Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 50-1 (2017) 524–530

Stochastic economic model predictive control for Markovian switching systems ⋆

Pantelis Sopasakis ⋆⋆, **Domagoj Herceg** ⋆,
Panagiotis Patrinos ⋆⋆ and **Alberto Bemporad** ⋆

⋆ *IMT School for Advanced Studies Lucca, Piazza San Ponziano 6, 55100 Lucca, Italy.*
⋆⋆ *KU Leuven, Department of Electrical Engineering (ESAT), STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics & Optimization in Engineering (OPTEC), Kasteelpark Arenberg 10, 3001 Leuven, Belgium.*

Abstract: The optimization of process economics within the model predictive control (MPC) formulation has given rise to a new control paradigm known as economic MPC (EMPC). Several authors have discussed the closed-loop properties of EMPC-controlled deterministic systems, however, little have uncertain systems been studied. In this paper we propose EMPC formulations for nonlinear Markovian switching systems which guarantee recursive feasibility, asymptotic performance bounds and constrained mean square (MS) stability.

Keywords: Stochastic control; Economic model predictive control; Markovian switching systems; Stochastic dissipativity.

1. INTRODUCTION

1.1 Background and motivation

Recently, a new approach to model predictive control (MPC) termed *economic model predictive control* (EMPC) has gained a lot of attention. Rather than minimizing a deviation from a prescribed (optimal/best) set-point or a tracking reference, the main objective in EMPC is to optimize a given economic cost functional (Angeli et al., 2012). Often, in engineering practice, the main objective is to devise control algorithms which asymptotically guarantee an economic operation of the controlled plant.

Already, a considerable body of theoretical results has been reported in the literature characterizing the asymptotic performance of EMPC. Perhaps *dissipativity* is the most salient notion in the pertinent literature which is shown to be a sufficient condition for proving optimal operation at a steady state and stability of EMPC formulations (Angeli et al., 2012). The same authors show that economic MPC has no worse an asymptotic average performance than the best admissible steady state operation (Müller et al., 2013).

The introduction of a, possibly non-quadratic and nonconvex, economic cost into the MPC framework disqualifies the standard stability analysis used in the MPC literature. Angeli et al. (2012) propose the use of a simple terminal constraint to guarantee stability of EMPC-controlled systems which is generalized by Amrit et al. (2011) using terminal set constraints. Fagiano and Teel (2013) use a generalized terminal state equality constraint where the target terminal state is left as a free variable to be optimized which increases the feasibility region of EMPC. This concept was further generalized to include a terminal region constraint (Müller et al., 2014). It was further shown that EMPC can achieve near-optimal operation without terminal constraints and costs for a sufficiently large prediction horizon (Grüne, 2013). Similar results exist for a system that is best operated at a periodic regime (Zanon et al., 2013). It is worth noting that this wealth of results concerns only deterministic systems.

In spite of the noticeable interest for the idea of EMPC there are very few theoretical results accounting for uncertainty, which is often relevant in a real-world operation. Bø and Johansen (2014) propose a scenario-based EMPC formulation for fault-tolerant constrained regulation and a similar approach is pursued by Lucia et al. (2014a). Lucia et al. (2014b) present a multi-stage scenario-based nonlinear MPC control strategy validated on a benchmark example, but no performance guarantees or stability analysis is provided. An interesting theoretical treatment is given by Bayer et al. (2014) where a tube-based EMPC formulation is proposed for constrained systems with bounded additive disturbances. Very recently Bayer et al. (2016) proposed a robust economic MPC formulation for linear systems with bounded additive uncertainty with known probability distribution.

1.2 Contributions

In this paper we endeavor to cover the theoretical gap in EMPC for an important class of stochastic systems — the Markovian switching systems. We first study the properties of an MPC formulation for Markovian switching systems where optimal steady states are mode-dependent. We propose an MPC scheme which is recursively feasible and satisfies an asymptotic performance bound. Assuming

⋆ This work was supported by the EU-funded H2020 research project DISIRE, grant agreement No. 636834. The work of the third author was supported by the KU Leuven Research Council under BOF/STG-15-043.

10.1016/j.ifacol.2017.08.056

that there is a common optimal steady state, we show that the MPC-controlled system is mean-square (MS) stable when a stochastic dissipativity condition is satisfied. We then formulate a variant of the MPC problem using mode-dependent terminal constraints and provide mean-square stability conditions and performance bounds. We then provide guidelines for the design of mean-square stabilizing predictive controllers for nonlinear systems imposing weak conditions on the system dynamics and the EMPC stage cost.

1.3 Notation and mathematical preliminaries

Let $\mathbb{R}$ and $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ denote the sets of real numbers, nonnegative reals, n-dimensional real vectors and n-by-m matrices. Let $\mathcal{B}_\delta$ be the ball of $\mathbb{R}$ of radius δ, that is $\mathcal{B}_\delta := \{x : \|x\| < \delta\}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *lower semicontinuous* if its epigraph, that is the set $\operatorname{epi} f = \{(x,\alpha) \in \mathbb{R}^{n+1} : f(x) \leq \alpha\}$, is closed. We say that $f : \mathbb{R}^n \to \mathbb{R}$ is *level-bounded* if its level sets, $\operatorname{lev}_\alpha f = \{x : f(x) \leq \alpha\}$, are bounded. We say that $f : \mathbb{R}^n \times \mathbb{R}^m \ni (x,u) \mapsto f(x,u) \in \mathbb{R}$ is *level-bounded in u locally uniformly in x* if for every $\bar{x}$ there is a neighborhood of $\bar{x}$, $V_{\bar{x}} \subseteq \mathbb{R}^n$, so that $\{(x,u) : x \in V_{\bar{x}}, f(x,u) \leq \alpha\}$ is bounded. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *β-smooth* if it is differentiable with β-Lipschitz gradient, that is $\|\nabla f(y) - \nabla f(x)\| \leq \beta\|y - x\|$ for all $x, y \in \mathbb{R}^n$; then, we have that $\|f(y) - f(x) - \nabla f(x)(y - x)\| \leq \frac{\beta}{2}\|y - x\|^2$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is positive definite around x_0 if $f(x_0) = 0$ and $f(x) > 0$ for $x \neq x_0$. $A \succcurlyeq 0$ denotes that A is a positive semidefinite matrix and $A \succ 0$ means that A is positive definite. We denote the transpose of a matrix A by $A^\top$.

2. STOCHASTIC ECONOMIC MODEL PREDICTIVE CONTROL

2.1 System dynamics

Consider the following Markovian switching system

$$x_{k+1} = f(x_k, u_k, \theta_k), \tag{1}$$

driven by the random parameter θ_k which is a time-homogeneous irreducible and aperiodic Markovian process with values in a finite set $\mathcal{N} = \{1, \ldots, \nu\}$ with transition matrix $P = (p_{ij}) \in \mathbb{R}^{\nu\times\nu}$ and *initial distribution* $v = (v_1, \ldots, v_\nu)$ (Costa et al., 2005). We assume that at time k we measure the full state x_k and the value of θ_k. Markov jump linear systems (MJLS) with additive disturbances are a special case of (1) with $f(x,u,\theta) = A_\theta x + B_\theta u + w_\theta$.

Let $\Omega := \prod_{k\in\mathbb{N}}(\mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N})$ and $\mathfrak{F}_k$ be the minimal σ-algebra over the Borel-measurable rectangles of Ω with k-dimensional base and $\mathfrak{F}$ be the minimal σ-algebra over all Borel-measurable rectangles. Define the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_k\}_{k\in\mathbb{N}}, \mathrm{P})$ where P is the unique product probability measure according to (Ash, 1972, Th. 2.7.2) with $\mathrm{P}(\theta_0 = i_0, \theta_1 = i_1, \ldots, \theta_k = i_k) = v_{i_0} p_{i_0 i_1} \cdots p_{i_{k-1} i_k}$ for any $i_0, i_1, \ldots, i_k \in \mathcal{N}$ and $k \in \mathbb{N}$, where θ_k is an $\mathfrak{F}_k$-adapted random variable from Ω to $\mathcal{N}$. We will use the notation $u \lhd \mathfrak{F}_k$ to denote that the random variable u is $\mathfrak{F}_k$-measurable.

Let $\mathbb{E}[\cdot]$ denote the expectation of a random variable with respect to P and $\mathbb{E}[\cdot|\mathfrak{F}_k]$ the conditional expectation. It can be shown (Tejada et al., 2010) that the augmented state (x_k, θ_k) contains all the probabilistic information relevant to the evolution of the Markovian switching system for all time instants $t > k$.

Definition 1. (Cover and bet node). For every node $i \in \mathcal{N}$, the *cover* of i is the set $\mathcal{C}(i) = \{j \in \mathcal{N} \mid p_{ij} > 0\}$. The *bet node* of an $i \in \mathcal{N}$ is a node $\operatorname{bet}(i) \in \mathcal{C}(i)$ with $p_{i\operatorname{bet}(i)} \geq p_{ij}$ for all $j \in \mathcal{C}(i)$.

A bet of a mode $\theta_k = i$ is one of the most likely successor modes θ_{k+1}.

System (1) is subject to the following joint state-input constraints

$$(x_k, u_k) \in Y_{\theta_k}. \tag{2}$$

Let $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}$ be a mode-dependent cost function.

Assumption 1. (Well-posedness). For each $\theta \in \mathcal{N}$, $\ell(\cdot,\cdot,\theta)$ are nonnegative, lower semicontinuous and level-bounded in u locally uniformly in x, $f(\cdot,\cdot,\theta)$ are continuous and sets Y_θ are nonempty and compact. The random process $\{\theta_k\}_k$ is an irreducible and aperiodic Markov chain.

Definition 2. (Optimal steady states). Given a stage cost function $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}$ which satisfies Assumption 1, a pair (x_s^θ, u_s^θ) is called an *optimal steady state* of (1) subject to (2) with respect to ℓ if it is a minimizer of the problem

$$\ell_s(\theta) := \min_{x,u} \{\ell(x,u,\theta) | f(x,u,\theta) = x, (x,u) \in Y_\theta\}$$

For reasons that will be better elucidated in the next section, we need to draw the following controllability assumption essentially requiring that if $x_k = x_s^i$ and $\theta_k = j$ then there is a control action $\bar{u}_s^{i,j}$ so that at time $k+1$ the state is steered to $x_{k+1} = x_s^{\operatorname{bet}(j)}$.

Assumption 3. (Controllability). In addition to Assumption 1, for all $i, j \in \mathcal{N}$ there is a control law $\bar{u}_s : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ with $\bar{u}_s(x_s^i, j) = \bar{u}_s^{i,j}$ so that $(x_s^i, \bar{u}_s^{i,j}) \in Y_j$ and $f(x_s^i, \bar{u}_s^{i,j}, j) = x_s^{\operatorname{bet}(j)}$.

2.2 Model predictive control

In this section we shall present a model predictive control framework for constrained Markovian switching systems with mode-dependent optimal steady-state points.

Let $u_k \lhd \mathfrak{F}_k$ and $\mathbf{u}_N = (u_0, \ldots, u_{N-1})$, and define

$$V_N(x_0, \theta_0, \mathbf{u}_N) = \mathbb{E}\left[V_f(x_N, \theta_N) + \sum_{j=0}^{N-1} \ell(x_j, u_j, \theta_j) \middle| \mathfrak{F}_0\right].$$

Here, we take $V_f = 0$ and let the state sequence satisfy (1).

We introduce the following stochastic economic model predictive control problem

$$\mathbb{P}(x,\theta) : V_N^\star(x,\theta) = \inf_{\mathbf{u}_N} V_N(x,\theta,\mathbf{u}_N), \tag{3a}$$

and for $k = 0, \ldots, N-1$, subject to

$$x_{k+1} = f(x_k, u_k, \theta_k) \tag{3b}$$
$$(x_k, u_k) \in Y_{\theta_k} \tag{3c}$$
$$(x_0, \theta_0) = (x, \theta) \tag{3d}$$
$$x_N = x_s^{\operatorname{bet}(\theta_{N-1})} \tag{3e}$$
$$u_k \lhd \mathfrak{F}_k. \tag{3f}$$

Because of Assumption 1 and in light of (Rockafellar and Wets, 2009, Thm. 1.17) the infimum in (3) is attainable and the corresponding set of minimizers is compact. Note that in the above formulation the minimization is carried out in a space of control policies $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$ where u_k are *causal* control laws — as required by (3f).

Let $\mathbf{u}^\star(x,\theta) = \{u_0^\star(x,\theta), \ldots, u_{N-1}^\star(x_{N-1}, \theta_{N-1})\}$ be an optimizer of (3). The receding horizon control law that accrues from this problem is $\kappa_N(x,\theta) := u_0^\star(x,\theta)$ and the closed-loop system satisfies

$$x_{k+1} = f(x_k, \kappa_N(x_k, \theta_k), \theta_k). \tag{4}$$

2.3 Recursive feasibility

We will now prove that the MPC problem in (3) is recursively feasible.

Proposition 4. Let $X_N \subseteq \mathbb{R}^n \times \mathcal{N}$ be the domain of problem $\mathbb{P}$. If Assumption 3 holds and problem $\mathbb{P}(x_k, \theta_k)$ is feasible, then problem $\mathbb{P}(x_{k+1}, \theta_{k+1})$, with $x_{k+1} = f(x_k, \kappa_N(x_k, \theta_k), \theta_k)$ and $\theta_{k+1} \in \mathcal{C}(\theta_k)$, is also feasible.

Proof. For given $(x,\theta) \in X_N$ let $\pi(x,\theta) = \{u_0^\star, \ldots, u_{N-1}^\star\}$ be an optimizer of $\mathbb{P}(x,\theta)$ and let $x^\star(x,\theta) = \{x, x_1^\star, \ldots, x_N^\star\}$ be the corresponding sequence of states. Because of (3e) we have that

$$x_N^\star = x_s^{\mathrm{bet}(\theta_{N-1})}.$$

Now take $x^+ = f(x, u_0^\star(x,\theta), \theta)$ and $\theta^+ \in \mathcal{C}(\theta)$. We need to show that $\mathbb{P}(x^+, \theta^+)$ is feasible. Take $\tilde{\pi}^+(x^+, \theta^+) := \{u_1^\star, \ldots, u_{N-1}^\star, u\}$ and let $u = \bar{u}_s(x_N, \theta_N)$. Then, by virtue of Assumption 3, $x_{N+1}^\star = x_s^{\mathrm{bet}(\theta_N)}$, so $\tilde{\pi}^+$ will satisfy the constraints of $\mathbb{P}(x^+, \theta^+)$. □

2.4 Performance assessment

We will now prove that the closed-loop system has a bounded expected asymptotic average cost (Thm. 6). First, we need to give the following result:

Lemma 5. Let Assumption 3 hold and let

$$\ell_N(\theta_k) := \mathbb{E}\Big[\ell(x_s^{\mathrm{bet}(\theta_{N-1})}, \bar{u}_s^{\mathrm{bet}(\theta_{N-1}),\theta_N}, \theta_N) \mid \theta_0 = \theta\Big]$$

and $\mathcal{L}V_N^\star(x_k, \theta_k) := \mathbb{E}[V_N^\star(x_{k+1}, \theta_{k+1}) - V_N^\star(x_k, \theta_k) | \mathfrak{F}_k]$; then, the following holds for all $(x_k, \theta_k) \in X_N$

$$\mathcal{L}V_N^\star(x_k, \theta_k) \leq \ell_N(\theta_k) - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k). \tag{5}$$

Proof. Let $(x,\theta) \in X_N$; then $\tilde{\pi}^+(x_{k+1}, \theta_{k+1})$ is feasible — but not necessarily optimal — for $\mathbb{P}(x_{k+1}, \theta_{k+1})$, therefore, $V_N^\star(x_{k+1}, \theta_{k+1}) \leq V_N(x_{k+1}, \theta_{k+1}, \tilde{\pi}^+(x_{k+1}, \theta_{k+1}))$. By the tower property of the conditional expectation we know that $\mathbb{E}[\mathbb{E}[\cdot \mid \mathfrak{F}_{k+1}] \mid \mathfrak{F}_k] = \mathbb{E}[\cdot \mid \mathfrak{F}_k]$ since $\mathfrak{F}_k \subseteq \mathfrak{F}_{k+1}$. We then have

$$\begin{aligned}
\mathcal{L}V_N^\star(x_k, \theta_k) &\leq \mathbb{E}\Bigg[\sum_{j=k+1}^{k+N-1} \ell(x_j, u_{j-k}^\star, \theta_j) + \ell(x_{k+N}, \bar{u}_s, \theta) - \\
&\qquad - \sum_{j=k}^{k+N-1} \ell(x_j, u_j, \theta_j) \mid \mathfrak{F}_k\Bigg] \\
&= \mathbb{E}\Big[\ell(x_s^{\mathrm{bet}(\theta_{k+N-1})}, \bar{u}_s^{\mathrm{bet}(\theta_{k+N-1}),\theta_{k+N}}, \theta_{k+N}) \\
&\qquad - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k) \mid \mathfrak{F}_k\Big] \\
&= \ell_N(\theta_k) - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k),
\end{aligned}$$

where $u_{j-k}^\star = u_{j-k}^\star(x_j, \theta_j)$ and this completes the proof. □

The irreducibility and aperiodicity assumptions (Assumption 1) imply the existence of a limiting probability vector $\pi = (\pi^1, \ldots, \pi^\nu) \in \mathbb{R}^\nu$ which satisfies $\pi P = \pi$ and does not depend on the initial distribution v (Levin et al., 2009).

Theorem 6. (Asymptotic performance). Let Assumption 3 hold and let $\{x_k\}_k$ be a sequence satisfying (4). Define the *asymptotic average cost* as the random variable

$$J := \mathbb{E}\Bigg[\limsup_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \ell(x_k, u_k, \theta_k)\Bigg] \tag{6a}$$

Then,

$$J \leq \ell_\infty := \sum_{i\in\mathcal{N}} \pi_i \ell_N(i). \tag{6b}$$

Proof. By taking asymptotic averages and the expectation with respect to $\mathfrak{F}_0$ on both sides of (5) and using Fatou's lemma we have

$$\begin{aligned}
&\mathbb{E}\Bigg[\liminf_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \mathcal{L}V_N^\star(x_k, \theta_k)\Bigg] \\
&\leq \mathbb{E}\Bigg[\liminf_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \ell_N(\theta_k) - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k)\Bigg] \\
&\leq \mathbb{E}\Bigg[\liminf_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \ell_N(\theta_k) \\
&\qquad - \limsup_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k)\Bigg] \\
&\leq \liminf_{T\to\infty} \mathbb{E}\Bigg[\tfrac{1}{T} \sum_{k=0}^{T-1} \ell_N(\theta_k)\Bigg] \\
&\qquad - \mathbb{E}\Bigg[\limsup_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k)\Bigg].
\end{aligned} \tag{7}$$

We now use the fact that $\mathbb{E}[\ell_N(\theta_k)] = \sum_{i\in\mathcal{N}} \pi_k^i \ell_N(i)$, where $\pi_k^i = \mathrm{P}[\theta_k = i]$ and since $\pi_k^i \to \pi^i$ as $k \to \infty$, we have that $\mathbb{E}[\ell_N(\theta_k)] \to \ell_\infty$ and the right hand side of (7) is equal to $\ell_\infty - J$.

Using (Patrinos et al., 2014, Lemma 19) and because of the fact that ℓ are nonnegative,

$$\mathbb{E}\left[\liminf_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1}\mathcal{L}V_N^\star(x_k,\theta_k)\right]$$
$$= \mathbb{E}\left[\liminf_{T\to\infty} \tfrac{1}{T}(V_N^\star(x_T,\theta_T) - V_N^\star(x_0,\theta_0))\right]$$
$$\geq \liminf_{T\to\infty}\left(-\tfrac{1}{T}V_N^\star(x_0,\theta_0)\right) = 0.$$

Combining the two results completes the proof. □

2.5 Mean-square stability

We will now study the conditions under which a Markovian system is mean square stable towards an equilibrium point.

Assumption 7. (Common optimal equilibrium). There exists one common optimal stationary point (x_s, u_s) for all modes which is the solution of the optimization problem in Definition 2. Without loss of generality we assume $x_s = 0$, $u_s = 0$.

Consider the following Markovian switching system

$$x_{k+1} = f(x_k, \theta_k), \tag{8}$$

and let $r_k = (\theta_0, \ldots, \theta_k)$ be an admissible *switching sequence* starting from θ_0. Let $\phi(k; x_0, r_k)$ be the trajectory of (8) with $\phi(0; x_0, r_0) = x_0$.

Definition 8. (Mean Square Stability). We say that (8) is *mean square stable* if $\mathbb{E}[\|\phi(k; x_0, r_k)\|^2] \to 0$, as $k \to \infty$ for all x_0 and θ_0.

We extend the notion of dissipativity to Markovian systems as follows

Definition 9. (Stochastic dissipativity). We say that system (8) is *stochastically dissipative* with respect to a stochastic supply rate $s : \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}$ if there is a function $\lambda : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}$, lower semicontinuous in the first argument, so that for all $x_k \in \mathbb{R}^n$ and $\theta_k \in \mathcal{N}$

$$\mathcal{L}\lambda(x_k, \theta_k) \leq s(x_k, u_k, \theta_k). \tag{9}$$

where $\mathcal{L}\lambda(x_k, \theta_k) := \mathbb{E}[\lambda(x_{k+1}, \theta_{k+1}) - \lambda(x_k, \theta_k) \mid \mathfrak{F}_k]$. We say that (1) is *strictly stochastically dissipative* with respect to s if there is a convex function $\rho : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}_+$, positive definite with respect to x_s, so that the left hand side of (9) is no larger than $s(x_k, u_k, \theta_k) - \rho(x_k, \theta_k)$.

Assumption 10. (Strict stochastic dissipativity). Function $\lambda(x_s, \theta)$ is independent of θ and let $\lambda_s := \lambda(x_s, \theta)$. In addition to Assumption 7, system (8) is strictly stochastically dissipative with supply rate $s(x, u, \theta) = \ell(x, u, \theta) - \ell_s$.

Let us define the *rotated stage cost* function as

$$L(x_k, u_k, \theta_k) := \ell(x_k, u_k, \theta_k) - \mathcal{L}\lambda(x_k, \theta_k). \tag{10}$$

We now define the *rotated cost function* $\widetilde{V}_N(x, \theta, \mathbf{u}_N)$ as follows

$$\widetilde{V}_N(x_0, \theta_0, \mathbf{u}_N) = \mathbb{E}\left[\sum_{j=0}^{N-1} L(x_j, u_j, \theta_j)\middle|\mathfrak{F}_0\right]$$

using again $V_f = 0$ and we introduce the rotated MPC problem

$$\bar{\mathbb{P}}(x, \theta) : \widetilde{V}_N^\star(x, \theta) = \inf_{\mathbf{u}_N} \widetilde{V}_N(x, \theta, \mathbf{u}_N), \tag{11}$$

subject to (3b)–(3f).

Lemma 11. Problem $\bar{\mathbb{P}}(x, \theta)$ is recursively feasible and it has the same set of minimizers as $\mathbb{P}(x, \theta)$. Let $\tilde{\kappa}_N$ be the receding horizon control law which accrues from $\bar{\mathbb{P}}(x, \theta)$. If Assumption 10 holds, then

$$\mathcal{L}\widetilde{V}_N^\star(x_k, \theta_k) \leq -\rho(x_k, \theta_k), \tag{12}$$

where $\rho : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}_+$ is a positive definite function in the first argument with respect to x_s.

Proof. Problems $\mathbb{P}$ and $\bar{\mathbb{P}}$ have the same set of constraints, therefore, they have the same feasibility domain and the recursive feasibility of $\bar{\mathbb{P}}$ follows from Prop. 4. The rotated cost function can be expanded as $\widetilde{V}_N(x_k, \theta_k, \mathbf{u}_N) = \mathbb{E}[\sum_{j=k}^{k+N-1} L(x_j, u_j, \theta_j) \mid \mathfrak{F}_k] = \mathbb{E}[\sum_{j=k}^{k+N-1} \ell(x_j, u_j, \theta_j) - \mathcal{L}\lambda(x_k, \theta_k) \mid \mathfrak{F}_k]$. We now use the fact that $\mathbb{E}[\sum_{j=k}^{k+N-1} \mathcal{L}\lambda(x_k, \theta_k) \mid \mathfrak{F}_k] = \mathbb{E}[\lambda(x_{k+N-1}, k+N-1) - \lambda(x_k, \theta_k) \mid \mathfrak{F}_k] = \lambda_s - \lambda(x_k, \theta_k)$. Therefore, $\widetilde{V}_N(x_k, \theta_k, \mathbf{u}_N) = V_N(x_k, \theta_k, \mathbf{u}_N) + \lambda(x_k, \theta_k) - \lambda_s$.

The rotated and original cost functions differ only by a constant so the two problems, $\mathbb{P}$ and $\bar{\mathbb{P}}$, share a common optimal sequence. Proceeding as in Lemma 5 the following holds

$$\mathcal{L}\tilde{V}_N^\star(x_k, \theta_k) \leq \ell_s - L(x_k, \tilde{\kappa}_N(x_k, \theta_k), \theta_k), \tag{13}$$

By tracing the arguments of Rawlings et al. (2012) we have that $L(x_k, u_k, \cdot) \geq \ell_s$. Combining (10) and Assumption 10 we arrive at

$$L(x_k, u_k, \theta_k) \geq \rho(x_k, \theta_k) + \ell_s, \tag{14}$$

which completes the proof. □

Next, we draw an additional assumption on $\rho(\cdot, \theta)$:

Assumption 12. (Quadratic lower bound). There exist a positive constant γ, such that $\rho(x, i) \geq \gamma\|x - x_s\|^2$ holds for all x.

Theorem 13. Suppose Assumption 12 is satisfied. Then, system (8) is MSS.

Proof. All assumptions required by (Patrinos et al., 2014, Theorem 24) are met and entail mean square stability. □

3. UNIFORM INVARIANCE AND TERMINAL CONSTRAINTS

In this section we relax the restrictive requirement $x_N = x_s^{\mathrm{bet}(\theta_{N-1})}$ and we instead replace it with a terminal constraint of the form $(x_N, \theta_N) \in X^f$ along with a terminal penalty function V_f and we derive conditions so that the controlled system is mean-square stable.

We will now make use of the following definition (Patrinos et al., 2014)

Definition 14. (Uniform positive invariance). A family of nonempty sets $C = \{C_i\}_{i\in\mathcal{N}}$ is said to be *uniformly positive invariant* (UPI) for the constrained Markovian switching system (8) if for every $x_k \in C_{\theta_k}$, $x_{k+1} \in C_{\theta_{k+1}}$.

As before, we assume that there is one stationary point ℓ_s and require, with a slight abuse of notation, that $\lambda_s = \lambda(x_s, \theta), V_f(x_s) = V_f(x_s, \theta)$ for all $\theta \in \mathcal{N}$. Now we make a central assumption regarding our exposition

Assumption 15. (Terminal control law). There exists a control law $\kappa_f : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ and a collection of sets $X^f = \{X_i^f\}_{i\in\mathcal{N}}$ so that

i. X^f is UPI for the closed-loop system controlled by κ_f and
ii. for all $(x, \theta) \in X^f$

$$\mathcal{L}V_f(x_k, \theta_l) \leq -\ell(x_k, \kappa_f(x_k, \theta_k), \theta_k) + \ell_s. \tag{15}$$

We now consider the following stochastic economic model predictive control problem:

$$\mathbb{P}_T(x,\theta) : V_N^\star(x,\theta) = \inf_{\mathbf{u}_N} V_N(x,\theta,\mathbf{u}_N) \tag{16a}$$

and for $k = 0, \ldots, N-1$, it is subject to

$$x_{k+1} = f(x_k, u_k, \theta_k) \tag{16b}$$
$$(x_k, u_k) \in Y_{\theta_k} \tag{16c}$$
$$(x_0, \theta_0) = (x, \theta) \tag{16d}$$
$$x_N \in X^f_{\theta_k} \tag{16e}$$
$$u_k \lhd \mathfrak{F}_k. \tag{16f}$$

Again, the same reasoning as in Section 2.2 applies regarding the existence of optimal solutions. Let $\hat{\mathbf{u}}^\star(x,\theta) = \{u_0^\star(x,\theta), \ldots, u_{N-1}^\star(x_{N-1}, \theta_{N-1})\}$ be an optimizer of (16). The receding horizon control law is given by $\hat{\kappa}_N(x,\theta) := u_0^\star(x,\theta)$.

In light of the state-input constraints (16c) we must require that the sets X_i^f in Assumption 15 are subsets of X_N, the feasibility domain of $\mathbb{P}_T$.

3.1 Recursive feasibility

Here, we will show that stochastic economic model predictive control problem (16) is recursively feasible.

Proposition 16. Let $X_N \subseteq \mathbb{R}^n \times \mathcal{N}$ be the feasibility domain of $\mathbb{P}_T$ and let Assumption 15-i hold. Then, X_N is UPI for the MPC-controlled system.

Proof. For given $(x,\theta) \in X_N$ let $\pi(x,\theta) = \{u_0^\star, \ldots, u_{N-1}^\star\}$ be an optimizer of $\mathbb{P}_T(x,\theta)$ and let $x^\star(x,\theta) = \{x, x_1^\star, \ldots, x_N^\star\}$ be the corresponding sequence of states. Because of (16e) we have $x_N^\star \in X^f_{\theta_N}$. Now take $x^+ = f(x, u_0^\star(x,\theta), \theta)$, $\theta^+ \in \mathcal{C}(\theta)$ and let $\tilde{\pi}^+(x^+, \theta^+) := \{u_1^\star, \ldots, u_{N-1}^\star, u_f\}$ where $u_f = \kappa_f(x_N, \theta_N)$. Then, since X^f is a UPI set, $(x_{N+1}, \theta_{N+1}) \in X^f$, so $\tilde{\pi}^+$ satisfies the constraints of $\mathbb{P}_T(x^+, \theta^+)$. □

3.2 Expected asymptotic average performance

We show next that the asymptotic average cost of the EMPC-controlled system with terminal constraints is no higher than the cost of the best stationary point.

Theorem 17. Let Assumption 15 hold and let $\{x_k\}_k$ be a sequence satisfying (4) with $\hat{\kappa}_N(x_k, \theta_k)$. Then, $J \leq \ell_s$.

Proof. Using the optimal solution $\pi(x,\theta)$ of (16) with initial conditions (x,θ) we construct a feasible shifted policy $\pi^+(x^+,\theta^+)$ as in the proof of the Prop. 16. Then $V_N^\star(x_{k+1}, \theta_{k+1}) \leq V_N(x^+, \tilde{\pi}^+, \theta^+)$ and

$$\begin{aligned}\mathcal{L}V_N^\star(x_k,\theta_k) = \mathbb{E}\Bigg[&\sum_{j=k+1}^{k+N-1} \ell(x_j, u_j^\star, \theta_j) \\ &+ \ell(x_{k+N}, \kappa_f(x_{k+N}, \theta_{k+N}), \theta_{k+N}) + V_f(x_{k+N+1}, \theta_{k+N+1}) \\ &- \sum_{j=k}^{k+N-1} \ell(x_j, u_j^\star, \theta_j) - V_f(x_{k+N}, \theta_{k+N}) \mid \mathfrak{F}_k\Bigg] \\ \leq{}& \ell_s - \ell(x, \hat{\kappa}_N(x,\theta), \theta).\end{aligned}$$

Here, we used the tower property and Assumption 15. Proceeding as in Thm. 6 we prove the assertion. □

3.3 Mean-square stability

In this section we will give conditions under which a Markovian system with terminal region constraint is mean-square stable towards a common equilibrium point. Once again, our main argument will be the equivalence between the original and a suitably *rotated* problem.

We define the following *rotated terminal function*

$$\tilde{V}_f(x_k, \theta_k) = V_f(x_k, \theta_k) + \lambda(x_k, \theta_k) - V_f(x_s) - \lambda_s. \tag{17}$$

Combining condition (9) (Definition 9) with the rotated stage cost we may easily derive

$$L(x_k, u_k, \theta_k) \geq \rho(x_k, \theta_k). \tag{18}$$

Lemma 18. Suppose Assumption 15 holds. Then

$$\mathcal{L}\tilde{V}_f(x_k, \theta_k) \leq -L(x_k, \kappa_f(x_k, \theta_k), \theta_k). \tag{19}$$

Proof. By adding $\mathcal{L}\lambda(x_k, \theta_k)$ to both sides of (15) we get

$$\begin{aligned}\mathcal{L}\tilde{V}_f(x_k,\theta_k) + \mathcal{L}\lambda(x_k,\theta_k) \leq{}& -\ell(x_k, \kappa_f(x_k,\theta_k), \theta_k) + \ell_s \\ &+ \mathcal{L}\lambda(x_k,\theta_k).\end{aligned}$$

The right hand side is equal to the rotated stage cost

$$\begin{aligned}\mathbb{E}\Bigg[&V_f(f(x_k, \kappa_f(x_k,\theta_k)), \theta_{k+1}) + \lambda(x_{k+1}, \theta_{k+1}) \\ &- V_f(x_k,\theta_k) - \lambda(x_k,\theta_k) \mid \mathfrak{F}_k\Bigg] \leq -L(x_k, \kappa_f(x_k,\theta_k), \theta_k).\end{aligned}$$

We add $V_f(x_s) + \lambda_s - V_f(x_s) - \lambda_s$ to the left hand side and, after rearranging, arrive at (19). □

Now, we introduce the *rotated stochastic economic MPC problem*

$$\bar{\mathbb{P}}_T(x,\theta) : \tilde{V}_N^\star(x,\theta) = \inf_{\mathbf{u}_N} \tilde{V}_N(x,\theta,\mathbf{u}_N) \tag{21}$$

subject to (16b)-(16f).

Theorem 19. Problem $\bar{\mathbb{P}}_T(x,\theta)$ is recursively feasible and has the same set of minimizers as $\mathbb{P}_T(x,\theta)$.

Proof. Problems $\mathbb{P}_T$ and $\bar{\mathbb{P}}_T$ have the same set of constraints, therefore, they have the same feasibility domains and the recursive feasibility of $\bar{\mathbb{P}}$ follows from Prop. 16. The rotated cost is $\tilde{V}_N(x_k, \theta_k, \mathbf{u}_k) = \mathbb{E}[\sum_{j=k}^{k+N-1} L(x_j, u_j, \theta_j) + \tilde{V}_f(x_N, u_N, \theta_N) \mid \mathfrak{F}_k] = \mathbb{E}[\sum_{j=k}^{N-1}(\ell(x_j, u_j, \theta_j) + \lambda(x_j, \theta_j) - \mathbb{E}[\lambda(x_{j+1}, \theta_{j+1} - \ell_s) \mid \mathfrak{F}_j]) + V_f(x_N, \theta_N) + \lambda(x_N, \theta_N) - V_f(x_s) - \lambda_s) \mid \mathfrak{F}_N] \mid \mathfrak{F}_k] = V_N(x, \mathbf{u}, \theta) + \lambda(x, \theta) - N\ell_s - V_f(x_s) - \lambda_s$. The two cost functions, V_N and $\tilde{V}_N$ differ by feedback-invariant quantities, hence, the optimal solutions of the two problems coincide. □

Theorem 20. Suppose Assumptions 12 and 15 are satisfied. Then, (4) is MSS with domain of attraction X_N.

Proof. It follows from (Patrinos et al., 2014, Thm. 24). □

3.4 Linearization-based design

In this section we demonstrate how to design a terminal cost function and give a terminal control law using local linearization around the origin. In other words, we give conditions under which Assumption 15-ii is satisfied, given that Assumption 15-i holds for a nonlinear system with a particular control law. In the next section we shall also demonstrate how to design an ellipsoidal set X^f such that it satisfies Assumption 15-i.

To simplify the notation let $\bar{\ell}(x,\theta) = \ell(x,\kappa_f(x,\theta),\theta) - \ell(0,0,\theta)$ for all $\theta \in \mathcal{N}$, be a shifted stage cost function. Define $\hat{f}_\theta(x) := f(x,\kappa_f(x,\theta),\theta)$, where $\kappa_f(x,\theta)$ is a terminal control law that we will introduce shortly. The evolution of the nonlinear system is described by $x_{k+1} = \hat{f}_\theta(x_k)$, for all $\theta \in \mathcal{N}$. To proceed we need the following assumption which is weaker than twice differentiability which is commonly used in the literature (Rawlings and Mayne, 2009).

Assumption 21. (Smoothness). Functions $\hat{f}_\theta(x)$ are β_f^θ-smooth and $\bar{\ell}(x,\theta)$ are β_ℓ^θ-smooth for all $\theta \in \mathcal{N}$.

Let

$$z_{k+1} = A_{\theta_k} z_k + B_{\theta_k} u_k \tag{22}$$

be the corresponding linearized Markovian jump linear systems (MJLS), where $A_i = \frac{\partial f_i}{\partial x}(0,0)$ and $B_i = \frac{\partial f_i}{\partial u}(0,0)$ for all $i \in \mathcal{N}$. Hereafter, we assume that

Assumption 22. The set of pairs $\{(A_i,B_i)\}_{i\in\mathcal{N}}$ is mean-square stabilizable.

Costa et al. (2005) provide conditions for Assumption 22 to hold. We recall the following result for MJLS (Patrinos et al., 2014)

Proposition 23. (MSS of MJLS). Consider system (22) subject to (2) in closed loop with $\kappa(x,i) = K_i x$. Suppose there is a UPI set X^f and matrices $P^f = \{P_i^f\}_{i\in\mathcal{N}}$ so that $P_i^f \succcurlyeq \Gamma_i^\top \mathscr{E}_i(P^f)\Gamma_i + Q_i^*$ with $\Gamma_i := A_i + B_i K_i$, $\mathscr{E}_i(P^f) := \sum_{j\in\mathcal{C}(i)} p_{ij} P_j^f$ and $Q_i^* = (Q_i^*)^\top \succ 0$. Then, the closed-loop system is MS stable in X^f.

Next, we will design a terminal cost function $V_f(x,\theta)$ which, under certain assumptions (see Thm. 25) satisfies a desired Lyapunov-type inequality (see Assumption 15-ii).

First, we design a quadratic cost function $\ell_q(x,\theta)$ which is an upper bound on the shifted cost.

Lemma 24. Let $\ell_q(x,\theta) := \frac{1}{2}x^\top Q_\theta^* x + q_\theta^\top x$ where $Q_\theta^* = (\alpha + \beta_\ell^\theta)I, q_\theta = \nabla\bar{\ell}(0,\theta)$. Then it holds that $\ell_q(x,\theta) \geq \bar{\ell}(x,\theta) + \frac{\alpha}{2}\|x\|^2$ for any $\alpha > 0$, for all $\theta \in \mathcal{N}$.

Proof. By Assumption 21 on $\bar{\ell}(x,\theta)$, we have that $|\bar{\ell}(x,\theta) - q_\theta^\top x| \leq \beta_l^\theta/2\|x\|^2$. Adding $\alpha/2\|x\|^2$ to both sides the assertion follows. □

We may now choose our terminal cost to be the following infinite sum

$$V_f(x,i) = \mathbb{E}\left[\sum_{k=0}^{\infty} \ell_q(x_k,\theta_k) \,\middle|\, \mathfrak{F}_0\right], \tag{23}$$

for the MJLS $x_{k+1} = \Gamma_{\theta_k} x_k$, with $x_0 = x, \theta_0 = \theta$.

Using the linearity of expectation we have $V_f(x,\theta) = \mathbb{E}\left[\sum_{k=0}^\infty \frac{1}{2}x_k^\top Q_{\theta_k}^* x_k\right] + \mathbb{E}\left[\sum_{k=0}^\infty q_{\theta_k}^\top x_k\right]$ and V_f can be written in the form

$$V_f(x,i) = \tfrac{1}{2}x^\top P_i^f x + p_i^\top x, \tag{24}$$

where P_i^f are computed as in Prop. 23 with = in lieu of $\succcurlyeq$ (Costa et al., 2005, Prop. 3.20). Because of the parametrization of Q_i^* in Lemma 24, we may choose $P_i^f = P_i^\beta + \alpha P_i^I$ and require that

$$P_i^I = I + \Gamma_i^\top \mathscr{E}_i(P^I)\Gamma_i, \tag{25a}$$

$$P_i^\beta = \beta_\ell^i I + \Gamma_i^\top \mathscr{E}_i(P^\beta)\Gamma_i. \tag{25b}$$

For convenience we re-introduce the operator $\mathcal{L}$, but this time with a distinction between nonlinear and linear systems:

i. $\mathcal{L}V_f(x_k,\theta_k) = \mathbb{E}[V_f(\hat{f}_{\theta_k}(x_k),\theta_{k+1}) - V_f(x_k,\theta_k) \mid \mathfrak{F}_k]$
ii. $\mathcal{L}V_f^{\text{lin}}(x_k,\theta_k) = \mathbb{E}[V_f(\Gamma_{\theta_k}x_k,\theta_{k+1}) - V_f(x_k,\theta_k) \mid \mathfrak{F}_k]$.

Parameter α will be used to bound the mismatch between $\mathcal{L}V_f(x_k,\theta_k)$ and $\mathcal{L}V_f^{\text{lin}}(x_k,\theta_k)$ and a method for choosing it is presented in the proof of the next theorem.

Theorem 25. Consider the control law $\kappa_f(x,i) = K_i x$ and let Assumptions 21 and 22 hold. Then $\mathcal{L}V_f(x,\theta) \leq -\bar{\ell}(x,\theta)$ for $x \in \mathcal{B}_\delta$ for some $\delta > 0$. If X^f satisfies Assumption 15-i with $X_i^f \subseteq \mathcal{B}_\delta$ and Assumption 12 is satisfied, the controlled system is locally mean square stable.

Proof. Let us introduce the shorthand $\Delta\mathcal{L}V_f(x_k,\theta_k) := \mathbb{E}[V_f(\hat{f}_{\theta_k}(x_k),\theta_{k+1}) - V_f(\Gamma_{\theta_k}x_k,\theta_{k+1}) \mid \mathfrak{F}_k]$. By the linearity of the conditional expectation $\mathcal{L}V_f(x,\theta) = \mathcal{L}V_f^{\text{lin}}(x,\theta) + \Delta\mathcal{L}V_f(x,\theta)$. By (23), the first term is $\mathcal{L}V_f^{\text{lin}}(x,\theta) = -\ell_q(x,\theta)$. The last term is $\Delta\mathcal{L}V_f(x,\theta) = \frac{1}{2}e(x,\theta)^\top\mathscr{E}_\theta(P^f)e(x,\theta) - (\Gamma_\theta x)^\top\mathscr{E}_\theta(P^f)e(x,\theta) + \mathscr{E}_\theta(p)^\top \cdot e(x,\theta)$, where $e(x,\theta) := \hat{f}_\theta(x) - \Gamma_\theta x$ is the linearization error. Under Assumption 21, $\|e(x,\theta)\| \leq \frac{\beta_f^\theta}{2}\|x\|^2$ and $\Delta\mathcal{L}V_f(x,\theta) \leq \frac{(\beta_f^\theta)^2}{8}\|\mathscr{E}_\theta(P^f)\|\|x\|^4 + \frac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathscr{E}_\theta(P^f)\|\|x\|^3 + \frac{\beta_f^\theta}{2}\|\mathscr{E}_\theta(p)\|\|x\|^2$. We need to show that $\Delta\mathcal{L}V_f(x,\theta)$ is upper bounded by $\frac{\alpha}{2}\|x\|^2$ in a region of the origin for adequately large α. Recall that $\mathscr{E}_\theta(P^f)$ depends on α as follows

$$\mathscr{E}_\theta(P^f) = \mathscr{E}_\theta(P^\beta) + \alpha\mathscr{E}_\theta(P^I). \tag{26}$$

Using the triangle inequality

$$\begin{aligned}\Delta\mathcal{L}V_f(x,\theta) \leq{}& \tfrac{(\beta_f^\theta)^2}{8}\|\mathscr{E}_\theta(P^\beta)\|\|x\|^4 \\ &+ \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathscr{E}_\theta(P^\beta)\|\|x\|^3 + \tfrac{\beta_f^\theta}{2}\|\mathscr{E}_\theta(p)\|\|x\|^2 \\ &+ \alpha\left(\tfrac{(\beta_f^\theta)^2}{8}\|\mathscr{E}_\theta(P^I)\|\|x\|^4 + \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathscr{E}_\theta(P^I)\|\|x\|^3\right)\end{aligned} \tag{27}$$

For the right hand side of the last inequality to be upper bounded by $\frac{\alpha}{2}\|x\|^2$ it suffices to take $x \in \mathcal{B}_\delta$ with $\delta > 0$ and

$$\max_{\theta\in\mathcal{N}} \tfrac{(\beta_f^\theta)^2}{8}\|\mathscr{E}_\theta(P^I)\|\delta^2 + \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathscr{E}_\theta(P^I)\|\delta < 1,$$

and α so that

$$\alpha \geq \max_{\theta\in\mathcal{N}} \frac{\frac{(\beta_f^\theta)^2}{8}\|\mathscr{E}_\theta(P^\beta)\|\delta^2 + \frac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathscr{E}_\theta(P^\beta)\|\delta + \frac{\beta_f^\theta}{2}\|\mathscr{E}_\theta(p)\|}{1 - \frac{(\beta_f^\theta)^2}{8}\|\mathscr{E}_i(P^I)\|\delta^2 + \frac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathscr{E}_\theta(P^I)\|\delta}.$$

Now for $x \in \mathcal{B}_\delta$ and α as above we have $\Delta\mathcal{L}V_f(x,\theta) \leq \frac{\alpha}{2}\|x\|^2$, and since $\mathcal{L}V_f(x,\theta) = -\ell_q(x,\theta) + \Delta\mathcal{L}V_f(x,\theta)$ we have $\mathcal{L}V_f(x,\theta) \leq -\ell_q(x,\theta) + \frac{\alpha}{2}\|x\|^2$, and employing Lemma 24 we obtain $\mathcal{L}V_f(x,\theta) \leq -\bar{\ell}(x,\theta)$. If Assumption 12 holds all assumptions of Thm. 20 are fulfilled and the controlled system is locally mean square stable. □

3.5 Computation of X^f

We demonstrate a possible way of finding X^f such that the requirements of Thm. 25 are satisfied. Take $X^f = \{X_i^f\}_{i\in\mathcal{N}}$ to be ellipsoidal of the form $X_i^f = \{x : x^\top P_i x \leq 1\}$. By Assumption 21, there exist constants $\gamma_i > 0$, $i \in \mathcal{N}$, such that

$$x_{k+1} = A_{\theta_k}x_k + B_{\theta_k}\kappa_f(x_k,\theta_k) + d_{k,\theta_k}, \tag{28}$$

with $\|d_{k,i}\|^2 \leq \gamma_i x_k^\top P_i^f x_k$ where $d_{k,i} = e(x_k,i)$ is the linearization error. For X^f to be UPI for the κ_f-controlled system it must satisfy

$$\max_{j\in\mathcal{C}(i)}\{x_{k+1}^\top P_j x_{k+1}\} \leq x_k^\top P_i x_k,\ \forall i \in \mathcal{N}$$

$$\Leftrightarrow \begin{bmatrix} x_k \\ d_{k,i}\end{bmatrix}^\top \begin{bmatrix} P_i - \Gamma_i^\top P_j \Gamma_i & -\Gamma_i^\top P_j \\ -P_j\Gamma_i & -P_j\end{bmatrix}\begin{bmatrix} x_k \\ d_{k,i}\end{bmatrix} \geq 0, \tag{29a}$$

for all $j \in \mathcal{C}(i)$ and $i \in \mathcal{N}$ whenever $d_{k,i}^\top d_{k,i} \leq \gamma_i x_k^\top P_i^f x_k$, or, for $i \in \mathcal{N}$

$$\begin{bmatrix} x_k \\ d_{k,i}\end{bmatrix}^\top \begin{bmatrix} \gamma_i P_i^f & \\ & -I\end{bmatrix}\begin{bmatrix} x_k \\ d_{k,i}\end{bmatrix} \geq 0. \tag{29b}$$

Using the S-lemma, (29b) implies (29a) so long as

$$\begin{bmatrix} P_i - \Gamma_i^\top P_j^f \Gamma_i & -\Gamma_i^\top P_j^f \\ -P_j\Gamma_i & -P_j^f\end{bmatrix} - \tau\begin{bmatrix} \gamma_i P_i^f & \\ & -I\end{bmatrix} \succcurlyeq 0 \tag{30}$$

for some $\tau \geq 0$ and for all $i \in \mathcal{N}$ and $j \in \mathcal{C}(i)$.

Introducing the variables $P_i^f = Z_i^{-1}$ and $K_i = Y_i Z_i^{-1}$, this is equivalent to the matrix inequality

$$\begin{bmatrix} Z_i & 0 & \tau Z_i & Z_i A_i^\top + Y_i^\top B_i^\top \\ 0 & \tau I & 0 & I \\ * & * & \tau\gamma_i^{-1} Z_i & 0 \\ * & * & 0 & Z_j\end{bmatrix} \succcurlyeq 0. \tag{31}$$

As required by Thm. 25, X_i^f must be in $\mathcal{B}_\delta$. This is equivalently written as

$$\begin{bmatrix} \delta I & P_i \\ P_i & I\end{bmatrix} \succcurlyeq 0. \tag{32}$$

We then choose P_i^f so as to satisfy (31) and (32) for all $i \in \mathcal{N}$ and $j \in \mathcal{C}(j)$. Note that (31) is a bilinear matrix inequality (BMI) with unknowns Z_i, Y_i and τ, but the bilinearity is only because of the term τZ_i.

4. CONCLUSIONS

This paper offers a theoretical framework for the control of Markovian switching systems using EMPC. We first studied a formulation with mode-dependent optimal steady states and terminal equality constraints for which we provided an upper bound on the expected asymptotic average cost (Thm. 6). We then studied an EMPC formulation with mode-dependent terminal region constraints and we provided design guidelines based on the system linearization assuming that the system dynamics and the stage cost function are β-smooth which are rather weak assumptions (Thm. 25).

REFERENCES

R. Amrit, J.B. Rawlings, and D. Angeli. Economic optimization using model predictive control with a terminal cost. *Ann. Rev. Contr.*, 35(2):178 – 186, 2011.

D. Angeli, R. Amrit, and J. B. Rawlings. On average performance and stability of economic model predictive control. *IEEE TAC*, 57(7):1615–1626, July 2012.

R.B. Ash. *Real analysis and probability.* Academic Press, 1972.

F.A. Bayer, M.A. Müller, and F. Allgöwer. Tube-based robust economic model predictive control. *J. Proc. Contr.*, 24(8):1237 – 1246, 2014.

F.A. Bayer, M. Lorenzen, M.A. Müller, and F. Allgöwer. Robust economic model predictive control using stochastic information. *Automatica*, 74:151–161, 2016.

T.I. Bø and T.A. Johansen. Dynamic safety constraints by scenario based economic model predictive control. *IFAC WC*, 47(3):9412 – 9418, 2014.

O.L.V. Costa, M.D. Fragoso, and R.P. Marques. *Discrete-time Markov Jump Linear Systems.* Springer, 2005.

L. Fagiano and A.R. Teel. Generalized terminal state constraint for model predictive control. *Automatica*, 49 (9):2622 – 2631, 2013.

L. Grüne. Economic receding horizon control without terminal constraints. *Automatica*, 49(3):725 – 734, 2013.

D.A. Levin, Y. Peres, and E.L. Wilmer. *Markov chains and mixing times.* AMS press, Rhode Island, 2009.

S. Lucia, J.A.E. Andersson, H. Brandt, A. Bouaswaig, M. Diehl, and S. Engell. Efficient robust economic nonlinear model predictive control of an industrial batch reactor. *IFAC WC*, 47(3):11093–11098, 2014a.

S. Lucia, J.A.E. Andersson, H. Brandt, M. Diehl, and S. Engell. Handling uncertainty in economic nonlinear model predictive control: A comparative case study. *J. Proc. Contr.*, 24(8):1247 – 1259, 2014b.

M.A. Müller, D. Angeli, and F. Allgöwer. On convergence of averagely constrained economic MPC and necessity of dissipativity for optimal steady-state operation. In *ACC*, pages 3141–3146, 2013.

M.A. Müller, D. Angeli, and F. Allgöwer. On the performance of economic model predictive control with self-tuning terminal cost. *J. Proc. Contr.*, 24(8):1179–1186, 2014.

P. Patrinos, P. Sopasakis, H. Sarmiveis, and A. Bemporad. Stochastic model predictive control for constrained discrete-time Markovian switching systems. *Automatica*, 50(10):2504–2514, October 2014.

J.B. Rawlings and D.Q. Mayne. *Model predictive control : theory and design.* Madison, Wis. Nob Hill Pub., 2009. ISBN 978-0-9759377-0-9.

J.B. Rawlings, D. Angeli, and C.N. Bates. Fundamentals of economic model predictive control. In *IEEE CDC*, pages 3851–3861, 2012.

R.T. Rockafellar and J.B. Wets. *Variational analysis.* Springer-Verlag, Berlin, 3rd edition, 2009.

A. Tejada, O.R. González, and W.S. Gray. On nonlinear discrete-time systems driven by Markov chains. *J. Frank. Inst.*, 347:795–805, 2010.

M. Zanon, G. Gros, and M. Diehl. A Lyapunov function for periodic economic optimizing model predictive control. In *IEEE CDC*, 2013.