INT. J. CONTROL, 2000, VOL. 73, NO. 8, 655–665

Output-feedback predictive control of constrained linear systems via set-membership state estimation

A. BEMPORAD†‡* and A. GARULLI‡

This paper combines model predictive control (MPC) and set-membership (SM) state estimation techniques for controlling systems subject to hard input and state constraints. Linear systems with unknown but bounded disturbances and partial state information are considered. The adopted approach guarantees that the constraints are satisfied for all the states which are compatible with the available information and for all the disturbances within given bounds. Properties of the proposed MPC-SM algorithm and simulation studies are reported.

1. Introduction

Two features frequently arise in many practical control problems: the necessity of satisfying input/state constraints and the presence of disturbances. The former are dictated by physical limitations of the actuators and by the necessity to keep some plant variables within safe limits; the latter by model inaccuracy and unmeasured noise. In recent years, several control techniques have been developed which are able to handle hard constraints (see e.g. Mayne and Polak 1993, Sussmann *et al.*, 1994). In particular, in the last decades industry has been attracted by model predictive control (MPC) (Sanchez 1976, Richalet *et al.*, 1978, Clarke *et al.* 1987, García *et al.* 1989, Mosca 1995, Lee and Cooley 1997, Qin and Badgewell 1997). MPC is based on the so-called *receding horizon* strategy. This consists of determining a sequence of control inputs that optimizes an open-loop *performance index*, according to a prediction of the future evolution of the system from the current time t. Then, only the first part of the optimal input sequence is actually applied to the system, until another sequence based on more recent measurements is newly computed. The involved prediction depends on the current state, the (unknown) future disturbances, and the selected control input. Several strategies, which have been developed for deterministic frameworks (Keerthi and Gilbert 1988, Mayne and Michalska 1990, Rawlings and Muske 1993, Zheng and Morari 1995) can be applied by neglecting the presence of state disturbances over the prediction horizon. However, this does not guarantee that state related constraints are actually satisfied.

More recently, Gilbert and Kolmanovsky (1995) and Bemporad and Mosca (1998) have developed computationally efficient techniques for solving constrained problems, by manipulating the reference trajectory (Bemporad *et al.* 1997, Bemporad 1998 b). In particular, Gilbert and Kolmanovsky (1995) guarantee constraint fulfilment also in the presence of input disturbances. However, these techniques require full state measurements. When these are not available, it is common practice to provide the predictor with an estimate generated by a state observer, e.g. a Kalman filter. Again, no guarantee of constraint fulfilment holds, due to a mismatch between the predicted evolution and the actual behaviour of the system.

Several approaches have been recently proposed in the literature in order to take into account such a mismatch, i.e. the presence of uncertainties in the model of the plant to be controlled (see Bemporad and Morari 1999 for a survey). Some schemes assume that the matrices of the plant can vary within given bounded sets and the state is fully available (state-space approach, see Kothare *et al.* 1996 and references therein), or suppose that the impulse/step response of the system lies within given uncertainty ranges (input/output approach, see Bemporad and Mosca 1998). In this paper we represent uncertainty and model errors as unmeasured input and output disturbances affecting a nominal model of the plant. Roughly speaking, this kind of modelling is typically used in Kalman filtering, where input disturbances are related to model uncertainty, and output disturbances to measurement noise.

This paper copes with full state information unavailability in the presence of model errors and measurement disturbances in a similar fashion. Instead of expressing their intensity in terms of stochastic properties, we assume that input disturbances and output noises are unknown but bounded. As shown in recent literature on robust control and identification, the description of uncertainty by additive terms that are known to be bounded in some norm is a reasonable choice, as it allows us to cope with both measurement noise and unmodelled dynamics (see e.g. Milanese and Vicino 1993, Mäkilä *et al.* 1995).

We adopt an approach which consists of: (i) considering the effect of the worst input disturbance sequence

Received November 1998. Revised October 1999.

* Author for correspondence. e-mail: bemporad@aut.ee.ethz.ch

† Automatic Control Laboratory, Swiss Federal Institute of Technology (ETH), ETH Zentrum, CH-8092 Zurich, Switzerland.

‡ Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma, 56, 53100 Siena, Italy.

International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2000 Taylor & Francis Ltd
http://www.tandf.co.uk/journals

over the prediction horizon (for this reason, we shall refer to a 'worst-case' approach, by meaning that constraint fulfilment is achieved *for all* possible disturbance realizations, and hence the *worst* disturbance realization is taken into account); (ii) handling state unavailability by using the so-called *set-membership* (SM) state estimation. This approach, first introduced by Schweppe (1968) and Bertsekas and Rhodes (1971), considers the *state uncertainty set*, i.e. the set of all state vectors compatible with model equations, uncertainty on the initial state, bounds on the disturbances, and output measurements available up to time t. Due to the tremendous amount of calculations required by the updating of the true state uncertainty set, many recursive approximation algorithms based on simple regions in the state space, e.g. ellipsoids (Schweppe 1968, Bertsekas and Rhodes 1971, Chernousko 1980, Filippova *et al.* 1994), have been proposed in the literature. In this paper, we adopt the minimum volume parallelotopic approximation developed by Chisci *et al.* (1996) and Vicino and Zappa (1996). The resulting set-membership estimation algorithm is particularly appealing for MPC, as it presents both good approximation capabilities and reasonable computational complexity.

The paper is organized as follows. In § 2 we formulate the worst-case MPC problem and give the basic assumptions. Section 3 shows how the posed infinite horizon optimization problem can be solved by considering only a finite number of constraints, and studies asymptotic properties of the overall feedback loop. The computations involved in the optimization algorithm are investigated in § 4. Finally, we report simulation results in § 5, and draw some conclusions in § 6.

2. Problem formulation and assumptions

By referring to the scenario depicted in figure 1, consider the following linear, discrete-time, time-invariant system

$$\left.\begin{aligned} x_p(t+1) &= A_p x_p(t) + B_p u_p(t) + \xi_p(t) \\ y(t) &= C_p x_p(t) + \zeta(t) \\ c(t) &= E_p x_p(t) + G_p u_p(t) \end{aligned}\right\} \tag{1}$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is the state vector and is supposed not to be directly measurable, $u_p(t) \in \mathbb{R}^{m_p}$ is the command input to the actuators, $y(t) \in \mathbb{R}^p$ the measured output which should track a desired reference $r(t) \in \mathbb{R}^p$, $c(t) \in \mathbb{R}^l$ is a vector to be constrained within the convex polyhedral set

$$\mathcal{C} = \{c \in \mathbb{R}^l : A_c c \leq \hat{B}_c\}, \qquad \hat{B}_c \in \mathbb{R}^q \tag{2}$$

$\xi_p(t) \in \mathbb{R}^{n_p}$ and $\zeta(t) \in \mathbb{R}^p$ are respectively unknown input and output disturbances, and $t = 0, 1, \ldots$. Typically, $\xi_p(t)$ accounts for unknown signals acting on the system, model errors and unmodelled dynamics (e.g. slowly time-varying drifts, mild non-linearities, etc.), while $\zeta(t)$ is measurement noise on the output $y(t)$.

For the linear plant (1), we assumed that the linear low-level controller

$$\left.\begin{aligned} x_c(t+1) &= A_c x_c(t) + B_c y(t) + F_c u(t) \\ u_p(t) &= C_c x_c(t) + D_c y(t) + K_c u(t) \end{aligned}\right\} \tag{3}$$

has already been designed *without taking care of the constraints*, for instance through standard control techniques such as PID, LQG or H_∞ control, to provide stability and noise attenuation properties. By considering $u(t) \in \mathbb{R}^m$ as a new input, the overall-closed loop system can be rewritten as

$$\left.\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + \xi(t) \\ y(t) &= Cx(t) + \zeta(t) \\ c(t) &= Ex(t) + Gu(t) + L\zeta(t) \end{aligned}\right\} \tag{4}$$

where

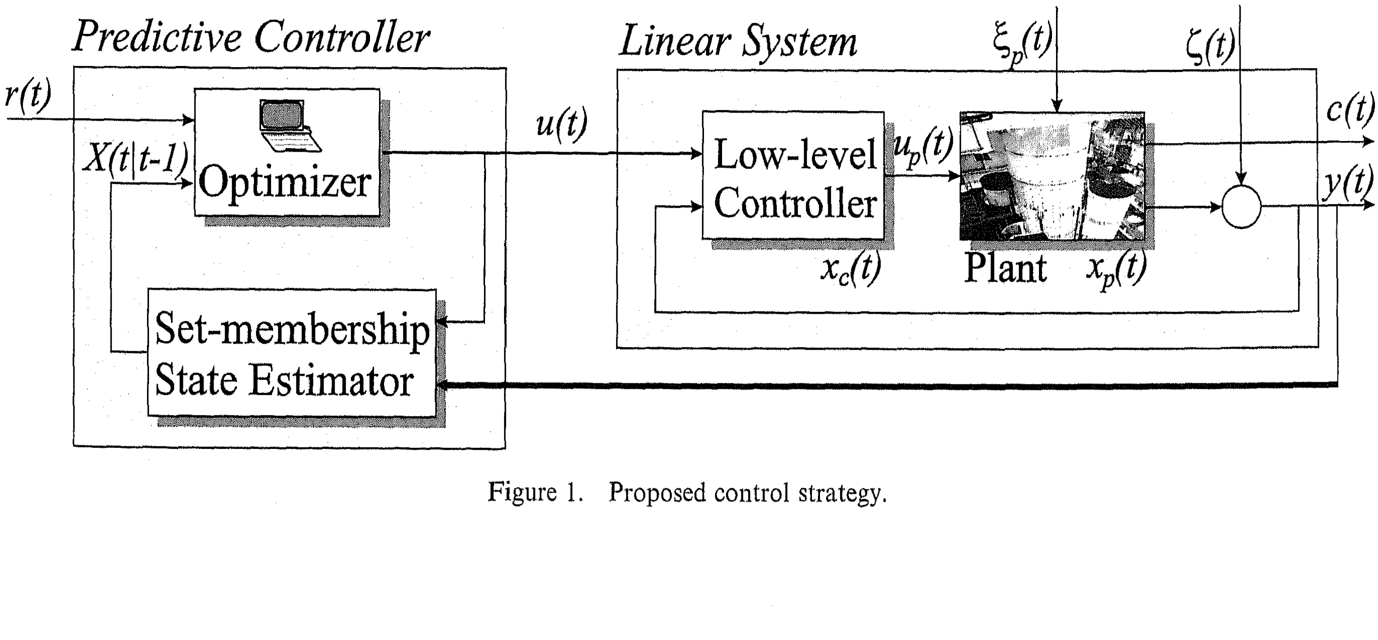

Figure 1. Proposed control strategy.

$$x(t) \triangleq \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} \in \mathbb{R}^n, \quad \xi(t) \triangleq \begin{bmatrix} B_p D_c \zeta(t) + \xi_p(t) \\ B_c \zeta(t) \end{bmatrix} \in \mathbb{R}^n$$

and the matrices in (4) can be derived by the matrices in (1)–(3). We assume that the precompensated linear system (4) satisfies the following.

Assumption 1: *A is asymptotically stable.*

Assumption 1 means that system (4), i.e. the feedback connection of the open-loop system (1) and the low-level controller (3), is asymptotically stable *in the absence of constraints*, and is indeed equivalent to stabilizability of the open-loop linear system (1). On the other hand, in the presence of constraints, for a certain set of initial conditions system (1) might not be stabilizable under the selected actuator constraints and bounded disturbances. As we will discuss later, this would lead to *infeasibility*, namely no control sequence $u(t)$ would be able to satisfy constraints (2), and in particular those constraints in (2) which come from actuator saturation (after the precompensation (3), the original input $u_p(t)$ becomes a state-dependent vector, which can be included in $c(t)$).

We assume that both $\xi(t)$ and $\zeta(t)$ are unknown but bounded, namely

$$\xi(t) \in \Xi \tag{5}$$

$$\zeta(t) \in \mathcal{Z} \tag{6}$$

$\forall t \geq 0$, where $\Xi, \mathcal{Z}$ are the hyper-rectangles

$$\Xi \triangleq \{\xi \in \mathbb{R}^n : \xi_i^- \leq \xi_i \leq \xi_i^+, \xi_i^- \leq 0 \leq \xi_i^+, i = 1, \ldots, n\}$$

$$\mathcal{Z} \triangleq \{\zeta \in \mathbb{R}^p : \zeta_i^- \leq \zeta_i \leq \zeta_i^+, \zeta_i^- \leq 0 \leq \zeta_i^+, i = 1, \ldots, p\}$$

and $\xi_i^-, \xi_i^+, \zeta_i^-, \zeta_i^+$ are given bounds.

The goal of this paper is to investigate a feedback control law such that the output $y(t)$ tracks a desired reference $r(t) \in \mathbb{R}^p$, while the vector $c(t)$ fulfils the constraint

$$c(t) \in C \tag{7}$$

for all possible disturbance realizations $\xi(t) \in \Xi$ and $\zeta(t) \in \mathcal{Z}$. Without loss of generality we can consider

$$c(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

and rewrite (2) as

$$C = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m} : A_c^x x + A_c^u u \leq B_c \right\} \tag{8}$$

where $A_c^x \triangleq A_c E$, $A_c^u \triangleq A_c G$, $B_c \triangleq \hat{B}_c - \max_{\zeta \in \mathcal{Z}} A_c L \zeta$. Note that this corresponds to setting $E = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $L = 0$ in (4). Hereafter, we shall assume that

Assumption 2: *C is bounded.*

Note that assuming that C is bounded is not restrictive in practice, since usually inputs and states are often bounded for physical reasons. The following developments will be meaningful if, in addition, C has a non-empty interior. Moreover, in order to compute a matrix pseudo-inversion later in (11), we assume that

Assumption 3: *The dc-gain matrix $H \triangleq C(I - A)^{-1}B$ has full rank,* $\operatorname{rank} H = \min\{m, p\}$.

According to the above setting, at a generic time t the available information on the state vector $x(t)$ is given by the model equation (4), the bounds on the input disturbances (5) and output noise (6), and the observed measurements $y(k)$, $k = 0, 1, \ldots, t$. Let us denote by $\mathcal{X}^*(t_1|t_2)$ the *state uncertainty set* of all state vectors $x(t_1)$ at time t_1, compatible with the dynamic equations (4), the bounds (5)–(6), and the measurements $\{y(0), y(1), \ldots, y(t_2)\}$ available at time t_2. If the *a priori information set* $\mathcal{X}^*(0|-1)$ is a bounded polytope containing the initial state $x(0)$, then the state uncertainty sets evolve according to the recursion

$$\mathcal{X}^*(t|t) = \mathcal{X}^*(t|t-1) \bigcap \mathcal{X}_y^*(t)$$

$$\mathcal{X}^*(t+1|t) = A\mathcal{X}^*(t|t) \oplus \{Bu(t)\} \oplus \Xi$$

where

$$\mathcal{X}_y^*(t) = \{x \in \mathbb{R}^n : y(t) - Cx \in \mathcal{Z}\}$$

is the set of states compatible with the single measurement $y(t)$, and $\oplus$ denotes the vector sum of sets. It is clear that the complexity of the polytopes $\mathcal{X}^*(t|t)$ and $\mathcal{X}^*(t+1|t)$ (i.e. the number of faces) grows with t, and therefore it is common practice to approximate these sets by simpler regions, the so-called *set-valued estimates* $\mathcal{X}(t|t)$ *and* $\mathcal{X}(t+1|t)$ respectively.

Here below, we base the MPC law on the set-valued estimate $\mathcal{X}(t|t-1)$. In this way, the input at time t is computed on the basis of the available information up to time $t-1$, so that the required computations can be performed over one full sample interval.

2.1. *MPC-SM controller*

As mentioned in the introduction, MPC consists of minimizing at each time t a performance index, which depends on the future evolution of the system, with respect to the sequence of future moves. Then, only the first move of the optimal sequence is applied, and the whole optimization is repeated at next time $t+1$. In order to use efficient optimization procedures, typically the sequence of future moves is parameterized by a finite number of variables, so that the optimization is performed in a finite dimensional space. We adopt the strategy proposed by Rawlings and Muske (1993) by limiting to N_ν the number of control degrees of freedom, and by defining the future control moves $\{u(t), u(t+1), \dots\}$ as

$$u(t+k) = \begin{cases} \nu(k) & \text{if } k = 0, \dots, N_\nu - 1 \\ \nu(N_\nu - 1) & \text{if } k = N_\nu, N_\nu + 1, \dots \end{cases} \tag{9}$$

Consequently, the optimization vector is

$$\mathcal{V} \triangleq \begin{bmatrix} \nu(N_\nu - 1) \\ \vdots \\ \nu(0) \end{bmatrix} \in \mathbb{R}^{N_\nu m}$$

In order to define an optimization problem which is based on the predicted evolution of the system, we denote by $c(t+k, x, \mathcal{V}, \mathcal{K}_k)$ the constrained vector at time $t+k$, evolved from $x(t) = x \in \mathcal{X}(t|t-1)$ by applying the input sequence $u(t+k) = \nu(k)$, $\forall k \geq 0$, and disturbance $\mathcal{K}_k \in \Xi_k$, where

$$\mathcal{K}_k \triangleq \begin{bmatrix} \xi(t+k-1) \\ \vdots \\ \xi(t) \end{bmatrix}$$

$$\Xi_k \triangleq \Xi \times \Xi \times \cdots \times \Xi \subseteq \mathbb{R}^{kn}$$

As optimization will be performed on line, we need to select a performance index $J(t, \mathcal{V})$ so that the minimization of $J(t, \mathcal{V})$ with respect to $\mathcal{V}$ is computationally viable and provides good tracking properties. A reasonable choice is to select the index

$$J(t, \mathcal{V}) = \sum_{k=0}^{N_\nu - 2} \|\nu(k) - \nu(N_\nu - 1)\|^2_{\Upsilon_1} + \|\nu(N_\nu - 1) - H^{\#} r(t)\|^2_{\Upsilon_2} \tag{10}$$

where $N_\nu \geq 1$ (the first sum in (10) is considered 0 for $N_\nu = 1$)

$$H^{\#} \triangleq \begin{cases} (H'H)^{-1}H & \text{if } m < p \\ H'(HH')^{-1} & \text{if } m \geq p \end{cases} \tag{11}$$

and Υ_1, $\Upsilon_2 > 0$ are symmetric weight matrices, $\|\nu\|^2_\Upsilon \triangleq \nu'\Upsilon\nu$. This choice is motivated by the two-fold objective of minimizing both the steady-state tracking error and the control energy. Since by (9), $\nu(N_\nu - 1)$ represents the final constant input on the prediction horizon, Υ_2 penalizes the predicted steady-state tracking error, while Υ_1 penalizes the deviations of the first $N_\nu - 1$ control moves from the steady-state input $\nu(N_\nu - 1)$. Note that if $m = p$, by Assumption 2 there exists H^{-1}, and therefore $\|\nu(N_\nu - 1) - H^{\#} r(t)\|^2_{\Upsilon_2} = \|H\nu - r\|^2_{\Upsilon_2^*}$, where $\Upsilon_2^* \triangleq (H^{-1})'\Upsilon_2 H^{-1}$.

At each time t, the selection of the optimal vector $\mathcal{V}_t$ proceeds as follows. Denote by $\Omega(t)$ the set of all vectors $\mathcal{V}$ leading to feasible evolutions of the constrained vector

$$\Omega(t) = \{\mathcal{V} \in \mathbb{R}^{N_\nu m} : c(t+k, x, \mathcal{V}, \mathcal{K}_k) \in \mathcal{C}, \ \forall x \in \mathcal{X}(t|t-1), \forall \mathcal{K}_k \in \Xi_k, \forall k \geq 0\} \tag{12}$$

If $\Omega(t)$ is non-empty, define

$$\mathcal{V}_t^* = \arg\min_{\mathcal{V} \in \Omega(t)} J(t, \mathcal{V}) \tag{13}$$

Then, by denoting by $\mathcal{V}_t^1$ the extension of the previous optimal vector $\mathcal{V}_{t-1}$, i.e.

$$\mathcal{V}_{t-1} = \begin{bmatrix} \nu_{t-1}(N_\nu - 1) \\ \nu_{t-1}(N_\nu - 2) \\ \vdots \\ \nu_{t-1}(1) \\ \nu_{t-1}(0) \end{bmatrix}, \mathcal{V}_t^1 = \begin{bmatrix} \nu_{t-1}(N_\nu - 1) \\ \nu_{t-1}(N_\nu - 1) \\ \nu_{t-1}(N_\nu - 2) \\ \vdots \\ \nu_{t-1}(1) \end{bmatrix} \tag{14}$$

we set

$$\mathcal{V}_t = \begin{cases} \mathcal{V}_t^* & \text{if } \Omega(t) \neq \varnothing \quad \text{and} \quad J(t, \mathcal{V}_t^*) < J(t, \mathcal{V}_t^1) - \epsilon(t) \\ \mathcal{V}_t^1 & \text{otherwise} \end{cases} \tag{15}$$

where $\epsilon(t) \triangleq \min\{\rho_1 J(t, \mathcal{V}_t^1), \rho_2\}$, and ρ_1, ρ_2 are fixed arbitrarily small positive scalars. Then, according to the receding horizon strategy described above, we set

$$u(t) = \nu_t(0) \tag{16}$$

The entire procedure is then repeated at time $t+1$.

Finally, in order to complete the above scheme, we make the following hypothesis on $\mathcal{X}(0|-1)$.

Assumption 4: *For the* a priori *information set* $\mathcal{X}(0|-1)$ *there exists a finite input sequence* $\mathcal{V}_{-1}$ *such that* $\mathcal{V}_{-1}^1 \in \Omega(0)$.

As mentioned above, stabilization of an unstable system with saturating actuators might not be possible for a given *a priori* information set $\mathcal{X}(0|-1)$. In this case, Assumption 4 would be violated, i.e. no feasible solution to the optimization problem (13) would exist at $t = 0$. In other words, the problem of stabilizing an unstable system with state and input constraints has been converted to a feasibility problem.

A brief summary of the MPC-SM algorithm is reported in table 1.

3. Constraint reduction and asymptotic properties

The optimization problem (13) involves an *infinite* number of linear constraints. However, in order to solve (13) via standard tools for Quadratic Programming, a *finite* number of constraints is desir-

Table 1. MPC-SM algorithm.

1. Solve the optimization problem (13).
2. If (13) is feasible and $J(t,\mathcal{V}_t^*) < J(t,\mathcal{V}_t^{\rm l}) - \epsilon(t)$, set $\mathcal{V}_t = \mathcal{V}_t^*$. Otherwise, set $\mathcal{V}_t = \mathcal{V}_t^{\rm l}$ as in (14).
3. Extract from $\mathcal{V}_t$ the first control move $v_t(0)$.
4. Apply $u(t) = v_t(0)$.
5. Compute $\mathcal{X}(t+1|t)$ as described in table 2.
6. $t \leftarrow t+1$.
7 Go to 1.

able. Next Proposition 1 shows that such a finite number exists, provided that an extra linear constraint on $\mathcal{V}$ is added.

Proposition 1: *There exist an index $k_o \geq N_v$ and $\delta > 0$ such that, if $\mathcal{V}$ satisfies*

$$[A_c^x(I-A)^{-1}B + A_c^u]v(N_v-1) \leq B_c - \delta\underline{1} \tag{17}$$

$\underline{1} = [1,\dots,1]'$, *then*

$$\begin{array}{l} c(t+k,x,\mathcal{V},\mathcal{K}_k) \in \mathcal{C}, \quad \forall x \in \mathcal{X}(t|t-1), \\ \forall \mathcal{K}_k \in \Xi_k, \quad \forall k = 0,\dots,k_o \end{array} \Rightarrow \mathcal{V} \in \Omega(t) \tag{18}$$

Proof: Without loss of generality set $t = 0$. Let $k \geq N_v$, $x \in \mathcal{X}(0|-1)$, $\mathcal{V} \in \Omega(0)$ such that (17) is satisfied, and $c(h,x,\mathcal{V},\mathcal{K}_h) \in \mathcal{C}$, $\forall h = 0,\dots,N_v$. Consider the state at time k

$$\begin{aligned} x(k,x,\mathcal{V},\mathcal{K}_k) &= A^{k-N_v}x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) \\ &\quad + \left[\sum_{i-0}^{k-N_v-1} A^iB\right]v(N_v-1) \\ &\quad + \sum_{i=0}^{k-N_v-1} A^i\xi(k-1-i) \end{aligned} \tag{19}$$

where

$$A_c^x x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) + A_c^u v(N_v-1) \leq B_c$$

By Assumption 2, there exist constants $\Delta_{\mathcal{C}}^x$ and $\Delta_{\mathcal{C}}^u$ such that

$$\|x\|_\infty \leq \Delta_{\mathcal{C}}^x, \qquad \|u\|_\infty \leq \Delta_{\mathcal{C}}^u, \qquad \forall \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{C}$$

Let $M > 0$ and $0 < \lambda < 1$ such that

$$\|A^k\|_\infty \leq M\lambda^k \tag{20}$$

where $\|A\|_\infty$ denotes the ∞-induced matrix norm, and let

$$N_k \triangleq \max_{\xi(i)\in\Xi} \| \sum_{i=0}^{k-N_v-1} A_c^x A^i \xi(i) \|_\infty \tag{21}$$

Since $0 \in \Xi$, by considering the case $\xi(k+1) = 0$, one can easily see that $N_{k+1} \geq N_k$. Hence, by (20) there exists

$$N_\infty \triangleq \lim_{k\to\infty} N_k < \infty \tag{22}$$

Since

$$\begin{aligned} &A^{k-N_v}x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) + \sum_{i=0}^{k-N_v-1} A^iBv(N_v-1) \\ &\qquad -(I-A)^{-1}Bv(N_v-1) \\ &= A^{k-N_v}x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) + \left[\sum_{i=0}^{k-N_v-1} A^i - \sum_{i=0}^{\infty} A^i\right]Bv(N_v-1) \\ &= A^{k-N_v}[x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) - (I-A)^{-1}Bv(N_v-1)] \end{aligned}$$

from (19) it follows that

$$\begin{aligned} &\|A_c^x[x(k,x,\mathcal{V},\mathcal{K}_k) - (I-A)^{-1}Bv(N_v-1)]\|_\infty \\ &\quad \leq \|A_c^x\|_\infty M\lambda^{k-N_v}(\Delta_{\mathcal{C}}^x + \|(I-A)^{-1}B\|_\infty \Delta_{\mathcal{C}}^u) + N_\infty \leq \delta \end{aligned} \tag{23}$$

for

$$\delta \geq N_\infty + \epsilon_3 \tag{24}$$

$$\begin{aligned} k &\geq k^* \\ &= \left\lceil N_v + \log_\lambda \frac{\epsilon_3}{M\ \|A_c^x\|_\infty\ (\Delta_{\mathcal{C}}^x + \|(I-A)^{-1}B\|_\infty\ \Delta_{\mathcal{C}}^u)} \right\rceil \end{aligned} \tag{25}$$

where $\epsilon_3 > 0$ is a chosen (small) number, and $\lceil r \rceil$ denotes the minimum integer greater than r. Hence, by (17) and (23)

$$\begin{aligned} &A_c^x x(k,x,\mathcal{V},\mathcal{K}_k) + A_c^u v(k) \\ &\quad = A_c^x[x(k,x,\mathcal{V},\mathcal{K}_k) - (I-A)^{-1}Bv(N_v-1)] \\ &\qquad + A_c^x(I-A)^{-1}Bv(N_v-1) + A_c^u v(N_v-1) \\ &\quad \leq A_c^x[x(k,x,\mathcal{V},\mathcal{K}_k) - (I-A)^{-1}Bv(N_v-1)] \\ &\qquad + B_c - \delta \begin{bmatrix} 1 \\ \underline{1} \\ 1 \end{bmatrix} \leq B_c \end{aligned}$$

or equivalently

$$\begin{bmatrix} x(k,x,\mathcal{V},\mathcal{K}_k) \\ v(k) \end{bmatrix} \in \mathcal{C}, \qquad \forall k \geq k^*$$

Then, there exist integers $k_o \leq k^*$ such that (18) is satisfied. □

Remark 1: Constraint (17) imposes that the predicted steady-state constrained vector, corresponding to the constant input level $v(N_v - 1)$ and $\xi(t) \equiv 0$, lies inside $\mathcal{C}$ by at least a fixed distance away from the border. Because of Assumption 1 and since the disturbances $\xi(t)$ are bounded, when $\xi(t) \neq 0$ after a finite time the c-trajectory will remain within a finite set. The idea is to select δ large enough so that this set lies inside $\mathcal{C}$. Then, it is clear that once the c-vector has entered this set, checking the fulfilment of the constraints is no longer necessary. In other words, there is an upper-bound to the number of possible active constraints. It is clear that N_∞, and consequently δ, might be large numbers. This mostly depends on the eigenvalues of A and the magnitude of Ξ. For instance, if A is nilpotent, the sum in (21) is only affected by the last n disturbance inputs $\xi(k - N_v - n), \ldots, \xi(k - N_v - 1)$. On the other hand, if A has eigenvalues close to the unit circle, there is an accumulation of the uncertainty which might lead to a very conservative value N_∞, and hence to large δ. We point out that simulation tests have shown that usually the constraints

$$c(t+k, x, \mathcal{V}, \mathcal{K}_k) \in \mathcal{C}, \qquad \forall x \in \mathcal{X}(t|t-1),$$
$$\forall \mathcal{K}_k \in \Xi_k, \qquad \forall k = 0, \ldots, k_o$$

are more stringent than the constraint (17). Unfortunately, a worst-case approach inescapably leads the set of admissible input moves to shrink, this reduction being above all proportional to the intensity of the process disturbance ξ. In the following we will assume that Ξ is small enough, so that the set defined by the inequality (17), with δ as in (24), is non-empty, and hence feasible solutions $\mathcal{V}$ exist.

Remark 2: The bound k^* in (25) is usually overestimated. This means that the minimum k_o such that (18) holds will be in general much smaller than k^*. Such a minimum k_o can be either estimated heuristically from the time-domain response of the system (4), or computed exactly as suggested in (Bemporad 1998a).

The next proposition describes the asymptotical behaviour of the overall control scheme.

Proposition 2: *Consider system* (4) *and a sequence of approximated state uncertainty sets* $\{X(t|t-1)\}_{t=0}^{\infty}$. *Let* $r(t) \equiv r$, $\forall t \geq t_r \geq 0$. *Then, the control strategy* (13)–(16), *based on the optimization of the performance index* (10) *in the presence of constraints* (12) *and* (17), *guarantees that* $\lim_{t\to+\infty} u(t) = \bar{u}$, *where* $\bar{u}$ *is a constant, and as* $t \to \infty$

$$x(t) \to \mathcal{X}_\Xi(\bar{u}) \triangleq \{x \in \mathbb{R}^n : x = (I-A)^{-1}B\bar{u} + \sum_{k=0}^{\infty} A^k \xi(k), \xi(k) \in \Xi\}$$

Proof: Let $\mathcal{L}(t) = J(t, \mathcal{V}_t)$, with $\mathcal{V}_t$ as in (15). Then, by (10), $\mathcal{L}(t)$ is a monotonically decreasing non-negative sequence

$$\mathcal{L}(t-1) - \mathcal{L}(t) \geq \mathcal{L}(t-1) - J(t, \mathcal{V}_t^1)$$
$$\geq \| v_{t-1}(0) - v_{t-1}(N_v - 1) \|_{\Upsilon_1}^2 \geq 0 \tag{26}$$

and converges as $t \to \infty$. Moreover

$$\lim_{t\to\infty} \| v_{t-1}(0) - v_{t-1}(N_v - 1) \|_{\Upsilon_1} = 0 \tag{27}$$

Let

$$t_\epsilon \triangleq \sup_{t \geqslant t_r} \{t : J(t, \mathcal{V}_t^*) < J(t, \mathcal{V}_t^1) - \epsilon(t)\} \tag{28}$$

If $t_\epsilon < +\infty$, then by (15)

$$\mathcal{V}_t = \mathcal{V}_t^1, \qquad \forall t \geq t_\epsilon$$

and therefore

$$u(t) = v(0) \equiv v_{t_\epsilon}(N_v - 1), \qquad \forall t \geq t_\epsilon + N_v - 1$$

Conversely, assume that $t_\epsilon = +\infty$, and define a subsequence $\{t_k\}_{k=0}^{\infty}$ such that $J(t_k, \mathcal{V}_{t_k}^*) < J(t_k, \mathcal{V}_{t_k}^1) - \epsilon(t_k)$. By (15)

$$\mathcal{L}(t_k - 1) - \mathcal{L}(t_k) \geq \| v_{t_k-1}(0) - v_{t_k-1}(N_v - 1) \|_{\Upsilon_1}^2 + \epsilon(t_k) \geq 0$$

Since $\mathcal{L}(t_k)$ converges as $k \to \infty$, and by (27), it results $\lim_{k\to+\infty} \epsilon(t_k) = \lim_{k\to+\infty} \rho_1 J(t_k, \mathcal{V}_{t_k}^1) = 0$. Then,

$$\mathcal{L}(t_k) = J(t_k, \mathcal{V}_{t_k}) < J(t_k, \mathcal{V}_{t_k}^1) - \epsilon(t_k) \to 0$$

and, being $\mathcal{L}(t)$ monotonic, it follows $\lim_{t\to\infty} \mathcal{L}(t) = 0$, or equivalently

$$\lim_{t\to\infty} \mathcal{V}_t = [v_r' \ldots v_r']'$$

where $v_r \triangleq H^{\#} r$. In particular, $\lim_{t\to\infty} u(t) = v_r$.

Assume for the moment that after a finite time $\xi(t) \equiv 0$. In this case, system (4) is converging input converging-state stable (Sontag 1995). In fact, let $\tilde{x}(t) \triangleq x(t) - (I-A)^{-1}B\bar{u}, \tilde{u}(t) \triangleq u(t) - \bar{u}$ and consider the system $\tilde{x}(t+1) = A\tilde{x}(t) + B\tilde{u}(t)$. By Assumption 1, there exist constants $\lambda < 1, k_1 > 0, k_2 > 0$ such that

$$\|\tilde{x}(t)\| \leq k_1 \lambda^{t-t_0} \|\tilde{x}(t_0)\| + k_2 g(t_0)$$

where $g(t_0) = \max_{t \geq t_0} \|\tilde{u}(t)\|$. Clearly, $g(t_0) \to 0$ as $t_0 \to \infty$. Let $\alpha > 0$, and define t_1 such that $k_2 g(t_0) \leq \alpha/2$, $\forall t_0 \geq t_1$, and $t_2 \geq t_1$ such that $k_1 \lambda^{t-t_1} \|\tilde{x}(t_0)\| \leq \alpha/2, \forall t \geq t_2$. Then, $\|\tilde{x}(t)\| \leq \alpha$, $\forall \geq t_2$. This proves that $\lim_{t\to\infty} x(t) = (I-A)^{-1}B\bar{u}$. By superimposing the effect of non-zero $\xi(t)$, it follows that $x(t)$ converges to the set $\mathcal{X}_\Xi(\bar{u})$. Notice that, when $\bar{u} = v_r$ and

$m \geq p, y(t)$ converges to the set $\{y \in \mathbb{R}^p: y = r + \zeta + \sum_{k=0}^{\infty} CA^k \xi(k), \xi(k) \in \Xi, \zeta \in \mathcal{Z}\}$. □

Remark 3: Even though Proposition 2 guarantees the convergence of the state to a set whose magnitude depends on Ξ, the desired set-point $C(I-A)^{-1}Bv_r$ is approached in steady-state only when $t_\epsilon = +\infty$ in (28). The verification of this condition is influenced by the constraints and the disturbance bounds Ξ and $\mathcal{Z}$. In fact, because of the adopted worst-case approach, too stringent constraints and/or too large Ξ and $\mathcal{Z}$ might prevent the control input $u(t)$ to reach the desired value v_r. In this case, the cost function (10) tends to pull $u(t)$ towards the safe level which is closest to v_r. The relation between $t_\epsilon, v_{t_\epsilon}(N_v - 1)$, r and the sets Ξ, $\mathcal{Z}$, and $\mathcal{C}$ clearly depends on the convergence properties of the adopted set-valued observer, whose analysis goes beyond the aims of this paper.

4. Constrained optimization algorithm

In this section, we derive the solution of the constrained optimization problem posed in § 2. The control algorithm must perform two main tasks:

(1) update the approximated state uncertainty set $\mathcal{X}(t|t-1)$;
(2) perform the constrained optimization (13), (17).

4.1. *Set-valued observer* $\mathcal{X}(t|t-1)$

The existence of a command input sequence which is able to guarantee the fulfilment of actual constraints (7), clearly depends on the quality of the approximation of the state uncertainty set. In fact, if the approximated state set $\mathcal{X}(t|t-1)$ is too large, an input sequence that achieves constraints fulfilment for every $x \in \mathcal{X}(t|t-1)$ might not exist. Conversely, when $\mathcal{X}(t|t-1)$ is too small, it might happen that the actual state vector $x(t) \notin \mathcal{X}(t|t-1)$, and hence feasibility cannot be guaranteed. It is easy to see that the control strategy (13)–(16) guarantees that $c(t) \in \mathcal{C}$, $\forall t \geq 0$, if and only if $\mathcal{X}(t|t-1) \supseteq \mathcal{X}^*(t|t-1)$, $\forall t \geq 0$. Hence, the approximated set must overbound the true uncertainty set and this bound should be as tight as possible.

In this paper, we will consider parallelotopes (Vicino and Zappa 1996) as approximating regions for the state uncertainty sets.

Definition 1: Let a non-singular matrix $T \in \mathbb{R}^{n \times n}$ and a vector $\hat{x} \in \mathbb{R}^n$ be given. Then

$$\mathcal{P}(T, \hat{x}) = \{x: x = \hat{x} + T\alpha, \|\alpha\|_\infty \leq 1\}$$

defines a *parallelotope in* $\mathbb{R}^n$, with centre $\hat{x}$ and edges parallel to the column vectors of T.

Recently, a recursive algorithm for the outer approximation of the uncertainty state set of a linear system through parallelotopic regions has been proposed by Chisci *et al.* (1996). At a generic time, t, the following two steps are performed

- *measurement update*: given the parallelotope $\mathcal{X}(t|t-1) = \mathcal{P}(t-1)$, compute a parallelotope $\bar{\mathcal{P}}$ outbounding $\mathcal{P}(t-1) \cap \mathcal{X}_y^*(t)$;
- *time update*: compute a parallelotope $\mathcal{P}(t)$ outbounding $A\bar{\mathcal{P}} \oplus \{Bu(t)\} \oplus \Xi$ and set $\mathcal{X}(t+1|t) = \mathcal{P}(t)$.

The iterations above are initialized by setting $\mathcal{X}(0|-1)$ equal to the given *a priori* information set $\mathcal{X}^*(0|-1)$. The recursive approximation is computed according to a minimum volume criterion. A brief summary of the algorithm is reported in table 2; more details can be found in Chisci *et al.* (1996).

Table 2. Recursive algorithm for computing the parallelotopic approximation of the state uncertainty set.

Initialization: Find a parallelotope $\mathcal{P}(0|-1)$ such that $\chi^*(0|-1) \subseteq \mathcal{P}(0|-1)$.

Recursion: For $t = 0, 1, \ldots$

- Set $\mathcal{P}_0(t|t) = \mathcal{P}(t|t-1)$.
- For $j = 0, 1, \ldots, p$,
  Compute the minimum volume parallelotope $\mathcal{P}_j(t|t)$ such that $\mathcal{P}_{j-1}(t|t) \cap \{x \in \mathbb{R}^n: \zeta_j^- \leq y_j(t) - C_j x \leq \zeta_j^+\} \subseteq \mathcal{P}_j(t|t)$, where C_j denotes the jth row of C.
- Set $\mathcal{P}_0(t+1|t) = \mathcal{P}_p(t|t)$.
- For $j = 0, 1, \ldots, m$,
  Compute the minimum volume parallelotope $\mathcal{P}_j(t+1|t)$ such that $A\mathcal{P}_{j-1}(t+1|t) \oplus \{Bu(t)\} \oplus \{x \in \mathbb{R}^n: \xi_j^- \leq x \leq \xi_j^+\} \subseteq \mathcal{P}_j(t+1|t)$.
- Set $\mathcal{P}(t+1|t) = \mathcal{P}_m(t+1|t)$.

4.2. *Computation of optimization matrices*

In order to solve the optimization problem, we need to express the cost function (10), the set $\Omega(t)$ in (12), and the additional constraint (17) in terms of the optimization vector $\mathcal{V}$. By letting

$$\Psi \triangleq \left[\begin{array}{c|ccc} (N_v-1)\Upsilon_1+\Upsilon_2 & -\Upsilon_1 & \cdots & -\Upsilon_1 \\ \hline -\Upsilon_1 & \Upsilon_1 & & \\ \vdots & & \ddots & \\ -\Upsilon_1 & & & \Upsilon_1 \end{array}\right]$$

the cost (10) can be rewritten as

$$J(t,\mathcal{V}) = \mathcal{V}'\Psi\mathcal{V} - 2r'(t)H^{\#'}\Upsilon_2[I_m \quad 0 \quad \cdots \quad 0] \times \mathcal{V} + r'(t)H^{\#'}\Upsilon_2 H^{\#} r(t) \tag{29}$$

By (8), the fulfilment of the constraints $c(t+k,\ x,\ \mathcal{V},\ \mathcal{K}_k) \in \mathcal{C}$, for every $x \in \mathcal{X}(t|t-1)$ and $\mathcal{K}_k \in \Xi_k$, over a finite horizon $k = 0,\dots,k_o$, can be expressed as

$$A_c^x x(t+k,x,\mathcal{V},\mathcal{K}_k) + A_c^u v(k) \le B_c$$

$$\forall x \in \mathcal{X}(t|t-1), \qquad \forall \mathcal{K}_k \in \Xi_k, \qquad \forall k = 0,\dots,k_o \tag{30}$$

where

$$x(t+k,x,\mathcal{V},\mathcal{K}_k) = A^k x + R_k^v M_k \mathcal{V} + R_k^{\xi}\mathcal{K}_k \tag{31}$$

and

$$R_k^v = [B \quad AB \quad \cdots \quad A^{k-1}B]$$

$$R_k^{\xi} = [I_n \quad A \quad \cdots \quad A^{k-1}]$$

$$M_k \triangleq \begin{cases} [0_{mk\times m(N_v-k)} \quad I_{mk}] & \text{if } k \le N_v \\ \left[\begin{array}{c|c} I_m & \\ \vdots & 0_{m(k-N_v)\times m(N_v-1)} \\ I_m & \\ \hline \multicolumn{2}{c}{I_{mN_v}} \end{array}\right] & \text{if } k > N_v \end{cases}$$

According to (31), after some algebraic manipulations, (30) can be rewritten as

$$\mathcal{A}^x x + \mathcal{A}^v\mathcal{V} + \mathcal{A}^{\xi}\mathcal{K}_{k_o} \le \mathcal{B}, \qquad \mathcal{B} \in \mathbb{R}^h$$

$$\forall x \in \mathcal{X}(t|t-1), \qquad \forall \mathcal{K}_{k_0} \in \Xi_{k_o} \tag{32}$$

where $h = q(k_o+1)$, and $\mathcal{A}^x \in \mathbb{R}^{h\times n}$, $\mathcal{A}^v \in \mathbb{R}^{h\times mN_v}$, $\mathcal{A}^{\xi} \in \mathbb{R}^{h\times nk_o}$ are suitably defined matrices.

Next Lemma 1, whose proof is straightforward, shows how to express (32) as a set of linear inequalities on the optimization vector $\mathcal{V}$.

Lemma 1: *Let*

$$V = \{v \in \mathbb{R}^v : P_1 v \le P_2\}, \qquad P_1 \in \mathbb{R}^{h\times v}, \qquad P_2 \in \mathbb{R}^h$$

be bounded and non-empty. Denote by $[P]^i$ *the* ith *row of* P *and by*

$$\max_{v\in V} Pv \triangleq \begin{bmatrix} \max_{v\in V}[P]^1 v \\ \vdots \\ \max_{v\in V}[P]^h v \end{bmatrix}$$

Then, the following sets

$$\mathcal{D} = \{w \in \mathbb{R}^w : P_3 v + P_4 w \le P_5, \forall v \in V\}$$

$$P_3 \in \mathbb{R}^{k\times v}, \qquad P_4 \in \mathbb{R}^{k\times w}, \qquad P_5 \in \mathbb{R}^k$$

and

$$\bar{\mathcal{D}} = \left\{ w \in \mathbb{R}^w : P_4 w \le P_5 - \max_{v\in V} P_3 v \right\}$$

are equal.

The above lemma proves that $\mathcal{V}$ satisfies the constraints (32) if and only if

$$\mathcal{A}^v\mathcal{V} \le \mathcal{B} - \max_{x\in\mathcal{X}(t|t-1)} \mathcal{A}^x x - \max_{\mathcal{K}_{k_o}\in\Xi_{k_o}} \mathcal{A}^{\xi}\mathcal{K}_{k_o} \tag{33}$$

Note that the second term in the RHS of (33) depends on the current approximated state uncertainty set $\mathcal{X}(t|t-1)$, and hence provides feedback from new output measurements. Finally, the additional constraint (17) can be rewritten as

$$[A_c^x(I-A)^{-1} \quad 0 \quad \cdots \quad 0]\mathcal{V} \le B_c - \underline{\delta} \tag{34}$$

4.3. *Computational complexity analysis*

Let us analyse the computational burden of the control strategy outlined above.

The complexity of the parallelotopic state observer described in §4.1 has been proved to be polynomial in the state dimension n. In particular, Chisci *et al.* (1996) have shown that the computation of the minimum volume parallelotope containing respectively the intersection of a parallelotope and a strip (measurement update) and the sum of a parallelotope and a segment (time update), both require $O(n^2)$ computations. As a result, the computational complexity of a single step of the parallelotopic approximation algorithm turns out to be $O(\max(n^3, n^2p, n^2m))$, which is comparable to that of the classical Kalman filter.

Turning to the optimization required by MPC, one has to minimize the positive definite quadratic cost (29), under the linear constraints (33)–(34), as shown in §4.2. This is a convex quadratic programming (QP) problem that can be efficiently solved by standard algorithms, once the two maximizations in (33) have been performed. Notice that the third term in the RHS of (33) does not depend on the system evolution and can be computed off-line through linear programming (LP). Therefore, at each time instant t one has to solve one convex N_v-dimensional QP and $q(k_o+1)$ LPs.

Summing up, the overall computational complexity of the proposed control strategy is suitable for on-line implementation. The quality of set approximation in SM state estimation and the length of the control horizon N_v (which determines the complexity of the quadratic program associated to MPC) are tuning parameters that can be selected by the control designer according to the available computer power.

5. Simulation results

The proposed control strategy has been investigated by simulations on the following second-order discrete time SISO system

$$\left.\begin{aligned} x(t+1) &= \begin{bmatrix} 1.6463 & -0.7866 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \xi(t) \\ y(t) &= [0.1404 \quad 0]x(t) + \zeta(t) \\ c(t) &= [-1.9313 \quad 2.2121]x(t) \end{aligned}\right\} \tag{35}$$

whose y- and c-step responses are depicted in figure 2 (dashed lines). The transfer function from the input u to the constrained variable c is underdamped and non-minimum phase.

In order to compress the dynamics of c within the range.

$$\mathcal{C} = [-1, 3]$$

and make the output y track the constant reference $r(t) \equiv 1$, we adopt the control law (10)–(17) along with the parameters $\Upsilon_1 = 1$, $\Upsilon_2 = 0.1$, $N_v = 2$, $\rho_1, \rho_2 = 10^{-6}$. Figure 2 shows the resulting trajectories (solid lines) when system (35) is affected by independent randomly generated disturbances $\|\xi(t)\|_\infty \leq 0.01$ and $|\zeta(t)| \leq 0.05$, for the *a priori* information set $\mathcal{X}(0|-1) = 0.25 \cdot [-1,1] \times [-1,1]$. Notice that the constraints are fulfilled at the price of a slower output response. The quantity N_∞ in (22) has been calculated by setting

$$A_c^x A^i \triangleq \begin{bmatrix} a_{i11} & a_{i12} \\ a_{i21} & a_{i22} \end{bmatrix}$$

and then

$$N_\infty = \left\| \sum_{i=0}^{\infty} \begin{bmatrix} |a_{i11}| & |a_{i12}| \\ |a_{i21}| & |a_{i22}| \end{bmatrix} \right\|_\infty \cdot \bar{\xi} \approx 0.2706$$

where $\bar{\xi} \triangleq \max_{i=1,\ldots,n}\{\max\{|\xi_i^-|, |\xi_i^+|\}\} = 0.01$. Then, if we set $\epsilon_3 = 0.01$, the value $\delta = 0.2806$ satisfies Proposition 1. By setting $\Delta_{\mathcal{C}}^x = 10$, $\Delta_{\mathcal{C}}^u = 2$, $M = 5.1107$, and $\lambda = 0.9032$, from (25) it results $k^* = 81$. On the other hand, as noticed in Remark 2, constraint horizons k_o which are smaller than the conservative upper-bound k^* are usually sufficient. In fact, in this example no difference has been noted in simulations for constraints horizon greter than $k_o = 16$ (see Remark 2).

Figure 3 shows the evolution of the parallelotopic state uncertainty sets $\mathcal{X}(t|t)$, and figure 4 reports $\mathcal{X}(t|t)$, $\mathcal{X}(t|t-1)$ during the in initial steps $t = 0, 1, 2$.

In figure 5 the effect of different bounds on the input disturbance is investigated. Due to the adopted worst-case approach, as the size of the disturbance increases, the constraints are fulfilled in a more conservative way and the output dynamics gets slower.

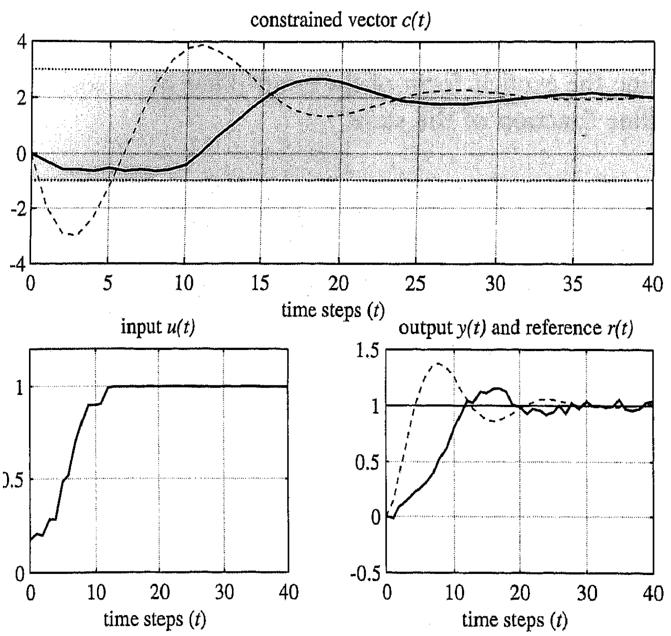

Figure 2. Closed loop behaviour (thick lines) and unconstrained response (dashed lines) for $r(t) \equiv 1$.

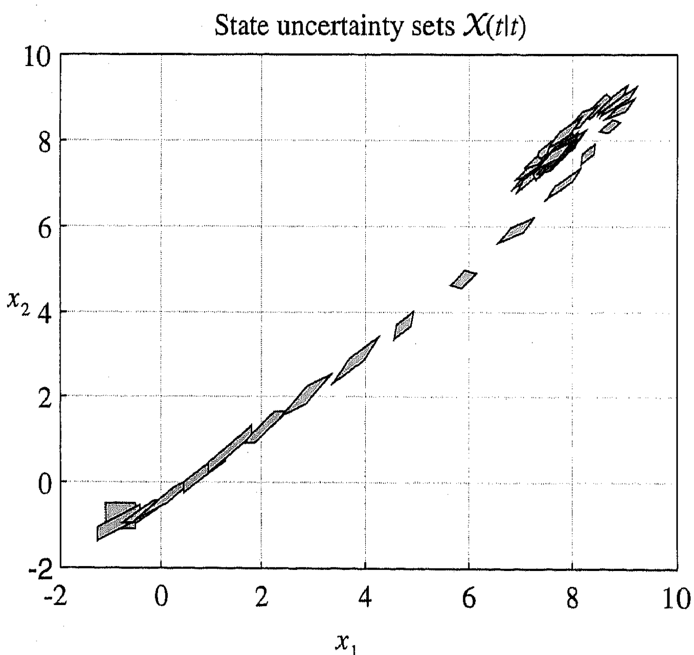

Figure 3. Evolution of the state uncertainty sets $\mathcal{X}(t|t)$.

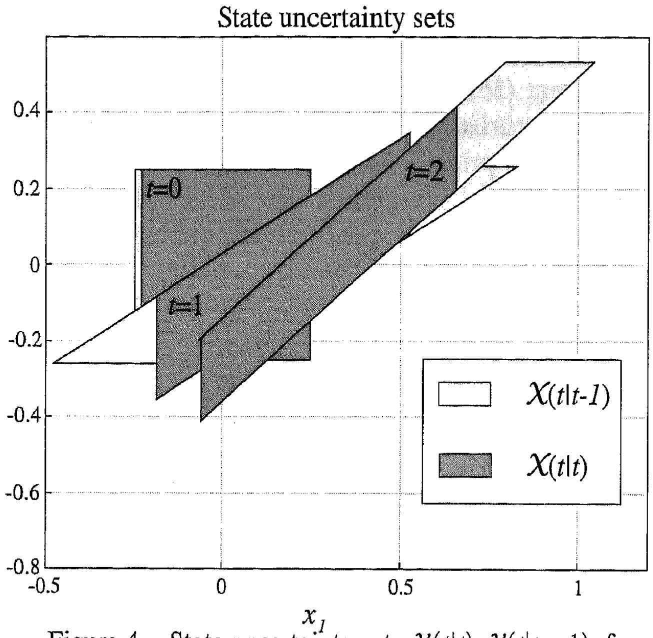

Figure 4. State uncertainty sets $\mathcal{X}(t|t), \mathcal{X}(t|t-1)$, for $t = 0, 1, 2$.

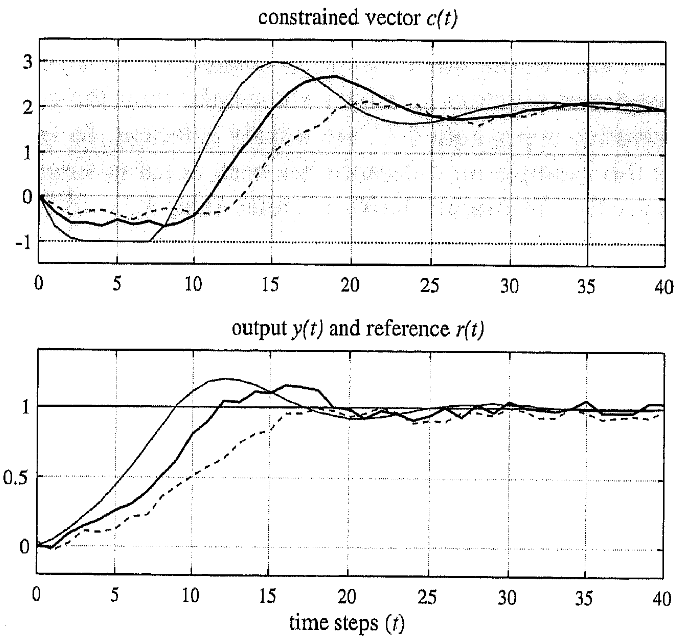

Figure 5. Effect of different input disturbance intensities: no disturbance and known initial state $x(0)$ (thin line); $\| \xi(t) \|_\infty \leq 0.01 (\| \zeta(t) \|_\infty \leq 0.05)$ (thick line); $\| \xi(t) \|_\infty \leq 0.04 (\| \zeta(t) \|_\infty \leq 0.05)$ (dashed line).

6. Conclusions

In this paper we have combined model predictive control and set-membership state estimation to enforce fulfilment of hard input and state constraints when only output measurements are available. The proposed approach is robust in that for the worst situation compatible with the available information, constraint fulfilment is guaranteed, and asymptotic stability properties of the system are preserved.

We believe that this may be a first step in a very promising research area. The use of set-membership techniques in estimation problems has been studied for quite a long time and efficient algorithms are now available. On the other hand, model predictive control is becoming more and more popular in complex industrial applications. However, robustness issues in MPC still need a deeper investigation and different techniques to deal with model uncertainties, disturbances and measurement noise must be analysed. In this paper, we have chosen the bounded-error paradigm which appears to be realistic in several situations of interest, as it allows to account for input and output disturbances and bounded model errors.

Encouraged by the promising simulation results, future research will concern reduction of conservativeness and improvement of performance in the overall control strategy. In particular, the proposed MPC-SM algorithm can be improved (i) by using different polytopic state uncertainty sets than parallelotopes, (ii) by selecting a cost function which depends on the current state vector, even though this might lead to solve non-convex minmax optimization problems, and (iii) by using *closed-loop* prediction schemes (Lee and Yu, 1997, Bemporad 1998 a). For the former problem, one may think of extending the techniques recently developed in Chisci *et al.* (1998) for parametric identification to state estimation, which allow one to trade off the complexity of the approximating region with the required computational burden. The effect of excessive conservativeness discussed above and shown in figure 5 might be partially mitigated by adopting closed-loop prediction schemes (Lee and Yu, 1997, Bemporad 1998 b), rather than the *open-loop* prediction approach of this paper.

Finally, significant results might come by coupling set-membership state estimation techniques with the multiparametric solution of the MPC optimization problem recently developed in Bemporad *et al.* (1999), where the authors provide an off-line algorithm to determine the explicit form of the control law as a piecewise affine function of the state.

References

Bemporad, A., 1998 a, Reducing conservativeness in predictive control of constrained systems with disturbances, *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, Florida, USA, pp. 1384–1391.

Bemporad, A., 1998 b, Reference governor for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, **43**, 415–419.

Bemporad, A., Casavola, A., and Mosca, E., 1997, Nonlinear control of constrained liner systems via predictive reference management. *IEEE Transactions on Automatic Control*, **AC-42**, 340–349.

Bemporad, A., and Morari, M., 1999, Robust model predictive control: A survey. In A. Garulli, A. Tesi and A. Vicino (Eds) *Robustness in Identification and Control*, number 245 in *Lecture Notes in Control and Information Sciences* (Springer-Verlag).

BEMPORAD, A., MORARI, M., DUA, V., and PISTIKOPOULOS, E. N., 1999, The explicit linear quadratic regulator for constrained systems. Technical Report AUT99-16, Automatic Control Lab, ETH Zürich, Switzerland.

BEMPORAD, A., and MOSCA, E., 1998, Fulfilling hard constraints in uncertain linear systems by reference managing. *Automatica*, **34**, 451–461.

BERTSEKAS, D., and RHODES, I., 1971, Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, **16**, 117–128.

CHERNOUSKO, F., 1980, Optimal guaranteed estimates of indeterminacies with the aid of ellipsoids. Parts i–iii, *Engineering Cybernetics*.

CHISCI, L., GARULLI, A., VICINO, A., and ZAPPA, G., 1998, Block recursive parallelotopic bounding in set membership identification. *Automatica*, **34**, 15–22.

CHISCI, L., GARULLI, A., and ZAPPA, G., 1996, Recursive state bounding by parallelotopes. *Automatica*, **43**, 1049–1056.

CLARKE, D., MOTHADI, C, and TUFFS, P., 1987, Generalized predictive control. *Proceedings of the IEE*, **140**, 247–354.

FILIPPOVA, T., KURZHANSKI, A., SUGIMOTO, K., and VALYI, I., 1994, Ellipsoidal state estimation for uncertain dynamical systems. *Bounding Approaches to System Identification* (Plinum Poers).

GARCÍA, C., PRETT, D., and MORARI, M., 1989, Model predictive control: theory and practice—a survey. *Automatica*, **25**, 335–348.

GILBERT, E., and KOLMANOVSKY, I., 1995, Discrete-time reference governors for systems with state and control constraints and disturbance inputs. *Proceedings of the 34th IEEE Conference on Decision and Control*, pp. 1189–1194.

KEERTHI, S., and GILBERT, E., 1988, Optimal infinite-horizon feedback control laws for a general class of constrained discrete-time systems: stability and moving-horizon approximation. *Journal of Operational Theory and Applications*, **57**, 265–293.

KOTHARE, M., BALAKRISHNAN, V., and MORARI, M., 1996, Robust constrained model predictive control using linear matrix inequalities. *Automatica*, **32**, 1361–1379.

LEE, J., and COOLEY, B.., 1997, Recent advances in model predictive control. *Chemical Process Control—V*, Vol. 93, No. 316. AlChe Symposuim Series—American Institute of Chemical Engineers, pp. 201–216 b.

LEE, J. H., and YU, Z., 1997, Worst-case formulations of model predictive control for systems with bounded parameters. *Automatica*, **33**, 763–781.

MÄKILÄ, P. M., PARTINGTON, J. R., and GUSTAFSSON, T. K., 1995, Worst-case control-relevant identification. *Automatica*, **31**, 1799–1819.

MAYNE, D., and MICHALSKA, H., 1990, Receding horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, **35**, 814–824.

MAYNE, D., and POLAK, E., 1993, Optimization based design and control. *Preprints 12th IFAC World Congress*. Vol. 3, Sydney, pp. 129–138.

MILANESE, M., and VICINO, A., 1993, Information-based complexity and nonparametric worst-case system identification. *Journal of Complexity*, **9**, 427–446.

MOSCA, E., 1995, *Optimal, Predictive, and Adaptive Control* (Englewood Cliffs, New York: Prentice Hall).

QIN, S., and BADGEWELL, T., 1997, An overview of industrial model predictive control technology. *Chemical Process Control—V*, Vol. 93, No. 316. AIChe Symposium Series—American Institute of Chemical Engineers, pp. 232–256.

RAWLINGS, J., and MUSKE, K., 1993, The stability of constrained receding-horizon control. *IEEE Transactions on Automatic Control*, **38**, 1512–1516.

RICHALET, J., RAULT, A., TESTUD, J. L., and PAPON, J., 1978, Model predictive heuristic control-application to industrial processes. *Automatica*, **14**, 413–428.

SANCHEZ, J. M. M., 1976, Adaptive predictive control system. US Patent No. 4,196,576.

SCHWEPPE, F., 1968, Recursive state estimation: unknown but bounded errors and system inputs. *IEEE Transactions on Automatic Control*, **13**, 22–28.

SONTAG, E., 1995, On the input-to-state stability property. *European Journal of Control*, **1**, 24–36.

SUSSMANN, H., SONTAG, E., and YANG, Y., 1994, A general result on the stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control*, **39**, 2411–2424.

VICINO, A., and ZAPPA, G., 1996, Sequential approximation of feasible parameter sets for identification with set membership uncertainty. *IEEE Transactions on Automatic Control*, **41**, 774–785.

ZHENG, A., and MORARI, M., 1995, Stability of model predictive control with mixed constraints. *IEEE Transactions on Automatic Control*, **40**, 1818–1823.