= (m_{t-1}^{c_{t-1}} + n_{t-1} p_{t-1}^{c_{t-1},i_{t-1}} (1-\epsilon_{\text{sell}}^{i_{t-1}} \\ &\quad -\beta_{\text{sell}}^{i_{t-1}})\, x_{t-1}^{c_{t-1},c_t} (1-\epsilon_{\text{fx}}^{c_{t-1},c_t}) - \beta_{\text{fx}}^{c_{t-1},c_t}. \end{aligned}$$

Finally, note that transaction costs may vary dependent on the assets involved.

Transition Dynamics

Given our assumption of being able to invest in currencies and noncurrency assets, there are six general types of transitions dependent on the investment at time $t-1$. For an introduction to *Markov decision processes* (MDP), see Puterman (2005). We initialize $z_0 = [0\ \ 0\ \ 0\ \ m_0^0\ \ 0\ \ m_0^0\ \ 0\ \ 0]$. Then, the transition dynamics are

$$z_t = \begin{cases} z_t^{(1)}, \text{ if } \{i_t : i_t = i_{t-1}, z_{t-1} \text{ with } i_{t-1} \in \mathcal{I}_{N_c}\}, \\ z_t^{(2)}, \text{ if } \{i_t : i_t \in \mathcal{I}_{N_c} \backslash \{i_{t-1}\}, z_{t-1} \text{ with } i_{t-1} \in \mathcal{I}_{N_c}\}, \\ z_t^{(3)}, \text{ if } \{i_t : i_t \in \mathcal{I}_{N_a}, z_{t-1} \text{ with } i_{t-1} \in \mathcal{I}_{N_c}\}, \\ z_t^{(4)}, \text{ if } \{i_t : i_t = i_{t-1}, z_{t-1} \text{ with } i_{t-1} \in \mathcal{I}_{N_a}\}, \\ z_t^{(5)}, \text{ if } \{i_t : i_t \in \mathcal{I}_{N_c}, z_{t-1} \text{ with } i_{t-1} \in \mathcal{I}_{N_a}\}, \\ z_t^{(6)}, \text{ if } \{i_t : i_t \in \mathcal{I}_{N_a} \backslash \{i_{t-1}\}, z_{t-1} \text{ with } i_{t-1} \in \mathcal{I}_{N_a}\}, \end{cases} \tag{2}$$

where $z_t^{(j)}$, $\forall j = 1, \ldots, 6$ is defined next, and our control variable u_{t-1} is the targeted investment identified by variable i_t (i.e., $u_{t-1} = i_t$). We have

$$\begin{aligned} z_t^{(1)} &= [i_{t-1}\ \ k_{t-1}\ \ j_{t-1}\ \ m_{t-1}^{c_{t-1}}\ \ 0\ \ w_{t-1}^0\ \ \tilde{d}_t\ \ c_{t-1}], \\ z_t^{(2)} &= [i_t\ \ k_{t-1}+1\ \ j_{t-1}\ \ \varphi(m_t^{c_t})\ \ 0\ \ m_t^{c_t} x_t^{c_t,0}\ \ \tilde{d}_t\ \ i_t], \\ z_t^{(3)} &= [i_t\ \ k_{t-1}+1\ \ j_{t-1}\ \ \tilde{m}_t^{c_t}\ \ \tilde{n}_t\ \ \tilde{w}_t^0\ \ \tilde{d}_t\ \ c(i_t)], \end{aligned}$$

with $c(i_t)$ denoting the currency of asset i_t and with

$$\begin{aligned} \tilde{d}_t &= \begin{cases} d_{t-1}+1, & \text{if } d_{t-1} < D-1, \\ 0 & \text{otherwise}, \end{cases} \\ \varphi(m_t^{c_t}) &= m_{t-1}^{c_{t-1}} x_{t-1}^{c_{t-1},c_t} (1-\epsilon_{\text{fx}}^{c_{t-1},c_t}) - \beta_{\text{fx}}^{c_{t-1},c_t}, \end{aligned}$$

and where variable D determines an overflow in d_t and will become relevant when later discussing the constraint of waiting a specific amount of time until the next admissible trade. Furthermore, $\tilde{m}_t^{c_t}$ and $\tilde{n}_t$ are obtained from solving

$$\max_{m_t^{c_t} \geq 0} \left\{ n_t : n_t = \frac{m_{t-1}^{c_{t-1}} x_{t-1}^{c_{t-1},c_t} (1-\epsilon_{\text{fx}}^{c_{t-1},c_t}) - \beta_{\text{fx}}^{c_{t-1},c_t} - \beta_{\text{buy}}^{i_t} - m_t^{c_t}}{p_{t-1}^{c_t,i_t} (1+\epsilon_{\text{buy}}^{i_t})}, n_t \in \mathbb{Z}_+ \right\}, \tag{3}$$

with $\tilde{m}_t^{c_t}$ denoting the optimizer and $\tilde{n}_t$ the corresponding optimal objective function value. Thus, given $m_{t-1}^{c_{t-1}}$, we find the largest possible positive integer number of assets we can purchase under the consideration of transaction costs. The (small) cash residual is then $\tilde{m}_t^{c_t} \geq 0$. Therefore, for the portfolio wealth at time t in euros (EUR), we obtain $\tilde{w}_t^0 = (\tilde{m}_t^{c_t} + \tilde{n}_t p_t^{c_t,i_t}) x_t^{c_t,0}$. Furthermore,

$$\begin{aligned} z_t^{(4)} &= [i_{t-1}\ \ k_{t-1}\ \ j_{t-1}\ \ m_{t-1}^{c_{t-1}}\ \ n_{t-1}\ \ w_{t-1}^0\ \ \tilde{d}_t\ \ c_{t-1}], \\ z_t^{(5)} &= [i_t\ \ k_{t-1}+1\ \ j_{t-1}\ \ \phi(m_t^{c_t})\ \ 0\ \ m_t^{c_t} x_t^{c_t,0}\ \ \tilde{d}_t\ \ i_t], \\ z_t^{(6)} &= [i_t\ \ k_{t-1}+1\ \ j_{t-1}\ \ \overline{m}_t^{c_t}\ \ \overline{n}_t\ \ \overline{w}_t^0\ \ \tilde{d}_t\ \ c(i_t)], \end{aligned}$$

with

$$\begin{aligned} \phi(m_t^{c_t}) &= (m_{t-1}^{c_{t-1}} + n_{t-1} p_{t-1}^{c_{t-1},i_{t-1}} (1-\epsilon_{\text{sell}}^{i_{t-1}}) \\ &\quad -\beta_{\text{sell}}^{i_{t-1}})\, x_{t-1}^{c_{t-1},c_t} (1-\epsilon_{\text{fx}}^{c_{t-1},c_t}) - \beta_{\text{fx}}^{c_{t-1},c_t}, \end{aligned}$$

and where $\overline{m}_t^{c_t}$ and $\overline{n}_t$ are obtained from solving

$$\max_{m_t^{c_t} \geq 0} \left\{ n_t : n_t = \frac{\phi(m_t^{c_t}) - \beta_{\text{buy}}^{i_t} - m_t^{c_t}}{p_{t-1}^{c_t,i_t} (1+\epsilon_{\text{buy}}^{i_t})},\ n_t \in \mathbb{Z}_+ \right\}, \tag{4}$$

with $\overline{m}_t^{c_t}$ denoting the optimizer and $\overline{n}_t$ the corresponding optimal objective function value. Then, for the portfolio wealth at time t in EUR, we obtain $\overline{w}_t^0 = (\overline{m}_t^{c_t} + \overline{n}_t p_t^{c_t,i_t}) x_t^{c_t,0}$.

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The solution to Equations 3 and 4 can be easily computed by setting $m_t^{c_t}$ initially zero, then rounding the corresponding real-valued n_t to the largest smaller integer, before then computing the cash residuals, respectively. The methodology of preserving a cash residual is implemented in order to enforce an integer-valued number of shares in assets.

Remarks about Optimality and Transition Dynamics Modeling

An investment trajectory is defined as a sequence of states z_t, $t = 0, 1, \ldots, N_t$. We wish to find an optimal (in the sense of wealth-maximizing) investment trajectory. Several remarks about the previously discussed problem formulation and transition dynamics modeling can be made.

First, suppose all of the initial money m_0^0 is fully allocated to the optimal investment trajectory, then there is no diversification present and, defining the final return as $r_{N_t} = (w_{N_t}^0 - m_0^0)/m_0^0$, the optimal investment trajectory never returns less than $r_{N_t} < 0\%$. This is because one feasible investment trajectory is to remain invested in the initial reference currency (EUR) for all $t = 0, 1, \ldots, N_t$. This can be taken into account as a heuristic for transition graph generation.

Second, transition dynamics modeling naturally results in cash residuals when investing in noncurrency assets. According to our modeling, the cash residuals are enforced to be in the currency of the purchased asset. This may be suboptimal when the noncurrency asset in which we invest is extremely expensive (e.g., worth thousands of euros), because the resulting cash residuals may then be very large. Then, in general it may be worthwhile to invest the cash residual into another asset that is more profitable than the "enforced" residual currency. Two comments are made. On the one hand, assets with such prices are rare in practice. On the other hand and, more importantly, in order to admit free investing of cash residuals, an extension of the state space (beyond eight variables) would be required such that any cash residual could be invested in any of the $N_c + N_a - 1$ assets. Then, $N_c + N_a - 1$ additional branches would need to be added to the transition graph, which in the most general case would also need further branching at subsequent stages. This considerably complicates the tracking of states and is therefore not applied in the following.

Third, transition dynamics 2 indicate an *all-or-nothing* strategy. At every time t, the investment at that time is maintained or, alternatively, reallocated to exactly *one*—the most profitable—currency or asset, whereby cash residuals are accounted for as described in the previous paragraph.

Fourth, let us briefly discuss the effect of absence of transaction costs on optimal trading frequency. For simplicity, let us consider the case of being able to invest in an asset of variable value (such as a stock) and holding cash in the currency in which the risky asset is traded. Relevant discrete-time dynamics can then be written as

$$w_t = m_t + n_t p_t, \quad \text{and} \quad m_t = m_{t-1} - n_t p_{t-1}, \tag{5}$$

with m_t the cash position, n_t the number of shares in the risky asset, p_t the price of the asset, and w_t the wealth at time t. At every time t, a decision about a reallocation of investments is made. For a final time period $t = 0, 1, ..., N_t$, we wish to maximize $w_{N_t} - w_0$, which can be expanded as

$$w_{N_t} - w_0 = \sum_{t=1}^{N_t} w_t - w_{t-1}. \tag{6}$$

To maximize Equation 6, we thus have to maximize the increments. Combining it with Equation 5, we write $w_t - w_{t-1} = n_t(p_t - p_{t-1}) - n_{t-1}p_{t-1}$, which therefore motivates the following optimal trading strategy, implemented at every $t - 1$. If $p_t > p_{t-1}$, maximize n_t and set $n_{t-1} = 0$ (i.e., allocate maximal resources toward the asset), and if $p_t \leq p_{t-1}$, set $n_t = 0$ and minimize n_{t-1} (i.e., sell the asset if held at $t - 1$ and allocate maximal resources toward the cash position). We profit on a price increase of the asset, and maintain our wealth on a price decrease.[1] Thus, it is optimal to trade upon any change of sign of $\Delta p_t = p_t - p_{t-1}$. This is visualized in Exhibit 1 and summarized in the following remark.

Remark 1. In absence of transaction costs, trading upon any change in sign of $\Delta p_t = p_t - p_{t-1}$ is the optimal trading policy.

Remark 1 implies that in the absence of transaction costs, high-frequency trading (Aldridge 2013; Bowen et al. 2010) is always the optimal trading policy. Furthermore, note that $\Delta w_t = w_t - w_{t-1} \geq 0$, $\forall t = 1, 2, \ldots, N_t$. Thus, in the absence of transaction costs, at every time step, there is at least no incremental decrease

[1] We here assume a *long-only* strategy. By the use of derivative contracts, we can increase wealth on a price decrease as well.

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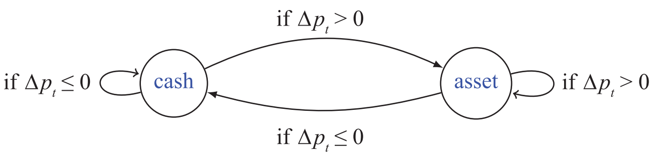
EXHIBIT 1

The Markov Decision Process when Optimally Trading Only Cash and an Asset in the Absence of Transaction Costs



Note: The optimal trading strategy is to trade upon any change of Δp_t – sign, that is, even if this is minimally small ($\Delta p_t \to 0$).

in wealth when employing the optimal trading policy. Naturally, when including nonzero transaction costs, this is in general not the case anymore, and we may (at least temporarily) have $\Delta w_t < 0$. In addition, the optimal trading frequency will be nontrivially affected. Quantitative examples for optimal trading frequencies under transaction costs are given later, in the Numerical Examples section.

Fifth, motivated by the previous paragraph and under the consideration of transaction costs, a valid question to address is when to sell and rebuy a noncurrency asset given a long-term trend but temporary dip in price. Selling and rebuying may optimize profit. The typical minimal decrease in price required for the strategy of selling and rebuying being optimal is approximately twice the proportional transaction cost level—"twice" because of selling *and* rebuying; "approximately" because of cash residuals due to the integer-valued number of assets and fixed transaction costs that need to be taken into account.

MULTISTAGE OPTIMIZATION WITHOUT DIVERSIFICATION CONSTRAINTS

Multistage Optimization

Multistage transition dynamics can be modeled in the form of a transition graph. We therefore assign a set $\mathcal{Z}_t$ of admissible states to every time stage t. For investment trajectory optimization without a diversification constraint, we employ *one* transition graph. For investment trajectory optimization with a diversification constraint, *multiple* transition graphs and $\mathcal{Z}_t^{(q)}$, $\forall q = 0, 1, \ldots, Q-1$, are defined and discussed in the next section. In contrast, for the remainder of this section, we dismiss superscript "(q)" and focus on optimization without a diversification constraint. We define the initial set $\mathcal{Z}_0 = \{z_0 : z_0 = [0 \;\; 0 \;\; 0 \;\; m_0^0 \;\; 0 \;\; m_0^0 \;\; 0 \;\; 0]\}$. In the following subsections, three constraints are discussed that affect transition graph generation.

Case 1: Unconstrained Trading Frequency

Remark 2. Suppose that following a particular investment trajectory, at time τ an investment state z_τ is reached with a particular i_τ, w_τ^0, and $j_\tau = i_{\tau-1}$. Suppose further that another investment trajectory exists resulting in the same asset (i.e., $\tilde{i}_\tau = i_\tau$), but in contrast with $\tilde{w}_\tau^0 > w_\tau^0$ and $\tilde{j}_\tau \neq j_\tau$. Then, the former investment trajectory can be dismissed from being a possible candidate segment for the optimal investment trajectory. This is because any trajectory continuing the latter investment trajectory will always outperform a continuation of the former investment trajectory for all $t > \tau$.

Remark 2 motivates a simple but efficient transition graph generation: First, branch from every state $z_{t-1} \in \mathcal{Z}_{t-1}$ to all possible states z_t at time t according transition dynamics 2, whereby we summarize the set of states at time $t-1$ from which z_t can be reached as $\mathcal{J}_{t-1}^{z_t}$; second, select the optimal transitions and thus determine $\mathcal{Z}_t$ according to

$$\mathcal{Z}_t = \left\{ z_t : \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\}, \; \forall i_t \in \mathcal{I} \right\}, \tag{7}$$

recalling the definition $j_t = i_{t-1}$ and thereby selecting the solutions with the highest value w_t^0, $\forall i_t \in \mathcal{I} = \{0, 1, \ldots N_c + N_a - 1\}$. The resulting transition graph holds a total of $N_z(t) = 1 + (N_c + N_a)t$ states up to time $t \geq 0$. For a time horizon N_t, the optimal investment strategy, here denoted by superscript "$*$", can then be reconstructed by proceeding backward as

$$\begin{aligned} z_{N_t}^* &= \left\{ z_{N_t} : i_{N_t}^* = \max_{i_{N_t}} \{w_{N_t}^0\}, \; i_{N_t} \in \mathcal{I} \right\}, \\ z_{t-1}^* &= \left\{ z_{t-1} : i_{t-1} = j_t^* \right\}, \quad \forall t = N_t, N_t - 1, \ldots, 1. \end{aligned} \tag{8}$$

The resulting investment trajectory is optimal because by construction of the transition graph as outlined, starting from z_0, exactly one wealth maximizing trajectory exists to every investment $i_t = 0, 1, \ldots, N_c + N_a - 1$

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for every time $t = 0, 1, \ldots, N_t$. By iterating backward, the optimal investment decisions at every time stage are determined.

Case 2: Bound on the Admissible Number of Trades

We constrain the investment trajectory to include at most $K \in \mathbb{Z}_+$ trades during $t = 0, 1, \ldots, N_t$, whereby we define a *trade* as any reallocation of an investment resulting in a change of the asset identification number i_t. A transition according $i_t = i_{t-1}$ is consequently no trade. The set of admissible states is generated as

$$\mathcal{Z}_t = \left\{ z_t : \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\}, \forall k_t < K \text{ and unique, and } \forall i_t \in \mathcal{I} \right\}. \tag{9}$$

Consequently, the resulting transition graph holds a total of $N_z(t) = 1 + \Sigma_{l=1}^{t}(N_c + N_a)\min(l, K)$ states. The reconstruction of optimal investment decisions is similar to Equation 8.

Note that the total number of states, $N_z(t)$, quickly reaches large numbers. We therefore introduce a heuristic to reduce $N_z(t)$ while not compromising optimality of the solution.

Proposition 1. *While not compromising the finding of an optimal investment trajectory, the set of admissible states $\mathcal{Z}_t$ of Equation 9 can be shrunken to $\widetilde{\mathcal{Z}}_t$ according to the following heuristic:*

1. *Initialize $\widetilde{\mathcal{Z}}_t = \mathcal{Z}_t$.*
2. ***For** every $i_t \in \mathcal{I}$ such that the corresponding $z_t \in \mathcal{Z}_t$ of Equation 9,*
3. *Compute $k_t^{\text{opt}}(i_t) = \{k_t : w_t^{0,\text{opt}}(i_t) = \max\{w_t^0\}$ s.t. corresponding $z_t \in \mathcal{Z}_t$ of Equation 9$\}$.*
4. *Shrink $\widetilde{\mathcal{Z}}_t = \widetilde{\mathcal{Z}}_t \backslash \{z_t : k_t > k_t^{\text{opt}}(i_t)\}$.*
5. ***End For***

Proof. W.l.o.g., suppose that for a given $i_t = i \in \mathcal{I}$, we have determined $k_t^{\text{opt}}(i_t)$. Let the associated state vector be denoted by $z_t^{\text{opt}}(i_t)$. Then, we can discard all z_t with $i_t = i$ and $k_t > k_t^{\text{opt}}(i_t)$, because $w_{t+\tau}^{0,\text{opt}}(i_t) \geq w_{t+\tau}^0, \forall \tau \geq 0$, and the admissible set for state $z_t^{\text{opt}}(i_t)$ is thus larger by at least the option of one additional trade, in comparison with the admissible set corresponding to all $z_t \in \mathcal{Z}_t$ of Equation 9 with $i_t = i$ and $k_t > k_t^{\text{opt}}(i_t)$.

Note that the total number of states, $N_z(N_t)$, cannot be predicted precisely as before. It is now data dependent instead. Quantitative implications are reported later in the Numerical Examples section.

Case 3: Waiting Period after Every Trade until the Next Trade

We constrain the investment trajectory to waiting of at least a specific time period D after every executed trade until the next trade. The set of admissible states is consequently generated as

$$\mathcal{Z}_t = \left\{ z_t : \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\}, \forall d_t < D \text{ and unique, and } \forall i_t \in \mathcal{I} \right\}. \tag{10}$$

The resulting transition graph holds a total of $N_z(t) = 1 + \Sigma_{l=1}^{t}(N_c + N_a - 1)\min(l, D) + 1 + \min(\max(0, l - D), D - 1)$ states. The reconstruction of optimal investment decisions is similar to Equation 8.

Similarly to the previous case, the total number of states, $N_z(t)$, quickly reaches large numbers. We therefore also introduce a heuristic to reduce $N_z(t)$ while not compromising optimality of the solution.

Proposition 2. *While not compromising the finding of an optimal investment trajectory, the set of admissible states $\mathcal{Z}_t$ of (10) can be shrunken to $\overline{\mathcal{Z}}_t$ according to the following heuristic:*

1. *Initialize $\overline{\mathcal{Z}}_t = \mathcal{Z}_t$.*
2. ***For** every $i_t \in \mathcal{I}$ such that the corresponding $z_t \in \mathcal{Z}_t$ of Equation 10,*
3. *Compute $d_t^{\text{opt}}(i_t) = \{d_t : w_t^{0,\text{opt}}(i_t) := \max\{w_t^0\}$ s.t. corresponding $z_t \in \mathcal{Z}_t$ of Equation 10$\}$.*
4. *Shrink $\overline{\mathcal{Z}}_t = \overline{\mathcal{Z}}_t \backslash \{z_t : 0 < d_t < d_t^{\text{opt}}(i_t)\}$.*
5. ***End For***

Proof. W.l.o.g., suppose for a given $i_t = i \in \mathcal{I}$, we have determined $d_t^{\text{opt}}(i_t)$. Let the associated state vector be denoted by $z_t^{\text{opt}}(i_t)$. Then, we can discard all z_t with $i_t = i$ and $0 < d_t < d_t^{\text{opt}}(i_t)$, because $w_{t+\tau}^{0,\text{opt}}(i_t) \geq w_{t+\tau}^0$, $\forall_\tau \geq 0$, and the admissible set for state $z_t^{\text{opt}}(i_t)$ is larger by being closer to a potential next trade by at least one trading sampling time, in comparison to the admissible set corresponding to all $z_t \in \mathcal{Z}_t$ of (10) with $i_t = i$ and $0 < d_t < d_t^{\text{opt}}(i_t)$.

Similar to the previous case, the total number of states, $N_z(N_t)$, cannot be predicted precisely because it is data dependent. Quantitative results are reported in the

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Numerical Examples section. This heuristic significantly reduces computational complexity in practice.

MULTISTAGE TRANSITION DYNAMICS OPTIMIZATION WITH A DIVERSIFICATION CONSTRAINT

In portfolio optimization, the introduction of diversification constraints is regarded as a measure to reduce drawdown risk. For our purpose of analysis of historical optimal trading, we first divide the initial wealth m_0 into Q parts of equal proportion. Then, we impose constraints on each of the corresponding Q investment trajectories. In the unconstrained case, all Q trajectories would coincide. In the constrained case, we distinguish between 1) constraints *between* multiple investment trajectories—diversification at only the initial time, diversification permitted at all times, asynchronous trading and synchronous trading—and 2) constraints *along* any specific investment trajectory— unconstrained trading frequency, at most K trades along the investment trajectory, and the enforcement of a waiting period after each executed trade.

We define a diversification constraint at a specific time t such that each of the states of the Q trajectories, $z_t \in \mathcal{Z}_t^{(q)}$, $\forall q = 0, \ldots, Q-1$, must be invested differently. Thus, each asset identification number $i_t^{(q)}$ must be different, $\forall t = 0, 1, \ldots, N_t$, $\forall q = 0, 1, \ldots, Q-1$.

We define the sets of admissible states $\mathcal{Z}_t^{(q)}$, $\forall t = 0, 1, \ldots, N_t$, and $\forall q = 0, 1, \ldots, Q-1$, sequentially and ordered according to optimality. Thus, $\mathcal{Z}_t^{(1)}$, $\forall t = 0, 1, \ldots, N_t$ is constructed accounting only for the optimal investment trajectory associated with $\mathcal{Z}_t^{(0)}$ (i.e., the set $\mathcal{Z}_t^{(0),*}$, $\forall t = 0, 1, \ldots, N_t$), whereas $\mathcal{Z}_t^{(q)}$ is constructed accounting for all of the optimal investment trajectories associated with $\mathcal{Z}_t^{(0),*}, \mathcal{Z}_t^{(1),*}, \ldots, \mathcal{Z}_t^{(q-1),*}$. Here, $\mathcal{Z}_t^{(q),*}$, $\forall q = 0, 1, \ldots, Q-1$ denotes the set of states at each time t that result from the reconstruction of optimal investment decisions along the optimal investment trajectory according to Equation 8. Thus, our methodology aims at being maximally invested in the investment trajectories ordered according to optimality.

Q Trajectories, Diversification for a Subset of Times and Asynchronous Trading

We define the subset of trading sampling times as $\mathcal{T}^{(q)} \subseteq \{0, 1, \ldots, N_t\}$, $\forall q = 0, 1, \ldots, Q-1$. For enforcement of diversification in form of Q trajectories, diversification for any subset of trading times and asynchronous trading, the sets of admissible states are initialized as

$$Z_0^{(q)} = \{z_0 : z_0 = [0\ \ 0\ \ 0\ \ m_0^{0,q}\ \ 0\ \ m_0^{0,q}\ \ 0\ \ 0]\},$$
$$\forall q = 0, 1, \ldots, Q-1. \tag{11}$$

For unconstrained trading frequency along an investment trajectory and $t > 0$, the sets of admissible states are thus generated according to

$$\mathcal{Z}_t^{(q)} = \Big\{ z_t : \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\},\ \forall i_t \in \mathcal{I} \text{ if } t \notin \mathcal{T}^{(q)}, \text{ or } \ldots$$
$$\forall i_t \in \mathcal{I} \setminus \left(\cup \left\{ i_t : i_t = i_t^{(r),*}, z_t^{(r),*} \in \mathcal{Z}_t^{(r),*} \right\}_{r=0}^{q-1} \right) \text{if } t \in \mathcal{T}^{(q)} \Big\}, \tag{12}$$

with $q = 0, 1, \ldots, Q-1$, and where $z_t^{(r),*} \in \mathcal{Z}_t^{(r)}$ denotes the optimal state at time t associated with investment trajectory r.

For the case of at most K admissible trades along any investment trajectory and $t > 0$, the sets of admissible states are generated according to

$$Z_t^{(q)} = \Big\{ z_t : \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\},\ \forall k_t < K \text{ and unique, and}$$
$$\forall i_t \in \mathcal{I} \text{ if } t \notin \mathcal{T}^{(q)}, \text{ or } \ldots$$
$$\forall k_t < K \text{ and unique, and}$$
$$\forall i_t \in \mathcal{I} \setminus \left(\cup \left\{ i_t : i_t = i_t^{(r),*}, z_t^{(r),*} \in \mathcal{Z}_t^{(r),*} \right\}_{r=0}^{q-1} \right) \text{ if } t \in \mathcal{T}^{(q)} \Big\}, \tag{13}$$

with $q = 0, 1, \ldots, Q-1$.

For the case of enforcing a waiting period after each executed trade along any investment trajectory and $t > 0$, the sets of admissible states are generated according to

$$\mathcal{Z}_t^{(q)} = \Big\{ z_t : \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\},\ \forall d_t < D \text{ and unique, and}$$
$$\forall i_t \in \mathcal{I} \text{ if } t \notin \mathcal{T}^{(q)}, \text{ or } \ldots$$
$$\forall d_t < D \text{ and unique, and}$$
$$\forall i_t \in \mathcal{I} \setminus \left(\cup \left\{ i_t : i_t = i_t^{(r),*}, z_t^{(r),*} \in \mathcal{Z}_t^{(r),*} \right\}_{r=0}^{q-1} \right) \text{ if } t \in \mathcal{T}^{(q)} \Big\}, \tag{14}$$

with $q = 0, 1, \ldots, Q-1$.

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Q Trajectories, Diversification for All Times and Synchronous Trading

Let us define a subset of trading sampling times as $\mathcal{T} \subseteq \{0, 1, \ldots, N_t\}$. This subset may, for example, indicate the sampling times at which trades were executed along the optimal investment trajectory associated with $\mathcal{Z}_t^{(0),*}$:

$$\mathcal{T} = \{t : i_t^{(0),*} \neq j_t^{(0),*},\ z_t^{(0),*} \in \mathcal{Z}_t^{(0),*},\ \forall t = 1, \ldots, N_t\}.$$

The set of a admissible states is initialized as in Equation 11. Then, for an unconstrained trading frequency along an investment trajectory and $t > 0$, the sets of admissible states are generated according to

$$\begin{aligned}\mathcal{Z}_t^{(q)} = \Big\{ & z_t : z_t = z_{t-1} \text{ if } t \notin \mathcal{T}, \text{ or } \ldots \\ & z_t \text{ s.t. } \max_{j_t \in \mathcal{J}_{t-1}^{z_t}} \{w_t^0\}, \\ & \forall i_t \in \mathcal{I} \setminus \left(\cup \left\{ i_t : i_t = i_t^{(r),*}, z_t^{(r),*} \in \mathcal{Z}_t^{(r),*} \right\}_{r=0}^{q-1} \right) \text{ if } t \in \mathcal{T} \Big\},\end{aligned} \tag{15}$$

with $q = 0, 1, \ldots, Q - 1$.

The case of at most K admissible trades along any investment trajectory as well as the case of enforcing a waiting period after each executed trade along any investment trajectory can then be defined analogously.

Remarks and Relevant Quantities for Interpretation

Note that the presented framework can also be extended to analyze alternative optimization criteria, such as determining a worst-case investment trajectory (*pessimization*) or the tracking of a target return reference trajectory (*index tracking*).

In order to interpret quantitative results in the following section, we define the *total return* (measured in percent) as $r_{N_t}^{\text{tot},(q)} = 100 \dfrac{w_{N_t}^0 - w_0^0}{w_0^0}, \quad \forall q \in \mathcal{Q}$. Similarly, we define the return at time t as $r_t^{\text{tot},(q)}$, $\forall q \in \mathcal{Q}$. We further report the total number of conducted trades as $K_{N_t}^{\text{tot}}$. The minimal time span between any two trades within time frame $t \in \mathcal{N}_t = \{0, 1, \ldots, N_t\}$ shall be denoted by $D_{N_t}^{\min}$. In addition, the average, minimal, and maximal percentage gain per conducted noncurrency asset trade is of interest. Stating the quantities with respect to our reference currency (EUR), we therefore first define the set

$$\begin{aligned}\Delta\mathcal{G}^{(q)} = \Big\{ & 100 \frac{w_\tau^0 - w_\eta^0}{w_\eta^0} : \text{with } \tau \text{ s.t. } \tau = t-1, \bar{i} \in \mathcal{I}_{N_a}, i_{t-1} \neq \bar{i}, i_t = \bar{i}, \\ & \text{with } \eta \text{ s.t. } \eta = t, \bar{i} \in \mathcal{I}_{N_a}, i_t = \bar{i}, i_{t+1} \neq \bar{i}, \\ & \text{and } \tau > \eta,\ z_t \in \mathcal{Z}_t^{(q),*}, \forall t \in \mathcal{N}_t, \forall q \in \mathcal{Q} \Big\},\end{aligned}$$

whereby $\bar{i}$ identifies an asset of interest. The average, minimum, and maximum shall then be denoted by $\text{avg}(\Delta\mathcal{G}^{(q)})$, $\min(\Delta\mathcal{G}^{(q)})$ and $\max(\Delta\mathcal{G}^{(q)})$, respectively. The associated trading times are summarized in

$$\begin{aligned}\Delta\mathcal{T}^{(q)} = \Big\{ & \tau - \eta : \text{with } \tau \text{ s.t. } \tau = t-1, \bar{i} \in \mathcal{I}_{N_a}, i_{t-1} \neq \bar{i}, i_t = \bar{i}, \\ & \text{with } \eta \text{ s.t. } \eta = t, \bar{i} \in \mathcal{I}_{N_a}, i_t = \bar{i}, i_{t+1} \neq \bar{i}, \\ & \text{and } \tau > \eta,\ z_t \in \mathcal{Z}_t^{(q),*}, \forall t \in \mathcal{N}_t, \forall q \in \mathcal{Q} \Big\},\end{aligned}$$

with corresponding $\text{avg}(\Delta\mathcal{T}^{(q)})$, $\min(\Delta\mathcal{T}^{(q)})$, and $\max(\Delta\mathcal{T}^{(q)})$ defined accordingly.

Then, we can partition quantities of interest into two groups: 1) overall performance measures and 2) quantities associated with noncurrency asset holdings along an investment-optimal q trajectory. We thus compactly summarize results in evaluation vectors and matrices:

$$e^{(q)} = \left[r_{N_t}^{\text{tot},(q)}\ K_{N_t}^{\text{tot},(q)}\ D_{N_t}^{\min,(q)} \right], \quad \forall q \in \mathcal{Q}, \tag{16}$$

$$E^{(q)} = \begin{bmatrix} \text{avg}(\Delta\mathcal{G}^{(q)}) & \min(\Delta\mathcal{G}^{(q)}) & \max(\Delta\mathcal{G}^{(q)}) \\ \text{avg}(\Delta\mathcal{T}^{(q)}) & \min(\Delta\mathcal{T}^{(q)}) & \max(\Delta\mathcal{T}^{(q)}) \end{bmatrix}, \quad \forall q \in \mathcal{Q}. \tag{17}$$

NUMERICAL EXAMPLES

To quantitatively evaluate results, three numerical examples are reported. For all examples, a time horizon of one year is chosen. The sampling time is selected as one day. Adjusted closing prices of both foreign exchange rates and stock indexes are retrieved from Yahoo! Finance (finance.yahoo.com). As a preprocessing step, all noncurrency assets are normalized to value 100 in their corresponding currency at time $t = 0$.

The first example treats optimal trading of euros, U.S. dollars, and the Nasdaq-100 Index (see Exhibit 2). This scenario is selected mainly to analyze currency effects. No diversification constraint is enforced, such

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EXHIBIT 2
Example 1—Currencies and Assets under Consideration

i	Yahoo! Finance Symbol	Interpretation	$c(i)$
0	—	EUR	0
1	EURUSD = x	USD	1
2	^ NDX	Nasdaq-100	1

Note: The currency in which asset i is traded is denoted by c(i); the reference currency of the Nasdaq-100 is U.S. dollars.

that we have $Q = 1$. The second example treats optimal trading of 16 different currencies and 15 different noncurrency assets. A diversification constraint is employed with $Q = 3$. The third example compares the results for an exemplary downtrending and an uptrending stock.

This section illustrates the effects of 1) different transaction cost levels and 2) various constraints on *a posteriori* optimal trading performance.

Example 1: EUR, USD, and Nasdaq-100

The results for numerical Example 1 are summarized in Exhibit 3. Different levels of transaction costs with variable proportional cost but constant fixed cost are considered. Exhibit 3 reports the evaluation quantities $e^{(0)}$ and $E^{(0)}$ for the different trading strategies. For the buy-and-hold strategy, only $e^{(0)}$ is reported. We assume proportional costs (in percentages) to be the same for buying and selling for both foreign exchange and asset trading (i.e., $\epsilon = \epsilon_{\text{buy}} = \epsilon_{\text{sell}}$). For $\epsilon = 0$, we also set $\beta = 0$. For all other cases, we set $\beta = 50$. Total returns ($r_{N_t}^{\text{tot},(q)}$) are shown in bold font for emphasis. The time span of interest is August 5, 2015, to August 3, 2016, and comprises 251 potential trading days.

Several observations can be made with respect to the results of Exhibit 3. First, even though only two currencies, EUR and USD (i.e., $i \in \mathcal{I}_{N_c} = \{0,1\}$), and one noncurrency asset (i.e., $i \in \mathcal{I}_{N_a} = \{2\}$), are traded long only, remarkable profits can be earned when optimally trading *a posteriori*. Even in case of (high) transaction costs with a proportional rate of 2%, the profits significantly outperform a one-year buy-and-hold strategy. Second, the influence of different levels of transaction costs is impressive. This holds specifically for unconstrained trading with respect to returns, optimal trading frequency, and percentage gains (average, minimum, and maximum) upon which the noncurrency asset is traded. Third, while the total return drops with increasing transaction cost levels, the remaining evaluation quantities remain approximately constant for the K trades strategy (here $K = 12$, i.e., 12 trades per year or one per month). Fourth, the results associated with the percentage gains upon which the noncurrency asset is traded were unexpected. Intuitively, they were thought to be higher. The same holds for optimal time periods between any two trades. Results from Example 1 encourage frequent trading. For example, for the case with a waiting constraint, trading is encouraged upon percentage gains of on average slightly less than 10% for all four levels of transaction costs.

Exhibit 4 further visualizes results. In order to compactly display multiple foreign exchange rates, we normalize w.r.t. the initial value at $t = 0$; see the subplot with label Δx_t^{norm}. For reference currency EUR, we set $\Delta x_t^{\text{norm}} = 0,\ \forall t = 0, 1, \ldots, N_t$. Analogously, we normalize noncurrency prices and additionally take currency effects into account by first converting prices to currency EUR; see the subplot with label $\Delta p_{t,\text{EUR}}^{\text{norm}}$. For a specific optimal investment trajectory, at every time t, an investment in exactly one currency or noncurrency asset is taken. Being invested in a noncurrency asset is indicated by the dark circles in Exhibit 4. Because noncurrency assets are associated with a specific currency, we also label them accordingly with dark circles. In contrast, an explicit investment in a currency is emphasized by white circles.

It is striking that despite an absence of clear trends in both the EUR/USD foreign exchange rate and the Nasdaq-100 stock index, significant profits can be made when optimally trading—even when employing a *long-only* strategy. The largest increases in return rates in currency EUR are achieved when the asset is increasing in value while the foreign exchange rate with reference euro is decreasing. Investments in USD are optimal when the EUR/USD foreign exchange rate is trending down and the Nasdaq-100 is decreasing likewise. Investments in EUR are in general optimal when the EUR/USD foreign exchange rate is trending up and the Nasdaq-100 is trending down.

Example 2: Global Investing and Including a Diversification Constraint

We consider 16 currencies and 15 noncurrency assets. Real-world data are obtained according to

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EXHIBIT 3

Summary of Quantitative Results of Example 1

Strategy	$\epsilon = 0$	$\epsilon = 0.5$	$\epsilon = 1$	$\epsilon = 2$
Buy and Hold	$\begin{bmatrix} \mathbf{-0.2} & 1 & 250 \end{bmatrix}$	$\begin{bmatrix} \mathbf{-1.2} & 1 & 250 \end{bmatrix}$	$\begin{bmatrix} \mathbf{-2.2} & 1 & 250 \end{bmatrix}$	$\begin{bmatrix} \mathbf{-4.2} & 1 & 250 \end{bmatrix}$
Unconstrained	$\begin{bmatrix} \mathbf{330.0} & 165 & 1 \end{bmatrix}$ $\begin{bmatrix} 2.0 & 0.1 & 11.0 \\ 1.8 & 1 & 5 \end{bmatrix}$	$\begin{bmatrix} \mathbf{134.6} & 59 & 1 \end{bmatrix}$ $\begin{bmatrix} 3.9 & 0.9 & 11.8 \\ 5.5 & 1 & 21 \end{bmatrix}$	$\begin{bmatrix} \mathbf{86.2} & 31 & 1 \end{bmatrix}$ $\begin{bmatrix} 5.9 & 0.4 & 16.6 \\ 12.2 & 1 & 45 \end{bmatrix}$	$\begin{bmatrix} \mathbf{50.1} & 14 & 1 \end{bmatrix}$ $\begin{bmatrix} 8.2 & 3.1 & 15.5 \\ 20.0 & 3 & 45 \end{bmatrix}$
≤12 Trades	$\begin{bmatrix} \mathbf{106.8} & 12 & 6 \end{bmatrix}$ $\begin{bmatrix} 11.9 & 7.1 & 17.8 \\ 23.7 & 11 & 45 \end{bmatrix}$	$\begin{bmatrix} \mathbf{87.9} & 12 & 6 \end{bmatrix}$ $\begin{bmatrix} 11.0 & 6.6 & 17.2 \\ 23.7 & 11 & 45 \end{bmatrix}$	$\begin{bmatrix} \mathbf{72.5} & 12 & 5 \end{bmatrix}$ $\begin{bmatrix} 10.1 & 5.8 & 16.6 \\ 23.8 & 11 & 45 \end{bmatrix}$	$\begin{bmatrix} \mathbf{49.6} & 12 & \mathbf{1} \end{bmatrix}$ $\begin{bmatrix} 9.2 & 5.0 & 15.5 \\ 23.7 & 11 & 45 \end{bmatrix}$
≥10 Days Waiting	$\begin{bmatrix} \mathbf{114.2} & 18 & 10 \end{bmatrix}$ $\begin{bmatrix} 9.1 & 2.7 & 16.6 \\ 15.3 & 10 & 21 \end{bmatrix}$	$\begin{bmatrix} \mathbf{84.8} & 17 & 10 \end{bmatrix}$ $\begin{bmatrix} 9.2 & 2.7 & 16.0 \\ 16.1 & 11 & 21 \end{bmatrix}$	$\begin{bmatrix} \mathbf{68.4} & 13 & 10 \end{bmatrix}$ $\begin{bmatrix} 9.8 & 6.0 & 15.4 \\ 22.3 & 11 & 45 \end{bmatrix}$	$\begin{bmatrix} \mathbf{46.5} & 10 & 11 \end{bmatrix}$ $\begin{bmatrix} 9.3 & 5.9 & 15.5 \\ 25.4 & 12 & 45 \end{bmatrix}$

Note: Total returns $\left(r_{N_t}^{\mathrm{tot},(q)}\right)$ are shown in bold font for emphasis.

EXHIBIT 4

Example 1—the Unconstrained Trading Case in the Absence of Any Transaction Costs

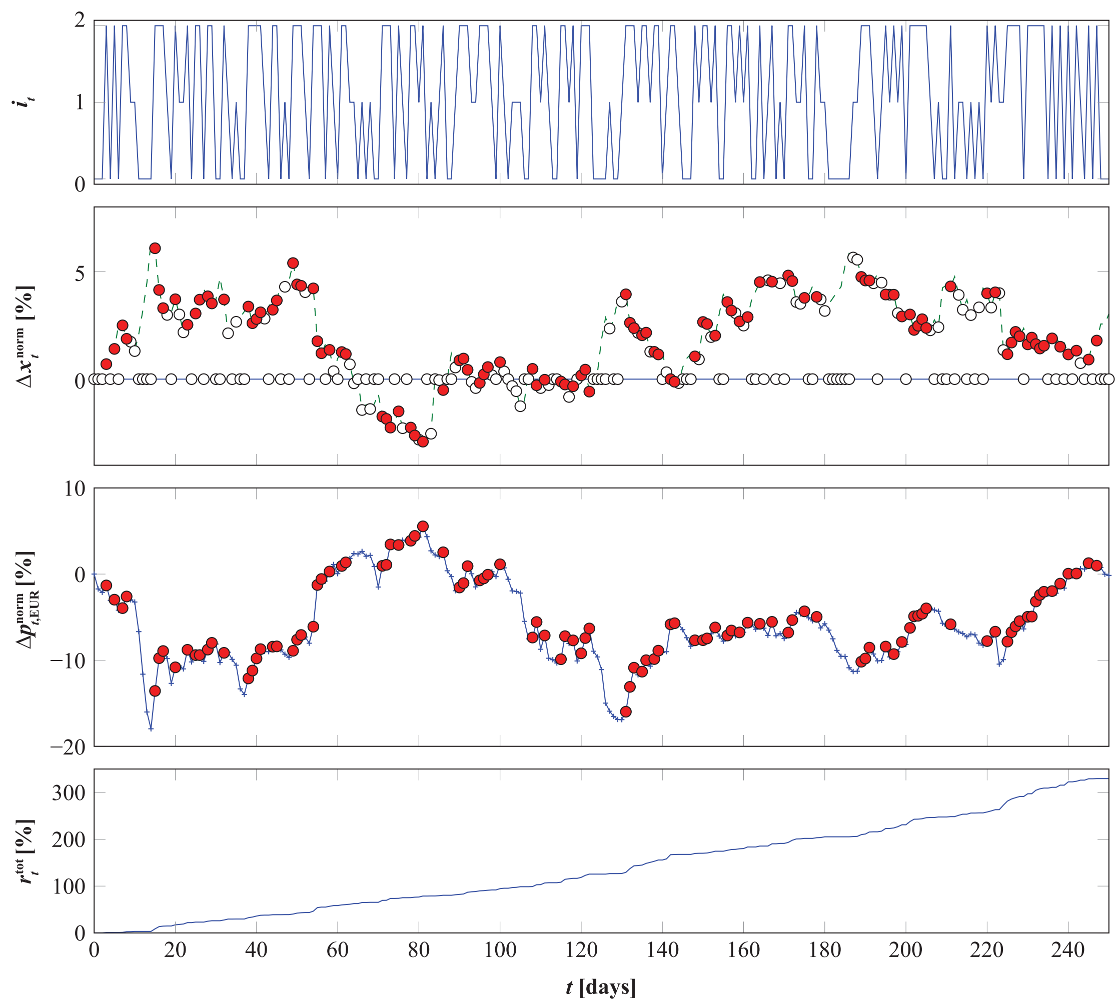

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Exhibit 5. We consider the time horizon August 5, 2015, to August 3, 2016. Because of different trading holidays in the different countries, a total of 199 trading days could be determined common to all assets. We diversify in three assets at every trading time t; that is, we set $Q = 3$.

We distinguish between two cases: *asynchronous* and *synchronous* trading. Exhibits 6 and 7 summarize the quantitative results for asynchronous and synchronous trading, respectively. The results for all Q trajectories are shown, and the Summary rows report the sum of returns of all Q trajectories. Results are further visualized in Exhibit 8. The black-dashed horizontal line in the corresponding top subplots denotes $N_c = 16$ to distinguish currency and noncurrency asset investments.

For performance comparison, we consider a buy-and-hold strategy, whereby an asset is bought initially and then held. The most performant noncurrency assets from Exhibit 5 for the time frame of interest were, in order, the IBOVESPA (BRA), the Dow Jones Russia GDR (RUS) and the S&P 500 (USA). Associated returns are reported in Exhibits 6 and 7, where we attribute the IBOVESPA to $q = 0$ and the other two assets to $q = 1$ and $q = 2$.

Interpretation of results is in line with earlier discussion. In particular, the influence of transaction costs and the encouragement of frequent trading upon relatively small percentage gains are recurrent.

A remark about computational complexity needs to be made. The total number of states, $N_z(N_t)$, without consideration of any heuristics is 6139, 71611, and 59905 for the three cases (respectively, unconstrained, constraint of at most $K = 12$ trades, and constraint of waiting at least $D = 10$ days between any two trades). These numbers can be computed according to the formulas stated earlier. Then, applying the previously discussed heuristics to the given Yahoo! Finance data trajectories, we measured (to give one example) $N_z(N_t) = 65{,}238$ and $N_z(N_t) = 33{,}161$ for the latter two cases, $q = 0$ and $\epsilon = 0$. Similar results are obtained for the other transaction cost levels and the other Q trajectories, resulting in overall computation times (for all $q = 0, 1, 2$) in the tens of minutes. In contrast, for the unconstrained case, overall computation times for the generation of all $Q = 3$ transition graphs were, on average, only slightly more than 10 seconds, thereby making the unconstrained case much more suitable for fast analysis of sets of multiple assets and foreign exchange rate trajectories.

Exhibit 5

Example 2—Identification of 16 Currencies and 15 Assets

i	Yahoo! Finance Symbol	Interpretation	$c(i)$
0	–	EUR	0
1	EURUSD = x	USD	1
2	EURJPY = x	JPY	2
3	EURGBP = x	GBP	3
4	EURCHF = x	CHF	4
5	EURCNY = x	CNY	5
6	EURDKK = x	DKK	6
7	EURHKD = x	HKD	7
8	EURNOK = x	NOK	8
9	EURRUB = x	RUB	9
10	EURBRL = x	BRL	10
11	EURAUD = x	AUD	11
12	EURCAD = x	CAD	12
13	EURTRY = x	TRY	13
14	EURZAR = x	ZAR	14
15	EURINR = x	INR	15
16	^ GDAXI	DAX (GER)	0
17	FTSEMIB.MI	FTSEMIB (ITA)	0
18	^ OSEAX	OSEAX (NOR)	8
19	CSSMI.SW	SMI (SUI)	4
20	^ NDX	Nasdaq-100 (USA)	1
21	^ GSPC	S&P 500 (USA)	1
22	^ N225	NIKKEI 225 (JPN)	2
23	^ HSI	Hang-Seng (HKG)	7
24	^ BVSP	IBOVESPA (BRA)	10
25	^ AORD	All Ordinaries (AUS)	11
26	^ GSPTSE	S&P/TSX (CAN)	12
27	AFS.PA	FTSE/JSE (RSA)	14
28	ARUS.P	Dow Jones Russia (RUS)	0
29	INR.PA	MSCI India (IND)	0
30	TUR.PA	Dow Jones Turkey (TUR)	0

Notes: Each currency is associated with a foreign exchange rate with respect to EUR. The currency in which an asset i is traded is denoted by c(i).

Additionally, the trajectories for $q = 0$ are identical for both time-asynchronous and time-synchronous trading. However, for the remaining investment trajectories with $q > 0$, the number of states is much lower for time-synchronous trading in comparison to the asynchronous case. For time-synchronous trading, $Q = 3$ and a bound on the total admissible number of trades, the total number of states is 63,803 for $q = 0$, but 39,713 and 23,549 for $q = 1$ and $q = 2$, respectively. All numerical experiments throughout this article were conducted on

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EXHIBIT 6

Summary of Quantitative Results of Example 2 for the First Case: Time-Asynchronous Trading with Diversification for All Times

	Strategy	$\epsilon = 0$	$\epsilon = 0.5$	$\epsilon = 1$	$\epsilon = 2$
$q = 0$	Buy and Hold	$\begin{bmatrix}\mathbf{18.2} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{17.0} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{15.9} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{13.6} & 1 & 199\end{bmatrix}$
		$\begin{bmatrix}\mathbf{17450.5} & 184 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{3092.0} & 126 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{1099.5} & 95 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{345.5} & 50 & 1\end{bmatrix}$
	Unconstrained	$\begin{bmatrix}3.3 & 0.1 & 19.5\\ 1.1 & 1 & 3\end{bmatrix}$	$\begin{bmatrix}4.3 & 0.2 & 21.9\\ 1.7 & 1 & 5\end{bmatrix}$	$\begin{bmatrix}5.9 & 0.7 & 24.7\\ 2.7 & 1 & 8\end{bmatrix}$	$\begin{bmatrix}8.6 & 1.8 & 26.2\\ 5.0 & 1 & 21\end{bmatrix}$
		$\begin{bmatrix}\mathbf{453.9} & 12 & 3\end{bmatrix}$	$\begin{bmatrix}\mathbf{377.9} & 12 & 3\end{bmatrix}$	$\begin{bmatrix}\mathbf{326.4} & 12 & 2\end{bmatrix}$	$\begin{bmatrix}\mathbf{258.7} & 12 & 2\end{bmatrix}$
	≤12 Trades	$\begin{bmatrix}19.0 & 5.6 & 56.7\\ 13.8 & 3 & 55\end{bmatrix}$	$\begin{bmatrix}12.0 & 7.2 & 17.8\\ 22.5 & 11 & 45\end{bmatrix}$	$\begin{bmatrix}20.8 & 5.4 & 54\\ 18.4 & 3 & 49\end{bmatrix}$	$\begin{bmatrix}22.3 & 5.4 & 50.1\\ 19.0 & 3 & 46\end{bmatrix}$
		$\begin{bmatrix}\mathbf{440.8} & 17 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{341.2} & 16 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{269.3} & 16 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{186.6} & 12 & 10\end{bmatrix}$
	≥10 Days Waiting	$\begin{bmatrix}12.9 & 3.8 & 33.5\\ 11.7 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}11.5 & 2.1 & 31.4\\ 11.7 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}11.1 & 1.8 & 29.5\\ 12.4 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}17.9 & 12.1 & 26.7\\ 18.5 & 11 & 36\end{bmatrix}$
$q = 1$	Buy and Hold	$\begin{bmatrix}\mathbf{3.3} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{2.8} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{2.3} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{173.2} & 38 & 1\end{bmatrix}$
		$\begin{bmatrix}\mathbf{3671.6} & 190 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{769.0} & 123 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{378.5} & 76 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{173.2} & 38 & 1\end{bmatrix}$
	Unconstrained	$\begin{bmatrix}2.4 & 0.02 & 7.5\\ 1.0 & 1 & 2\end{bmatrix}$	$\begin{bmatrix}3.3 & 0.4 & 9.0\\ 1.8 & 1 & 6\end{bmatrix}$	$\begin{bmatrix}4.8 & 1.5 & 9.0\\ 3.3 & 1 & 9\end{bmatrix}$	$\begin{bmatrix}8.0 & 2.8 & 22.5\\ 7.0 & 2 & 19\end{bmatrix}$
		$\begin{bmatrix}\mathbf{227.0} & 12 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{212.2} & 12 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{173.0} & 12 & 4\end{bmatrix}$	$\begin{bmatrix}\mathbf{129.6} & 12 & 3\end{bmatrix}$
	≤12 Trades	$\begin{bmatrix}12.8 & 0.5 & 36.0\\ 15.5 & 1 & 43\end{bmatrix}$	$\begin{bmatrix}14.5 & 2.6 & 34.5\\ 16.2 & 1 & 43\end{bmatrix}$	$\begin{bmatrix}13.8 & 2.6 & 33.2\\ 18.3 & 7 & 43\end{bmatrix}$	$\begin{bmatrix}18.5 & 9.5 & 30.5\\ 23.7 & 7 & 43\end{bmatrix}$
		$\begin{bmatrix}\mathbf{267.5} & 17 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{200.5} & 17 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{153.2} & 17 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{113.3} & 15 & 10\end{bmatrix}$
	≥10 Days Waiting	$\begin{bmatrix}10.4 & 2.0 & 18.4\\ 11.2 & 1 & 15\end{bmatrix}$	$\begin{bmatrix}9.8 & 3.8 & 17.2\\ 11.5 & 10 & 14\end{bmatrix}$	$\begin{bmatrix}9.2 & 1.5 & 14.5\\ 11.0 & 8 & 14\end{bmatrix}$	$\begin{bmatrix}9.1 & 1.2 & 22.1\\ 14.3 & 6 & 33\end{bmatrix}$
$q = 2$	Buy and Hold	$\begin{bmatrix}\mathbf{1.7} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{0.6} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{-0.4} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{-2.4} & 1 & 199\end{bmatrix}$
		$\begin{bmatrix}\mathbf{1882.1} & 191 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{459.7} & 113 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{229.3} & 66 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{102.9} & 37 & 1\end{bmatrix}$
	Unconstrained	$\begin{bmatrix}2.0 & 2e– & 6.9\\ 1.0 & 1 & 8\end{bmatrix}$	$\begin{bmatrix}3.0 & 0.4 & 11.4\\ 2.0 & 1 & 8\end{bmatrix}$	$\begin{bmatrix}4.9 & 1.2 & 12.0\\ 3.7 & 1 & 10\end{bmatrix}$	$\begin{bmatrix}6.4 & 2.9 & 10.0\\ 6.7 & 1 & 16\end{bmatrix}$
		$\begin{bmatrix}\mathbf{186.3} & 12 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{158.0} & 12 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{137.9} & 12 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{89.0} & 12 & 3\end{bmatrix}$
	≤12 Trades	$\begin{bmatrix}12.9 & 7.2 & 31.8\\ 16.6 & 1 & 67\end{bmatrix}$	$\begin{bmatrix}11.8 & 5.8 & 30.6\\ 15.3 & 1 & 49\end{bmatrix}$	$\begin{bmatrix}11.9 & 7.5 & 24.6\\ 18.0 & 5 & 37\end{bmatrix}$	$\begin{bmatrix}10.7 & 5.9 & 18.3\\ 18.0 & 8 & 36\end{bmatrix}$
		$\begin{bmatrix}\mathbf{215.2} & 18 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{164.1} & 18 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{116.2} & 15 & 10\end{bmatrix}$	$\begin{bmatrix}\mathbf{70.7} & 11 & 10\end{bmatrix}$
	≥10 Days Waiting	$\begin{bmatrix}9.2 & 3.3 & 13.9\\ 10.8 & 9 & 14\end{bmatrix}$	$\begin{bmatrix}7.5 & 2.5 & 12.4\\ 10.6 & 8 & 14\end{bmatrix}$	$\begin{bmatrix}9.0 & 4.9 & 12.3\\ 13.3 & 10 & 20\end{bmatrix}$	$\begin{bmatrix}9.7 & 5.8 & 18.3\\ 17.6 & 10 & 29\end{bmatrix}$
Summary					
	Buy and Hold	**23.2**	**20.4**	**17.8**	**12.5**
	Unconstrained	**23,004.2**	**4,320.7**	**1,707.3**	**630.6**
	≤12 Trades	**867.2**	**748.2**	**637.3**	**477.3**
	≥10 Days Waiting	**923.5**	**705.8**	**538.7**	**370.6**

Note: Total returns $\left(r_{N_t}^{\mathrm{tot},(q)}\right)$ *are shown in bold font for emphasis.*

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EXHIBIT 7

Summary of Quantitative Results of Example 2 for the Second Case: Time-Synchronized Trading with Diversification for All Times

	Strategy	$\epsilon = 0$	$\epsilon = 0.5$	$\epsilon = 1$	$\epsilon = 2$
$q = 0$	Buy and Hold	$\begin{bmatrix}\mathbf{18.2} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{17.0} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{15.9} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{13.6} & 1 & 199\end{bmatrix}$
	Unconstrained	$\begin{bmatrix}\mathbf{17450.5} & 184 & 1\end{bmatrix}$ $\begin{bmatrix}3.3 & 0.1 & 19.5\\ 1.1 & 1 & 3\end{bmatrix}$	$\begin{bmatrix}\mathbf{3092.0} & 126 & 1\end{bmatrix}$ $\begin{bmatrix}4.3 & 0.2 & 21.9\\ 1.7 & 1 & 5\end{bmatrix}$	$\begin{bmatrix}\mathbf{1099.5} & 95 & 1\end{bmatrix}$ $\begin{bmatrix}5.9 & 0.7 & 24.7\\ 2.7 & 1 & 8\end{bmatrix}$	$\begin{bmatrix}\mathbf{345.5} & 50 & 1\end{bmatrix}$ $\begin{bmatrix}8.6 & 1.8 & 26.2\\ 5.0 & 1 & 21\end{bmatrix}$
	≤12 Trades	$\begin{bmatrix}\mathbf{454.3} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}19.0 & 7.0 & 56.7\\ 13.8 & 2 & 55\end{bmatrix}$	$\begin{bmatrix}\mathbf{377.9} & 12 & 3\end{bmatrix}$ $\begin{bmatrix}19.0 & 5.4 & 54.3\\ 15.3 & 3 & 55\end{bmatrix}$	$\begin{bmatrix}\mathbf{326.4} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}20.8 & 5.4 & 54.0\\ 18.4 & 3 & 49\end{bmatrix}$	$\begin{bmatrix}\mathbf{258.7} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}22.3 & 5.4 & 50.1\\ 19.0 & 3 & 46\end{bmatrix}$
	≥10 Days Waiting	$\begin{bmatrix}\mathbf{440.8} & 17 & 10\end{bmatrix}$ $\begin{bmatrix}12.9 & 3.8 & 33.5\\ 11.7 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}\mathbf{341.2} & 16 & 10\end{bmatrix}$ $\begin{bmatrix}11.5 & 2.1 & 31.4\\ 11.7 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}\mathbf{269.3} & 16 & 10\end{bmatrix}$ $\begin{bmatrix}11.1 & 1.8 & 29.5\\ 12.4 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}\mathbf{186.6} & 12 & 10\end{bmatrix}$ $\begin{bmatrix}17.9 & 12.1 & 26.7\\ 18.5 & 11 & 36\end{bmatrix}$
$q = 1$	Buy and Hold	$\begin{bmatrix}\mathbf{1.7} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{0.6} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{-0.4} & 1 & 199\end{bmatrix}$	$\begin{bmatrix}\mathbf{-2.4} & 1 & 199\end{bmatrix}$
	Unconstrained	$\begin{bmatrix}\mathbf{3009.4} & 177 & 1\end{bmatrix}$ $\begin{bmatrix}2.4 & 0.02 & 7.8\\ 1.1 & 1 & 3\end{bmatrix}$	$\begin{bmatrix}\mathbf{523.5} & 89 & 1\end{bmatrix}$ $\begin{bmatrix}3.6 & 0.4 & 13.0\\ 2.5 & 1 & 8\end{bmatrix}$	$\begin{bmatrix}\mathbf{259.9} & 46 & 1\end{bmatrix}$ $\begin{bmatrix}6.1 & 0.6 & 18.6\\ 5.8 & 1 & 17\end{bmatrix}$	$\begin{bmatrix}\mathbf{137.8} & 34 & 1\end{bmatrix}$ $\begin{bmatrix}9.1 & 2.9 & 25.9\\ 10.3 & 2 & 23\end{bmatrix}$
	≤12 Trades	$\begin{bmatrix}\mathbf{142.8} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}8.9 & 5.0 & 18.0\\ 11.3 & 2 & 55\end{bmatrix}$	$\begin{bmatrix}\mathbf{112.7} & 12 & 3\end{bmatrix}$ $\begin{bmatrix}8.2 & 0.8 & 16.8\\ 13.5 & 4 & 55\end{bmatrix}$	$\begin{bmatrix}\mathbf{87.7} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}9.8 & 5.4 & 19.7\\ 14.8 & 4 & 49\end{bmatrix}$	$\begin{bmatrix}\mathbf{67.3} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}11.1 & 3.9 & 22.4\\ 15.3 & 4 & 46\end{bmatrix}$
	≥10 Days Waiting	$\begin{bmatrix}\mathbf{179.1} & 17 & 10\end{bmatrix}$ $\begin{bmatrix}7.6 & 0.7 & 15.7\\ 11.7 & 5 & 16\end{bmatrix}$	$\begin{bmatrix}\mathbf{135.9} & 14 & 10\end{bmatrix}$ $\begin{bmatrix}8.1 & 0.8 & 15.1\\ 12.8 & 5 & 24\end{bmatrix}$	$\begin{bmatrix}\mathbf{116.0} & 12 & 10\end{bmatrix}$ $\begin{bmatrix}9.0 & 2.1 & 23.4\\ 16.6 & 10 & 41\end{bmatrix}$	$\begin{bmatrix}\mathbf{73.2} & 9 & 10\end{bmatrix}$ $\begin{bmatrix}11.3 & 6.0 & 14.8\\ 19.8 & 11 & 32\end{bmatrix}$
$q = 2$	Buy and Hold	$\begin{bmatrix}\mathbf{3.3} & 1 & 250\end{bmatrix}$	$\begin{bmatrix}\mathbf{2.1} & 1 & 250\end{bmatrix}$	$\begin{bmatrix}\mathbf{1.1} & 1 & 250\end{bmatrix}$	$\begin{bmatrix}\mathbf{-0.9} & 1 & 250\end{bmatrix}$
	Unconstrained	$\begin{bmatrix}\mathbf{1640.8} & 177 & 1\end{bmatrix}$ $\begin{bmatrix}2.1 & 0.02 & 10.6\\ 1.1 & 1 & 4\end{bmatrix}$	$\begin{bmatrix}\mathbf{342.9} & 91 & 1\end{bmatrix}$ $\begin{bmatrix}3.2 & 0.1 & 9.5\\ 2.4 & 1 & 6\end{bmatrix}$	$\begin{bmatrix}\mathbf{170.3} & 45 & 1\end{bmatrix}$ $\begin{bmatrix}5.8 & 0.4 & 11.4\\ 6.1 & 1 & 21\end{bmatrix}$	$\begin{bmatrix}\mathbf{87.9} & 20 & 1\end{bmatrix}$ $\begin{bmatrix}7.9 & 3.9 & 11.7\\ 11.0 & 3 & 22\end{bmatrix}$
	≤12 Trades	$\begin{bmatrix}\mathbf{114.8} & 12 & 2\end{bmatrix}$ $\begin{bmatrix}8.2 & 4.6 & 14.6\\ 11.6 & 2 & 55\end{bmatrix}$	$\begin{bmatrix}\mathbf{92.2} & 11 & 3\end{bmatrix}$ $\begin{bmatrix}9.4 & 0.6 & 24.6\\ 15.7 & 3 & 59\end{bmatrix}$	$\begin{bmatrix}\mathbf{74.4} & 11 & 2\end{bmatrix}$ $\begin{bmatrix}10.3 & 1.8 & 28.2\\ 18.3 & 3 & 53\end{bmatrix}$	$\begin{bmatrix}\mathbf{50.0} & 7 & 6\end{bmatrix}$ $\begin{bmatrix}15.9 & 3.8 & 25.6\\ 36.0 & 6 & 53\end{bmatrix}$
	≥10 Days Waiting	$\begin{bmatrix}\mathbf{134.2} & 16 & 10\end{bmatrix}$ $\begin{bmatrix}6.4 & 0.3 & 11.7\\ 12.4 & 10 & 16\end{bmatrix}$	$\begin{bmatrix}\mathbf{102.9} & 13 & 10\end{bmatrix}$ $\begin{bmatrix}8.4 & 0.5 & 12.2\\ 17.3 & 11 & 33\end{bmatrix}$	$\begin{bmatrix}\mathbf{87.8} & 12 & 11\end{bmatrix}$ $\begin{bmatrix}7.3 & 4.5 & 10.1\\ 14.7 & 13 & 21\end{bmatrix}$	$\begin{bmatrix}\mathbf{52.8} & 7 & 10\end{bmatrix}$ $\begin{bmatrix}13.9 & 8.1 & 19.3\\ 26.7 & 25 & 32\end{bmatrix}$
Summary					
	Buy and Hold	**23.2**	**20.4**	**17.8**	**12.5**
	Unconstrained	**22,100.7**	**3,958.4**	**1,529.7**	**580.2**
	≤12 Trades	**711.9**	**582.8**	**488.5**	**376.0**
	≥10 Days Waiting	**754.1**	**580.0**	**473.1**	**312.3**

Note: Total returns $\left(r_{N_t}^{\mathrm{tot},(q)}\right)$ *are shown in bold font for emphasis.*

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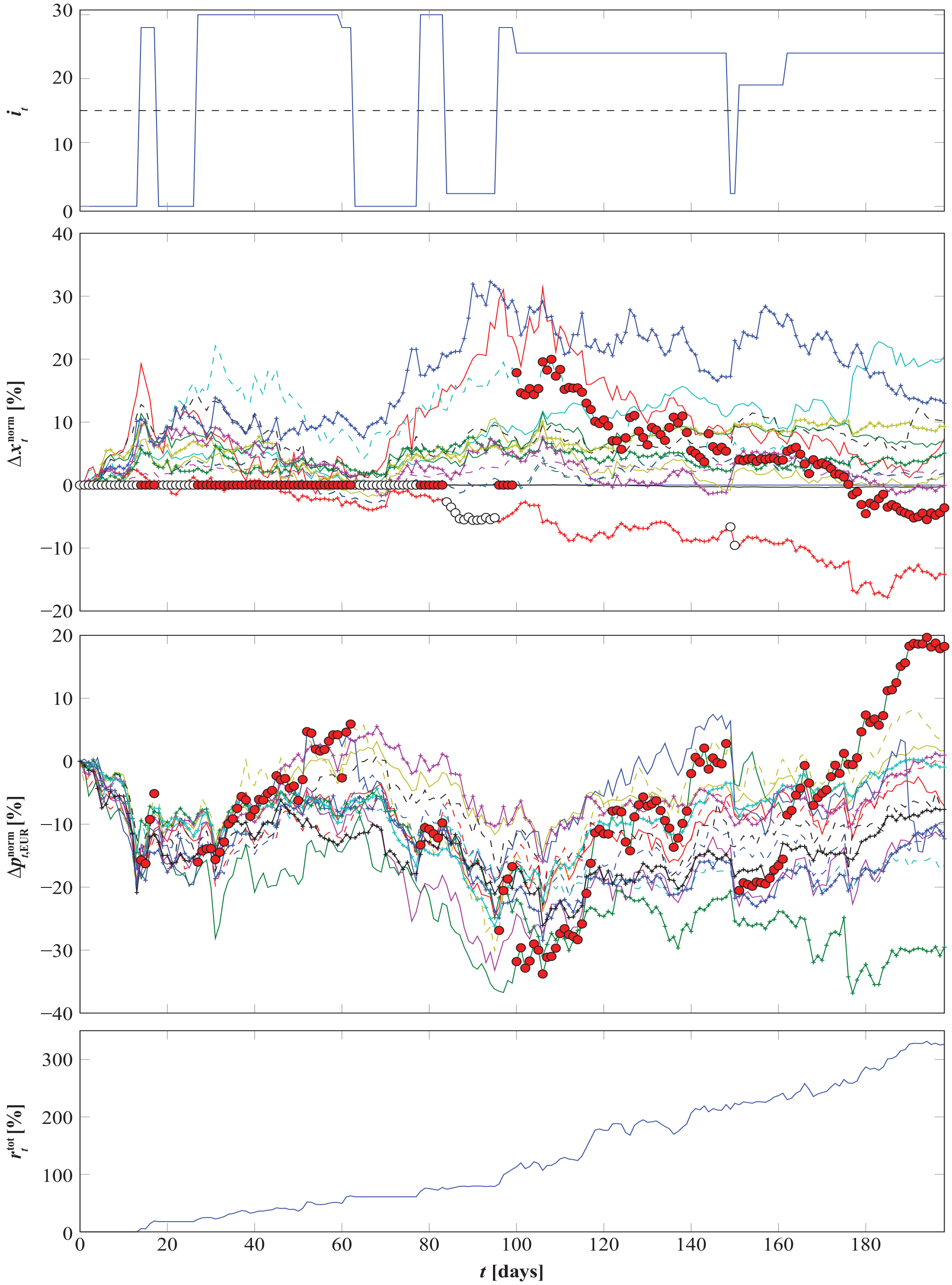
EXHIBIT 8

Example 2—Results for q = 0, at Most K = 12 Admissible Trades, and Transaction Cost Level $\epsilon = 1$



Note: For q = 0, the results for asynchronous and synchronous trading are identical. Furthermore, similarly to Exhibit 4, the normalized evolutions of 16 foreign exchange rates and 15 noncurrency asset prices are displayed in the two middle subplots, respectively. At every time t, an investment in exactly one currency or noncurrency asset is taken. Being invested in a noncurrency asset is indicated by dark circles. Because noncurrency assets are associated with a specific currency, we also label them accordingly with dark circles. In contrast, an explicit investment in a currency is emphasized by white circles.

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EXHIBIT 9

Summary of Quantitative Results of Example 3

DKB.DE	$\epsilon = 1$	ADS.DE	$\epsilon = 1$
Buy and Hold	$\begin{bmatrix}\mathbf{-61.4} & 1 & 260\end{bmatrix}$	Buy and Hold	$\begin{bmatrix}\mathbf{95.6} & 1 & 260\end{bmatrix}$
	$\begin{bmatrix}\mathbf{382.8} & 51 & 1\end{bmatrix}$		$\begin{bmatrix}\mathbf{249.4} & 43 & 1\end{bmatrix}$
Unconstrained	$\begin{bmatrix}7.5 & 1.5 & 21.8 \\ 4.2 & 1 & 11\end{bmatrix}$	Unconstrained	$\begin{bmatrix}7.0 & 1.0 & 20.1 \\ 7.5 & 2 & 24\end{bmatrix}$

Note: This exhibit shows a comparison of a downtrending (DBK.DE) and an uptrending stock (ADS.DE) for the time period between August 10, 2015, and August 8, 2016.

a laptop running Ubuntu 14.04 equipped with an Intel Core i7 CPU @ 2.80GHz × 8, 15.6 GB of memory, and using Python 2.7.

Example 3: A Downtrending and an Uptrending Stock

The ultimate example compares achievable performances for an exemplary downtrending and an uptrending stock. The exemplary downtrending stock is of Deutsche Bank AG (Yahoo! Finance Symbol: DKB.DE). The exemplary uptrending stock is of Adidas AG (Yahoo! Finance Symbol: ADS.DE). Both stocks are listed in the German stock index (DAX). The time frame considered is August 10, 2015, to August 8, 2016. There are 260 potential trading days. Both stocks are traded in currency EUR. We thus find optimal investment trajectories when 1) trading DKB.DE and EUR, and 2) trading ADS.DE and EUR. We assume proportional costs of 1% identical for buying and selling (i.e., $\epsilon = \epsilon_{\text{buy}} = \epsilon_{\text{sell}}$). We set $\beta = 50$. Results are summarized in Exhibits 9 and 10.

Unexpectedly and remarkably, the yearly return associated with the optimal investment trajectory of the downtrending stock is higher than its uptrending counterpart: 382.8% versus 249.2%. Importantly, note that the corresponding buy-and-hold returns are −61.4% and 95.6%, respectively. While overall downtrending, the price of DKB.DE indicates temporary steep price increases. Furthermore, these occur mostly toward the second half of the time period of interest and thus imply stronger return growth due to the already compounded portfolio wealth that is available for investing at that time (instead of the initial m_0). Naturally, without *a posteriori* knowledge of price evolutions, an uptrending stock such as ADS.DE offers the advantage that missing the right selling dates is less important. Interestingly, both downtrending and uptrending are traded optimally upon similar short-term average price increases: 7.5% and 7%. Similarly, the optimal holding periods of the stocks are short with on average 4.2 and 7.8 days, respectively.

CONCLUSION

We developed a simple graph-based method for *a posteriori* (historical) multivariate, multistage optimal trading under transaction costs and a diversification constraint. Three variants were discussed, including unconstrained trading frequency, a fixed number of total admissible trades, and the waiting of a specific time period after every executed trade until the next trade. Findings were evaluated quantitatively on real-world data.

The results illustrated that transaction cost levels are decisive for achievable performance and significantly influence optimal trading frequency. Quantitative results further indicated optimal trading upon occasion rather than on fixed trading intervals and, dependent on transaction cost levels, upon single- to low double-digit percentage gains with respect to the reference currency and exploiting short-term trends. Achievable returns for optimized trading are incomparably outperforming buy-and-hold strategies. Naturally, these returns are very difficult to achieve in practice without knowledge of future price and foreign exchange rate evolutions.

The fundamental motivation and possibly best application of this work is to use it for 1) the preparatory and *automated labeling* of financial time-series data, which is almost unlimitedly available, and where transaction cost level ϵ can then be regarded as a hyper-parameter

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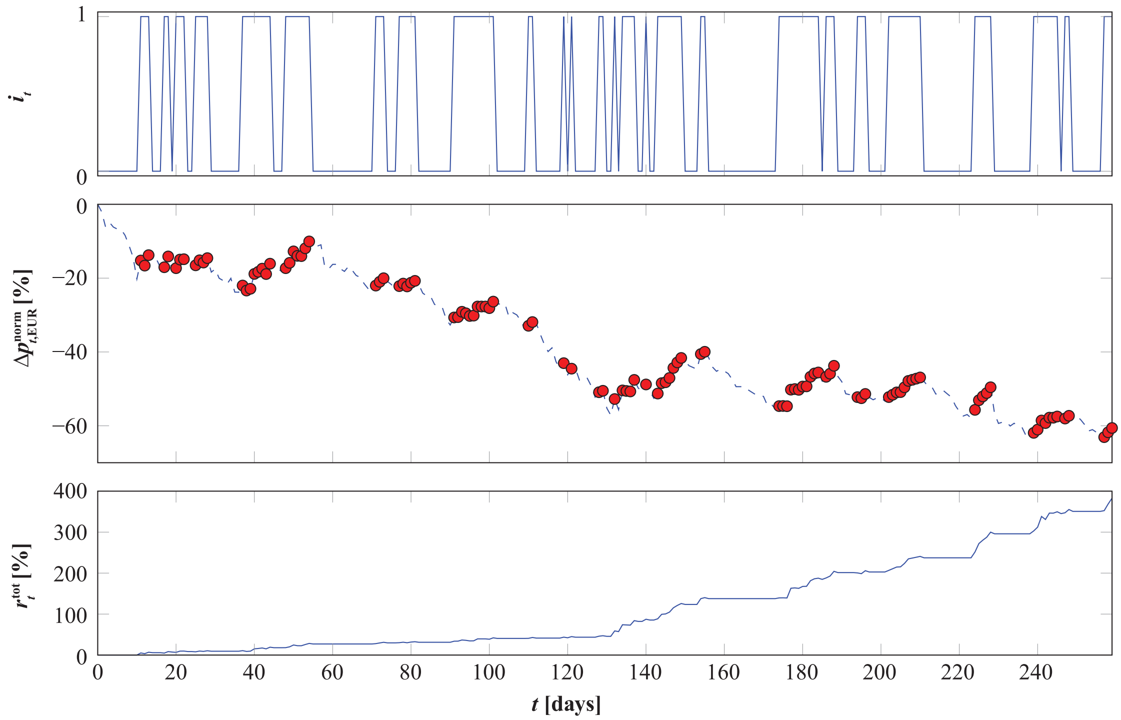
Exhibit 10

The Downtrending Stock (DKB.DE) of Example 3 for the Unconstrained Trading Case with Proportional Transaction Costs of 1%



Note: In the middle subplot the normalized evolution of DKB.DE is displayed (dashed line). At every time t, an investment can be taken in either EUR or DKB.DE. The time instances of optimally being invested in DKB.DE are emphasized by dark circles.

for the desired tuning of labeled data, before 2) developing supervised machine learning applications for algorithmic trading and screening systems. This is the subject of ongoing work.

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