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Dynamic Output Disturbance Models for Robust Constraint Satisfaction

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Abstract: Robust control methods such as tube-based robust model predictive control (MPC) schemes, developed to provide robust constraint satisfaction guarantees, require an uncertain model of the controlled plant. In this paper, we present a method to identify such models, along with a robust MPC scheme with reduced conservativeness tailored to employ them. We consider input-output models in which uncertainty is modeled as an additive disturbance on the output. Reduction of conservativeness is achieved by identifying the dynamics generating the disturbance. Standard linear system identification methods are used in the model development procedure, with residuals from the identification process extracted to characterize uncertainty in a set-membership setting. The effectiveness of a using dynamic output disturbance models is demonstrated through simulations.

Keywords: Uncertainty descriptions, Predictive control, Constrained control, Bounded error identification, Data-based control

1. INTRODUCTION

Model predictive control (MPC) schemes use a model of the plant being controlled to predict and optimize its future evolution. Accurate models can help MPC make better decisions with respect to performance and constraint satisfaction. However, not only high-fidelity models can make the MPC schemes unwieldy, but they may also impossible to get due to nondeterminism. Uncertain models represent a good trade-off between model simplicity and taking unmodeled dynamics and disturbances into account. In such models, inaccuracies with respect to a nominal model of the plant are characterized as uncertainties, and the corresponding robust MPC schemes explicitly use this uncertainty characterization while performing predictions, especially to enforce constraints under all circumstances.

In this work, we present a method to characterize uncertainty in a linear model of the plant, in a set-membership (SM) setting. An early survey of such techniques was presented in (Walter and Piet-Lahanier, 1990). Further methods were discussed in (Kosut et al., 1992), and more recently in (Terzi et al., 2018) and (Mohammadi et al., 2015). In all these works, a set-based characterization of the nominal model parameters was used to capture uncertainty.

Uncertain models obtained by following these methods can be used in min-max MPC schemes, such as that presented in (Kothare et al., 1996). Tube-based robust MPC schemes like those presented in (Chisci et al., 2001) and (Limon et al., 2010) provide an attractive alternative, since they use an optimization problem whose complexity is the same as that of nominal MPC, but with robust constraint satisfaction guarantees. Uncertain models developed for these schemes typically require (a) a state observer to estimate the initial state of the uncertain model, (b) a bounded disturbance characterization of uncertainty. This paper presents a method to develop uncertain models that satisfy both these requirements.

We deal with the problem of state estimation by working with input-output models. Assuming perfect measurements of input and output signals are available, these models allow us to construct the state vector exactly using past values of inputs and outputs, forgoing the need for a state observer. A detailed study of such models can be found in (Ljung, 1986). We first identify a nominal input-output model of the plant, and model its uncertainty as the model error acting as an additive disturbance on the output. Then, we identify a dynamical model causing this error, *along with the set of disturbances driving it.* This completes the characterization of a bounded disturbance form of uncertainty.

The idea follows along the lines of the error model presented in (Garulli and Reinelt, 2000) and (Ljung, 2000), but diverges from them since we model this error to be caused by an external source, rather than by the plant input. We remark that the presented methods can easily be extended to account for the contribution of plant inputs to the uncertainty, though we skip this aspect in this paper.

Alongside the identification procedure, we also present a tube-based MPC scheme that is tailored for the uncertain model parameterization we discuss. These models can also be used seamlessly within a robust reference governor framework, further details of which can be found in (Garone et al., 2017). Dynamic output disturbance models have previously been discussed in literature, e.g., in (Huusom et al., 2010) and (Ohshima et al., 1995), but within the context of offset-free tracking MPC schemes, without any discussion on robust constraint satisfaction.

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This paper is structured as follows. We introduce the uncertain model structure and discuss an identification procedure in Section 2. We present the specialized tube-based robust MPC scheme in Section 3. In Section 4, we present simulation results and discuss the merits of the chosen model parameterization. Conclusions and possible future directions are given in Section 5.

Notation: The operator $\oplus$ denotes Minkowski sum, $\ominus$ set difference. The symbol $\mathbb{I}_1^p$ denotes the set of indices $\{1,..,p\}$. Boldface $\mathbf{0}$ and $\mathbf{1}$ denote matrices of all 0's and 1's, with dimension given by the context.

2. IDENTIFICATION FOR ROBUST MODEL PREDICTIVE CONTROL

In this paper we consider uncertain systems of the form

$$x(t+1) = f_\mathrm{s}(x(t), u(t)),$$
$$y(t) = g_\mathrm{s}(x(t), v(t)),$$

with input $u(t) \in \mathbb{R}^{n_u}$, state $x(t) \in \mathbb{R}^{n_x}$, measured output $y(t) \in \mathbb{R}^{n_y}$, and bounded unmeasured output disturbance $v(t) \in \mathcal{R}^{n_v} \subset \mathbb{R}^{n_v}$ generated by a stationary process. We are interested in designing a controller which satisfies input and output constraints for all possible realizations of the disturbance $v(t)$. To that end, in Section 3 we will use tube-based robust Model Predictive Control (MPC) (Chisci et al., 2001). In the rest of this section, we will introduce an uncertain model $\mathcal{G}_\mathrm{M}$ of the plant $\mathcal{G}_\mathrm{P}$, and present methods to identify it from input-output data obtained by experiments.

2.1 Model Structure

We consider a *nominal* input-output model $\mathcal{G}_\mathrm{N}$ of the plant, given by

$$\bar{y}(t) = \sum_{i=1}^{n_\mathbf{A}} \mathbf{A}_i \bar{y}(t-i) + \sum_{i=1}^{n_\mathbf{B}} \mathbf{B}_i u(t-i). \qquad (1)$$

Due to the disturbance and model inaccuracy, in general the output behavior of the nominal model $\mathcal{G}_\mathrm{N}$ does not match the one of the plant $\mathcal{G}_P$ exactly.

A standard approach when developing uncertainty prediction models for robust MPC consists of modeling the nominal output error $\Delta y(t) = y(t) - \bar{y}(t)$ as an additive (bounded) disturbance. The uncertainty set $\mathbb{D}_\Delta$ satisfying $\Delta y(t) \in \mathbb{D}_\Delta$, $\forall t$, is then used to predict future uncertainty and adequately formulate the constraints in MPC. While this approach is viable, it can be conservative, since in many cases of interest, the output error is not directly caused by an additive disturbance that can change arbitrarily from one step t to the next within $\mathbb{D}_\Delta$, but rather has its own dynamics.

With the aim of reducing such conservativeness, we therefore model the output error $\Delta y(t) = y(t) - \bar{y}(t)$ as the output of a dynamical system, characterized by a *disturbance* model $\mathcal{G}_\mathrm{D}$ driven by a disturbance signal $w(t)$. Hence, the uncertain plant model $\mathcal{G}_\mathrm{M}$ is given by the composition of models $\mathcal{G}_\mathrm{N}$ and $\mathcal{G}_\mathrm{D}$, as schematically shown in Figure 1.

2.2 Identification Procedure

Identifying model $\mathcal{G}_\mathrm{N}$ requires a dataset $\mathbb{D}_\mathrm{P} = \{u(t), y(t),\ t = 0,..,N\}$, generated by suitably designed open-loop experiments (Ljung, 1986). For example, one can identify $\mathbf{A}$, $\mathbf{B}$ by the prediction-error method (PEM) (Ljung, 1986):

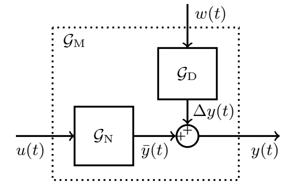

Fig. 1. System model

$$\min_{\mathbf{A},\mathbf{B}} \sum_{t=n_\mathbf{A}+1}^{N} \left\| y(t) - \sum_{i=1}^{n_\mathbf{A}} \mathbf{A}_i y(t-i) - \sum_{i=1}^{n_\mathbf{B}} \mathbf{B}_i u(t-i) \right\|_2^2. \qquad (2)$$

However, depending on the underlying assumptions, other identification strategies or model structures might be better suited. For more details see, e.g., (Palma and Magni, 2004).

To identify model $\mathcal{G}_\mathrm{D}$, we first build a dataset $\mathbb{D}_\Delta$ of the output errors $\Delta y(t)$, starting from the dataset $\mathbb{D}_\mathrm{P}$ as

$$\mathbb{D}_\Delta = \{\Delta y(t) = y(t) - \bar{y}(t),\ t = 0,..,N,\ y(t) \in \mathbb{D}_\mathrm{P}\}. \quad (3)$$

We simulate model (1) with initial condition $\bar{y}(t) = y(t)$, $\forall t \in [0, n_\mathbf{A}]$ and input $u(t)$, $t = 0, \dots, N-1$.

Using $\mathbb{D}_\Delta$, we identify $\mathcal{G}_\mathrm{D}$ as an autoregressive model, described by the linear difference equations

$$\Delta y(t) = \sum_{i=1}^{n_\mathbf{C}} \mathbf{C}_i \Delta y(t-i) + w(t-1). \qquad (4)$$

For the identification of $\mathcal{G}_\mathrm{D}$, similar considerations as for $\mathcal{G}_\mathrm{N}$ apply. PEM reads as

$$\min_{\mathbf{C}} \sum_{t=n_\mathbf{C}+1}^{N} \left\| \Delta y(t) - \sum_{i=1}^{n_\mathbf{C}} \mathbf{C}_i \Delta y(t-i) \right\|_2^2. \qquad (5)$$

Knowledge on $\mathcal{G}_\mathrm{N}$ and $\mathcal{G}_\mathrm{D}$ is not sufficient to predict a future plant output, since the set $\mathbb{W}$ of disturbances $w(t)$ is not known a priori. While w cannot be measured, the identification procedure performed to identify (4) also returns residuals $\{w(t), t \in [n_\mathbf{C}, N-1]\}$, which can be used to perform a simple set-membership identification. Equivalently, $w(t)$ can be computed using the same procedure used for $\Delta y(t)$.

A natural choice for set-membership identification is

$$\hat{\mathbb{W}} := \mathrm{Conv}\{w(t),\ \forall t \in [n_\mathbf{C}, N]\}. \qquad (6)$$

Note, however, that such a choice intrinsically underestimates the true uncertainty set $\mathbb{W}$, since the samples only cover the full set for $N \to \infty$. A thorough discussion on the best method for estimating $\mathbb{W}$ is out of the scope of this paper.

Knowing the nominal model $\mathcal{G}_\mathrm{N}$, the disturbance model $\mathcal{G}_D$ and the set of disturbances $\mathbb{W}$, we can make set-valued predictions of the future plant output $y(t+s)$ for a sequence of future inputs $\{u(t+m), m = 0,..,s-1\}$, as required to properly formulate robust MPC. We summarize the procedure to identify these in Algorithm 1.

Remark 1. The autoregressive parametrization of $\mathcal{G}_\mathrm{D}$ in (4) implicitly assumes that the process generating $\mathbb{D}_\Delta$ is time-invariant. One can extend the proposed method by

Algorithm 1 Identification of $\mathcal{G}_{\mathrm{M}}$

procedure IDENTIFY $\mathcal{G}_{\mathrm{N}}$, $\mathcal{G}_{\mathrm{D}}$ AND $\mathbb{W}$
  Build dataset $\mathbb{D}_{\mathrm{P}}$ through experiments
  Identify $\mathbf{A}$, $\mathbf{B}$, e.g., by solving (2)
  Build dataset $\mathbb{D}_{\Delta}$ as (3)
  Identify $\mathbf{C}$, e.g., by solving (5)
  Extract residuals w and identify $\mathbb{W}$
end procedure

borrowing ideas from adaptive MPC techniques (Tanaskovic et al., 2014) by computing the model parameters and disturbance sets $\mathbb{W}$ online.

3. ROBUST MODEL PREDICTIVE CONTROLLER

Since robust MPC schemes are typically formulated in state space, we will first write the plant model in state space form and then present the robust MPC formulation.

The state vectors of $\mathcal{G}_{\mathrm{N}}$ and $\mathcal{G}_{\mathrm{D}}$ are given by

$$x_{\mathrm{n}}(t) = \begin{bmatrix} \bar{y}(t) \\ \vdots \\ \bar{y}(t-n_{\mathbf{A}}+1) \\ u(t-1) \\ \vdots \\ u(t-n_{\mathbf{B}}+1) \end{bmatrix}, \qquad x_{\mathrm{d}}(t) = \begin{bmatrix} \Delta y(t) \\ \vdots \\ \Delta y(t-n_{\mathbf{C}}+1) \end{bmatrix}.$$

with dynamics

$$\begin{aligned} x_{\mathrm{n}}(t+1) &= A_{\mathrm{n}} x_n(t) + B_{\mathrm{n}} u(t), \\ \bar{y}(t) &= C_{\mathrm{n}} x_{\mathrm{n}}(t), \end{aligned} \tag{7a}$$

$$\begin{aligned} x_{\mathrm{d}}(t+1) &= A_{\mathrm{d}} x_{\mathrm{d}}(t) + B_{\mathrm{d}} w(t), \\ \Delta y(t) &= C_{\mathrm{d}} x_{\mathrm{d}}(t). \end{aligned} \tag{7b}$$

where matrices $(A_{\mathrm{n}}, B_{\mathrm{n}}, C_{\mathrm{n}}), (A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}})$ are obtained from the coefficients of (1) and (4), respectively. Since the nominal state-space model (7a) stems from the input-output form (1), a state estimator is not needed, since full knowledge on the state is obtained from the current and past output measurements, and the past inputs fed to the system. This presents an advantage over conventional MPC schemes, which also model uncertainty on the initial state using a state estimator to perform predictions (Chisci and Zappa, 2002). Once the measurement $y(t)$ becomes available, we update $\bar{y}(t) = y(t)$, such that $\Delta y(t) = 0$, entailing that $x_{\mathrm{d}}(t) = 0$. Note, however, that, given the information available at time t, $x_{\mathrm{d}}(\tau)$ evolves according to the dynamics in (7b) for $\tau > t$, with $w \in \mathbb{W}$.

We assume that the real plant $\mathcal{G}_P$ must satisfy the following constraints

$$(y(t), u(t)) \in \mathbb{C} := \{(y,u) : [H_y \ H_u] \begin{bmatrix} y^\top & u^\top \end{bmatrix}^\top \le h\}, \ \forall t, \tag{8}$$

such that $\mathbb{C}$ is a closed convex polytope with the origin in its interior. Before proceeding with the design of the controller, we make the following assumptions.

Assumption 1. a) The eigenvalues of A_{n} and A_{d} are strictly within the unit circle; b)The disturbance set $\mathbb{W}$ is compact, convex, and contains the origin in its interior.

We define the set $\Delta\mathcal{Y}(s)$ of all possible output prediction error realizations at time $t+s$ as

$$\Delta\mathcal{Y}(s) := \bigoplus_{m=0}^{s-1} C_{\mathrm{d}} A_{\mathrm{d}}^m B_{\mathrm{d}} \mathbb{W}. \tag{9}$$

Assumption 2. Sets $\mathbb{C}$ and $\mathbb{W}$ satisfy $H_y \Delta\mathcal{Y}(\infty) \subset \mathbb{C}$.

Remark 2. Note that Assumption 1 is not restrictive, since it requires stabilizability of the nominal system. If the nominal system is unstable, one can introduce a stabilizing linear feedback controller, such that the closed-loop dynamics are stable. Since the developments of this paper hold for any pre-stabilized system, we directly assume that A_{n} is the state transition matrix of the pre-stabilized system.

We design a robust linear MPC (RMPC) using a tube-based approach, that reads the current nominal model state $x_{\mathrm{n}}(t)$ and current output $y(t)$, and solves the optimization problem

$$\min_{\mathbf{z}} \sum_{s=1}^{L-1} \|\bar{y}(t+s|t) - \mathbf{y}\|_Q^2 + \|u(t+s|t) - u(t+s-1|t)\|_R^2 \tag{10a}$$

$$\begin{aligned} \text{s.t.} \ & x_{\mathrm{n}}(t|t) = x_{\mathrm{n}}(t), && (10b) \\ & x_{\mathrm{n}}(t+s|t) = A_{\mathrm{n}} x_{\mathrm{n}}(t+s-1|t) + B_{\mathrm{n}} u(t+s-1|t), && \\ & && s \in \mathbb{I}_1^{L-1}, (10c) \\ & \bar{y}(t+s|t) = C_{\mathrm{n}} x_{\mathrm{n}}(t+s|t), && s \in \mathbb{I}_1^{L-1}, (10d) \\ & H_y \bar{y}(t+s|t) + H_u u(t+s|t) \le h(s), && s \in \mathbb{I}_1^{L-1}, (10e) \\ & H_u u(t|t) \le h - H_y y(t), && (10f) \\ & G x_{\mathrm{n}}(t+L|t) \le g, && (10g) \end{aligned}$$

where $x_{\mathrm{n}}(t+s|t)$, $\bar{y}(t+s|t)$ and $u(t+s|t)$ are the predicted state, output and input respectively of the nominal model at timestep $t+s$, given the model state $x_{\mathrm{n}}(t)$. The optimization variables of this problem are $\mathbf{z} := (u(t|t), x_{\mathrm{n}}(t+1|t), \ldots, u(t+L-1|t), x_{\mathrm{n}}(t+L|t))$. The objective of this controller is to drive the nominal predicted output $\bar{y}(t+s|t)$ towards a reference $\mathbf{y}$. The first input $u(t|t)$, that is the only one applied to the plant $\mathcal{G}_P$, is constrained by (10f) as a function of the current output $y(t)$. RMPC solves Problem (10) at every time instant t, given the current state $x_{\mathrm{n}}(t)$. In this paper we focus on optimizing the nominal cost, as in tube-based RMPC. Nevertheless, the disturbance model can also be deployed with min-max RMPC

While the MPC formulation (10) seems to neglect the uncertainty, it implicitly accounts for it by a constraint tightening procedure which yields $h(s) \le h$. In order to compute $h(s)$, we exploit the facts that (a) $x_{\mathrm{d}}(t) = \mathbf{0}$, and (b) $w(t+s|t) \in \mathbb{W}$, to compute

$$h^i(s) = h^i - \max_{\Delta y \in \Delta\mathcal{Y}(s)} H_y^i \Delta y = h^i - \sum_{m=0}^{s-1} \max_{w \in \mathbb{W}} H_y^i C_{\mathrm{d}} A_{\mathrm{d}}^m B_{\mathrm{d}} w, \tag{11}$$

for each $s = 1, .., L-1$. If the set $\mathbb{W}$ is a polytope, the maximization problems in (11) can be formulated as linear programs, which are solved offline before building optimization problem (10).

Finally, terminal constraints (10g) are introduced to ensure that the problem remains feasible beyond the prediction horizon, i.e., for all time $t+s$ with $s \ge L$. The terminal constraint set $\mathbb{X}_{\mathrm{f}} := \{x_{\mathrm{n}} : G x_{\mathrm{n}} \le g\}$ is formulated as a robust positive invariant (RPI) set:

$$\mathbb{X}_{\mathrm{f}} = \{\, x \mid A_{\mathrm{n}} x \in \mathbb{X}_{\mathrm{f}}, H_y (C_{\mathrm{n}} x + \Delta y(s)) \le h, \forall\, s \,\};$$

i.e., a set of nominal model states, such that for all possible future disturbance sequences $\{w(t+s) \in \mathbb{W}\}$, the plant constraints (8) are respected and the states remain inside the set $\mathbb{X}_{\mathrm{f}}$. In order to make the domain of the RMPC problem as large as possible, $\mathbb{X}_{\mathrm{f}}$ is commonly chosen as the maximum RPI set, which we suggest to compute as in the following two stages: First, we assume $x_{\mathrm{d}}(t+L) = \mathbf{0}$. Then, we account for uncertainty in $x_{\mathrm{d}}(t+L)$ by tightening the obtained constraints further, as detailed below:

- Under the assumption that $x_{\mathrm{d}}(t+L) = \mathbf{0}$, we define the RPI set
  $\hat{\mathbb{X}}_{\mathrm{f}} = \{\, x_{\mathrm{n}} \,|\, A_{\mathrm{n}} x_{\mathrm{n}} \in \hat{\mathbb{X}}_{\mathrm{f}}, H_y y(s) \leq h,\ x_{\mathrm{d}}(0) = \mathbf{0},\ \forall\, s \,\}.$
  Then, we define the set $\hat{\mathbb{X}}(s)$ of nominal states such that the plant constraints (8) are satisfied over the time interval $[t, t+s]$:
  $$\hat{\mathbb{X}}(0) := \{x_{\mathrm{n}}(t) : H_y C_{\mathrm{n}} x_{\mathrm{n}}(t) \leq h(0)\},$$
  $$\hat{\mathbb{X}}(s) := \{x_{\mathrm{n}}(t) \in \hat{\mathbb{X}}(s-1) : H_y C_{\mathrm{n}} A_{\mathrm{n}}^s x_{\mathrm{n}}(t) \leq h(s)\}.$$
  Under Assumptions 1 and 2, there exists a *finite-determination index* s^* such that $\hat{\mathbb{X}}_{\mathrm{f}} = \hat{\mathbb{X}}(s^*) = \hat{\mathbb{X}}(s^*+1)$. (Kolmanovsky and Gilbert, 1998)
- The uncertainty set $\mathbb{X}_{\mathrm{d}}(L)$ of the disturbance model state at time $t+L$ is given by $\mathbb{X}_{\mathrm{d}}(L) = \bigoplus_{m=0}^{L-1} A_{\mathrm{d}}^m B_{\mathrm{d}} \mathbb{W}$. Then, the actual set of nominal states satisfying plant constraints (8) in time interval $[t, t+s]$ accounting for this uncertainty are obtained by tightening $\hat{\mathbb{X}}(s)$ as
  $$\mathbb{X}(s) := \hat{\mathbb{X}}(s) \ominus H_y C_{\mathrm{d}} A_{\mathrm{d}}^s \mathbb{X}_{\mathrm{d}}(L)$$

By computing $q^i(s) = \max_{x_{\mathrm{d}} \in \mathbb{X}_{\mathrm{d}}(L)} H_y^i C_{\mathrm{d}} A_{\mathrm{d}}^s x_{\mathrm{d}}$ for each $s = 0, .., s^*$, the matrices G and g defining $\mathbb{X}_{\mathrm{f}}$ are hence given by

$$G = \begin{bmatrix} H_y C_{\mathrm{n}} \\ H_y C_{\mathrm{n}} A_{\mathrm{n}} \\ \vdots \\ \vdots \\ H_y C_{\mathrm{n}} A_{\mathrm{n}}^{s^*} \end{bmatrix}, \qquad g = \begin{bmatrix} h(0) - q(0) \\ h(1) - q(1) \\ \vdots \\ \vdots \\ h(s^*) - q(s^*) \end{bmatrix}.$$

The role of Assumption (2) is now clear: it guarantees that $\mathbb{X}_{\mathrm{f}} \neq \emptyset$ and (10e) defines a nonempty set. In turn, these two conditions ensure that the domain of Problem (10) is nonempty.

4. SIMULATION RESULTS

We present simulation results obtained with two examples. In the first one, the real plant $\mathcal{G}_P$ is a single-input single-output (SISO) system matching the assumed structure exactly. In the second one, we consider a 2-input 2-output nonlinear precompensated mass-spring-damper system. Simulations were performed using MATLAB, and the identification procedures were carried out using the System Identification Toolbox™.

4.1 SISO Plant with Output Disturbances

We consider the SISO plant $\mathcal{G}_P$ described by

$$\begin{aligned} y_u(t) &= 0.6 y_u(t-1) - 0.6 y_u(t-2) + u(t-1), \\ y_v(t) &= 0.8 y_v(t-1) - 0.8 y_v(t-2) + v(t-1), \\ y(t) &= y_u(t) + y_v(t). \end{aligned} \tag{12}$$

with simulations performed by uniformly sampling $v(t) \in [-2, 2]$. For ease of presentation, we assume that the model structure is known a priori

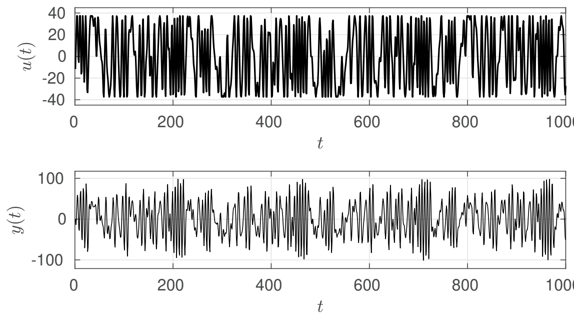

Fig. 2. $\mathbb{D}_{\mathrm{P}}$ for plant (12).

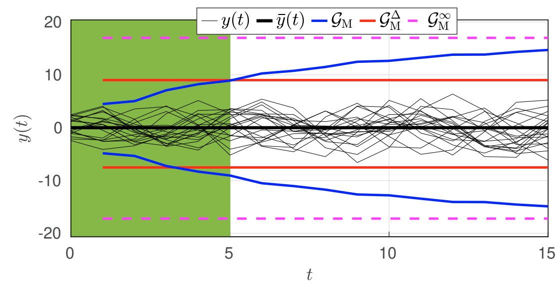

Fig. 3. Comparison of predictive performance of uncertain models.

Using the dataset $\mathbb{D}_{\mathrm{P}}$ shown in Figure 2, we solve (2) to identify the nominal model $\mathcal{G}_{\mathrm{N}}$ as

$$\bar{y}(t) = 0.5978 \bar{y}(t-1) - 0.5985 \bar{y}(t-2) + 0.9981 u(t-1).$$

We consider the following two disturbance models that accompany $\mathcal{G}_{\mathrm{N}}$ to perform set-valued predictions of future plant output:

(1) Dynamic disturbance model $\mathcal{G}_{\mathrm{D}}$: The design of this model follows the procedure described in Section 2. We identify it as
$$\Delta y(t) = 0.7932 \Delta y(t-1) - 0.791 \Delta y(t-2) + w(t-1),$$
with disturbance set $\mathbb{W} = \{-2.1687 \leq w(t) \leq 2.1492\}$. We use $(A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}})$ to represent the state-space matrices corresponding to this model. The uncertain model $\mathcal{G}_{\mathrm{M}}$ is composed of $\mathcal{G}_{\mathrm{N}}$ and $\mathcal{G}_{\mathrm{D}}$, which performs the predictions as $y(t) \in \{\bar{y}(t)\} \oplus \Delta\mathcal{Y}(t)$.

(2) Static disturbance model $\mathcal{G}_{\Delta}$: This model considers the bounds on the output error $\Delta y(t)$ obtained from $\mathbb{D}_{\Delta}$, taken into account as an additive disturbance on the output. These bounds are identified as $\mathbb{W}_{\Delta} = \{-8.3751 \leq \Delta y(t) \leq 8.6271\}$. The uncertain model $\mathcal{G}_{\mathrm{M}}^{\Delta}$ is composed of $\mathcal{G}_{\mathrm{N}}$ and $\mathcal{G}_{\Delta}$, which performs the predictions as $y(t) \in \{\bar{y}(t)\} \oplus \mathbb{W}_{\Delta}$.

We compare the predictive performance of these two uncertain models in Figure 3, for different realizations of $v(t)$. The plant and the models start from initial conditions set to $\mathbf{0}$, and no input $u(t)$ is applied. We also plot the sets $y(t) \in \{\bar{y}(t)\} \oplus \Delta\mathcal{Y}(\infty)$ obtained by simulating the model $\mathcal{G}_{\mathrm{M}}^{\infty}$, which represents the asymptotic behavior of $\mathcal{G}_{\mathrm{M}}$.

We can see from Figure 3 that $\mathcal{G}_{\mathrm{M}}$ gives tighter predictions than $\mathcal{G}_{\mathrm{M}}^{\Delta}$ for the initial few steps (indicated in green). This is expected, since these predictions take the dynamics of the disturbance model into account. Since MPC schemes usually exploit tighter predictions to reduce conservative-

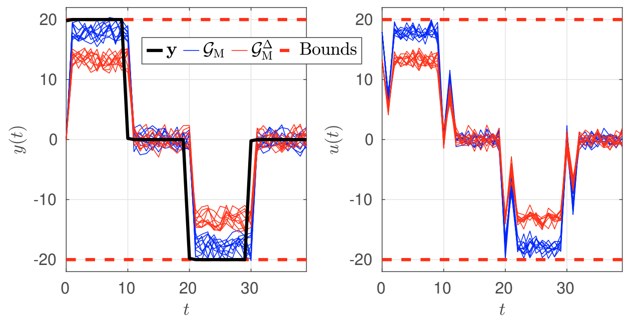

Fig. 4. Comparison of robust MPC performance for different realizations of $v(t)$.

ness, we expect that a robust MPC designed using $\mathcal{G}_{\mathrm{M}}$ would be less conservative than that using $\mathcal{G}_{\mathrm{M}}^{\Delta}$. This is evidenced in Figure 4, in which the tracking performance of robust MPC controllers designed using the uncertain models $\mathcal{G}_{\mathrm{M}}$ and $\mathcal{G}_{\mathrm{M}}^{\Delta}$ are compared. The figure shows that $\mathcal{G}_{\mathrm{M}}$ consistently offers less conservative performance for different realizations of $v(t)$. Infact, the average sum of stage costs over the realizations is 158.5289 with $\mathcal{G}_{\mathrm{M}}$, and 971.4325 with $\mathcal{G}_{\mathrm{M}}^{\Delta}$. The parameters of the controllers are

$$\mathbb{C} = \left\{ \begin{bmatrix} -20 \\ -20 \end{bmatrix} \le \begin{bmatrix} y \\ u \end{bmatrix} \le \begin{bmatrix} 20 \\ 20 \end{bmatrix} \right\}, L = 15, Q = 1, R = 10^{-4}.$$

The slight constraint violation in the case of $\mathcal{G}_{\mathrm{M}}$ can be attributed to the underestimation of the true set $\mathbb{W}$ from a finite set of data. If knowledge on the noise distribution is available, it seems possible to estimate the constraint violation probability by borrowing ideas from (Bujarbaruah et al., 2019).

In addition to being conservative in predictions over short horizons, the uncertain model $\mathcal{G}_{\mathrm{M}}^{\Delta}$ might be describing overly optimistic bounds on $\Delta y(t)$ over longer horizons. This is because given $\mathcal{G}_{\mathrm{M}}$ and $\mathbb{W}$, we can only obtain $\mathbb{W}_{\Delta} \subset \Delta\mathcal{Y}(s) \subset \Delta\mathcal{Y}(\infty)$ for some integer $s > 0$ from a finite dataset.

4.2 Nonlinear plant

In this example, we consider the following nonlinear spring-mass-damper system described by the discretized dynamics

$$\begin{bmatrix} \mathbf{p}(t+1) \\ \mathbf{v}(t+1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(t) + T_s \mathbf{v}(t) \\ \mathbf{v}(t) - \frac{T_s}{m}(c\mathbf{v}(t) + K\mathbf{p}(t)) + \frac{T_s}{m}F(t) - \frac{T_s}{m}\hat{K}\mathbf{p}(t)^3 \end{bmatrix} \tag{13}$$

with a feedback controller $F(t) = K_1 u(t) - K_2 y(t)$ to be the plant $\mathcal{G}_P$. The parameters of this system are mass $m = 0.5Kg$, damping coefficient $c = 3Ns/m$, stiffness coefficient $K = 5N/m$, nonlinear stiffness coefficient $\hat{K} = 0.05N/m^3$, and $T_s = 0.05s$. The controller coefficients are $K_1 = [10.2564\ 4.4115]$ and $K_2 = [5.2564\ 1.4115]$. The output vector is $y(t) = [\mathbf{p}(t)\ \mathbf{v}(t)]^\top$. Though this plant does not match the assumed structure, we demonstrate that a dynamic disturbance model could still be used to obtain reduced conservativeness when the effect of the nonlinearities do not dominate.

The dataset $\mathbb{D}_{\mathrm{P}}$ used to identify the models $\mathcal{G}_{\mathrm{N}}$, $\mathcal{G}_{\mathrm{D}}$ and disturbance set $\mathbb{W}$ following Algorithm 1 is shown in Figure 5. The identified parameters are

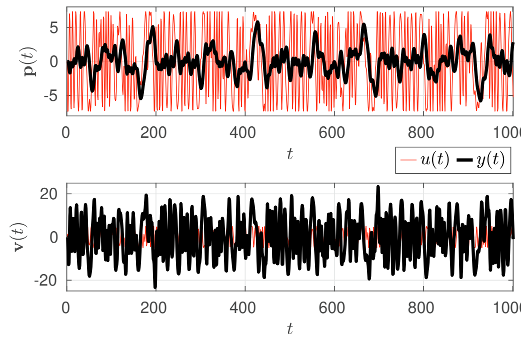

Fig. 5. $\mathbb{D}_{\mathrm{P}}$ for system (13) with feedback controller $F(t) = K_1 u(t) - K_2 y(t)$.

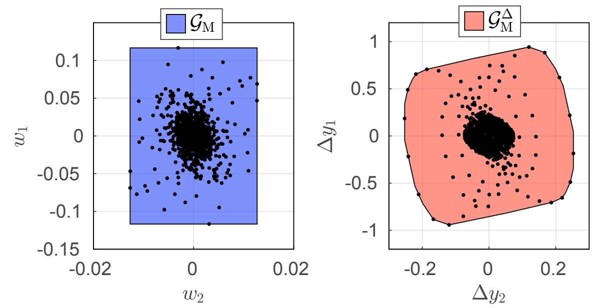

Fig. 6. Disturbance input sets for $\mathcal{G}_{\mathrm{M}}$ and $\mathcal{G}_{\mathrm{M}}^{\Delta}$.

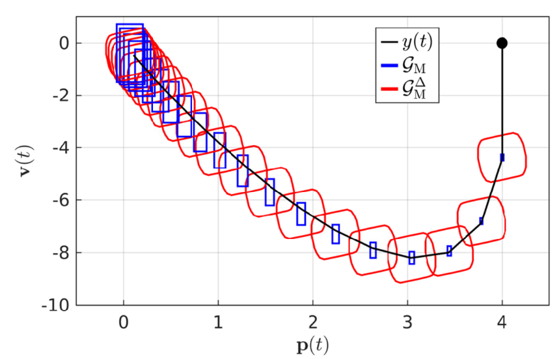

Fig. 7. Comparison of predictive performance of uncertain models.

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0.05 \\ -1.096 & 0.5561 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ 1.024 & 0.4432 \end{bmatrix}, \mathbf{C}_1 = \begin{bmatrix} 1.917 & 0 \\ 0 & 1.799 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.00288 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.002 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} -0.9509 & 0 \\ 0 & -0.9265 \end{bmatrix}.$$

As discussed in the previous example, we also consider the static output disturbance model $\mathcal{G}_{\Delta}$, whose disturbance set $\mathbb{W}_{\Delta}$ is constructed using the prediction error samples $\Delta y(t)$ directly. These two sets are shown in Figure 6, and the corresponding predictive performance in the output space are compared in Figure 7.

We synthesize RMPC controllers using these two models, whose tracking performance is compared in Figure 8. We use the parameters $\mathbb{C} = \{(y,u) : -8 \le y_2 \le 8,\ -80 \le K_1 u - K_2 y \le 80\}$, $L = 20$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 0.001$. As in the first example, less conservative performance is obtained when the dynamic disturbance model is used.

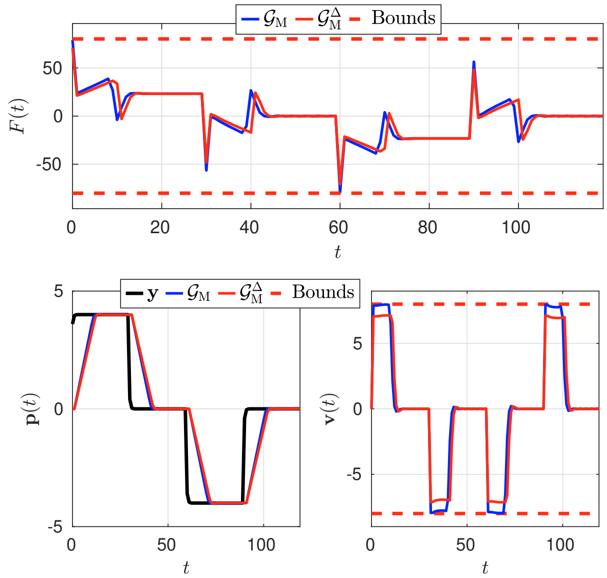

Fig. 8. Comparison of force input and robust MPC closed-loop performance.

The sum of stage costs is 238.351 with $\mathcal{G}_{\mathrm{M}}$, and 264.868 with $\mathcal{G}_{\mathrm{M}}^{\Delta}$.

5. CONCLUSIONS

We presented a procedure to identify an uncertain model of a plant described by an output disturbance superimposed to a nominal output. While we have used well-assessed identification tools taken from literature, we derived datasets in a way that the output uncertainty can be captured and modeled, and used the resulting uncertainty model in a tube-based robust MPC scheme. We have demonstrated the effectiveness of the proposed procedure in reducing conservativeness in two simulation examples.

Several topics of future research are open, such as: (a) Extension to cases where the exact model states are unknown (noisy settings and generic state-space models); (b) Extension to models with coupling of nominal and disturbance-model states; (c) Extension to cases like decentralized control, where additional information on the output disturbance is available; (d) A theoretical analysis of the proposed procedure. Our method can also be extended in the direction of identification for stochastic MPC along the lines of (Shang and You, 2018), since ARX and AR model identification procedures are inherently probabilistic methods.

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